A note on constructing digraphs with prescribed properties

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Abstract

Let n be a non-negative integer and k be a positive integer. A digraph D is said to have property $Q(n,k)$ if every subset of n vertices of D is dominated by at least k other vertices. For $q \equiv 5 \pmod 8$ a prime power, we define the quadruple Payley digraph $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite field $\mathbf{F}_q$. Vertex a is joined to vertex b by an arc if and only if $a - b = y^4$ for some $y \in \mathbf{F}_q$. In this paper, we show that for sufficiently large q, the digraph $D_q^{(4)}$ has property $Q(n,k)$.

1. Introduction

In this paper, our graphs are directed. For our purpose, all digraphs are finite and strict. If (x, y) is an arc in a digraph D, then we say vertex x **dominates** vertex y. A set of vertices A dominates a set of vertices B if every vertex of A dominates every vertex of B. A digraph D is said to have property Q(n,k) if every subset of n vertices of D is dominated by at least k other vertices. Further, a digraph D is said to have property Q(m,n,k) if for any set of m + n distinct vertices of D there exist at least k other vertices each of which dominates the first m vertices and is dominated by the latter n vertices.

A special digraph arises in round robin tournaments. More precisely, consider a tournament T_q with q players 1, 2, ..., q in which there are no draws. This gives rise to a digraphs in which either (a, b) or (b, a) is an arc for each pair a, b. Tournaments with property Q(n, k) have been studied by Ananchuen and Caccetta [2] Bollobás [3] and Graham and Spencer [4].

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Graham and Spencer [4] defined the following tournament. Let $p \equiv 3 \pmod 4$ be a prime. The vertices of digraph D_p are $\{0, 1, \ldots, p-1\}$ and D_p contains the arc (a, b) if and only if $a - b$ is a quadratic residue modulo p. The digraph D_p is sometimes referred to as the **Paley tournament**. Graham and Spencer [4] proved that D_p has property Q(n, 1) whenever $p > n^2 2^{2n-2}$. Bollobás [3] extended these results to prime powers. More specifically, if $q \equiv 3 \pmod 4$ is a prime power, the Paley tournament D_q is defined as follows. The vertex set of D_q are the elements of the finite field $\mathbf{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadratic residue in $\mathbf{F}_q$. Bollobás [3] noted that D_q has property Q(n,1) whenever

$$q > \{(n-2)2^{n-1} + 1\}\sqrt{q} + n2^{n-1}.$$

In [2], Ananchuen and Caccetta proved that D_q has property Q(n,k) whenever

$$q > \{(n-3)2^{n-1} + 2\}\sqrt{q} + k2^n - 1.$$

Ananchuen and Caccetta [2] proved that D_q has property Q(m,n,k) for every

$$q > \{(t-3)2^{t-1} + 2\}\sqrt{q} + (t + 2k - 1)2^{t-1} - 1,$$

where $t = m + n$.

By using higher order residues on finite fields we can generate other classes of digraphs. Let $q \equiv 5 \pmod 8$ be a prime power. Define the quadruple Paley digraph $D_q^{(4)}$ as follows. The vertices of $D_q^{(4)}$ are the elements of the finite fields $\mathbf{F}_q$. Vertex a joins to vertex b by an arc if and only if $a - b$ is a quadruple in $\mathbf{F}_q$; that is $a - b = y^4$ for some $y \in \mathbf{F}_q$. Since $q \equiv 5 \pmod 8$ is a prime power, -1 is not a quadruple in $\mathbf{F}_q$. The condition -1 is not a quadruple in $\mathbf{F}_q$ is needed to ensure that (b, a) is not defined to be an arc when (a, b) is defined to be an arc. Consequently, $D_q^{(4)}$ is well-defined. However, $D_q^{(4)}$ is not a tournament. The figure below displays the digraph $D_{13}^{(4)}$.

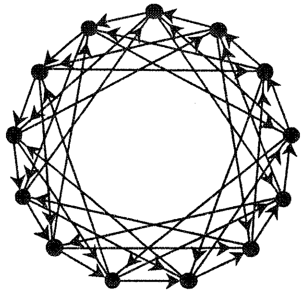

Figure 1. Paley digraph $D_{13}^{(4)}$

In this paper, we will show that $D_q^{(4)}$ has property Q(n,k) whenever

$$q > [1 + (3n-4)4^{n-1}]\sqrt{q} + (4k-3)4^{n-1},$$

and has property Q(m,n,k) whenever

$$q > (t2^{t-1} - 2^t + 1)3^m\sqrt{q} + (t + 4k - 4)3^{-n}4^{t-1},$$

where $t = m + n$.

In the next section we present some preliminary results on finite fields which we mark use of in the proof of our main results.

2. PRELIMINARIES

We make use of the following basic notation and terminology.

Let $\mathbf{F}_q$ be a finite field of order q where q is a prime power. A **character** χ of $\mathbf{F}_q^*$, the multiplicative group of the non–zero elements of $\mathbf{F}_q$, is a map from $\mathbf{F}_q^*$ to the multiplicative group of complex numbers with $|\chi(x)| = 1$ for all $x \in \mathbf{F}_q^*$ and with $\chi(xy) = \chi(x)\chi(y)$ for any $x, y \in \mathbf{F}_q^*$. Among the characters of $\mathbf{F}_q^*$, we have the **trivial character** χ_o defined by $\chi_o(x) = 1$ for all $x \in \mathbf{F}_q^*$; all other characters of $\mathbf{F}_q^*$ are called **nontrivial**. A character χ is of **order** d if $\chi^d = \chi_o$ and d is the smallest positive integer with this property. It will be convenient to extent the definition of nontrivial character χ to the whole $\mathbf{F}_q$ by defining $\chi(0) = 0$. For χ_o we define $\chi_o(0) = 1$.

Let g be a fixed primitive element of the finite field $\mathbf{F}_q$; that is g is a generator of the cyclic group $\mathbf{F}_q^*$. Define a function β by

$$\beta(g^t) = i^t,$$

where $i^2 = -1$. Therefore, β is a quadruple character, character of order 4, of $\mathbf{F}_q$. The values of β are the elements of the set $\{1, -1, i, -i\}$. Observe that β^3 is also a quadruple character while β^2 is a quadratic character. Moreover, if a is not a quadruple of an element of $\mathbf{F}_q^*$, then $\beta(a) + \beta^2(a) + \beta^3(a) = -1$.

The following lemmas were proved in [1].

Lemma 2.1. Let β be a quadruple character of $\mathbf{F}_q$ and let A be a subset of n vertices of $\mathbf{F}_q$. Put

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \geq q - [1 + (3n-4)4^{n-1}]\sqrt{q}.$$ ❑

Lemma 2.2. Let β be a quadruple character of $\mathbf{F}_q$ and let A and B be disjoint subsets of $\mathbf{F}_q$. Put

$$g = \sum_{x \in F_q} \prod_{a \in A} \{1 + \beta(x-a) + \beta^2(x-a) + \beta^3(x-a)\} \prod_{b \in B} \{3 - \beta(x-b) - \beta^2(x-b) - \beta^3(x-b)\}.$$

As usual, an empty product is defined to be 1. Then

$$g \geq 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q},$$

where $|A| = m$, $|B| = n$ and $t = m + n$. ❑

We conclude this section by noting that for $q \equiv 5 \pmod 8$ a prime power, there exists a quadruple character β of $\mathbf{F}_q$ and $\beta(-a) = -\beta(a)$ for all $a \in \mathbf{F}_q$. Furthermore, if a and b are any vertices of $G_q^{(4)}$, then for t = 1 and 3

$$\beta^{t}(a-b)=\begin{cases}1, & \text{if a dominates b,}\\ 0, & \text{if } a=b,\\ -1, i \text{ or } -i, & \text{otherwise.}\end{cases}$$

Note that β^2 is a quadratic character; that is

$$\beta^2(a-b)=\begin{cases}1, & \text{if } a-b \text{ is a qudratic ressidue,}\\ 0, & \text{if } a=b,\\ -1, & \text{otherwise.}\end{cases}$$

3. Results

Our first result concerns quadruple Paley digraphs having property Q(n,k).

Theorem 3.1. Let $q \equiv 5 \pmod 8$ be a prime power and k a positive integer. If

$$q > [1+(3n-4)4^{n-1}]\sqrt{q} + (4k-3)4^{n-1}, \tag{3.1}$$

then $D_q^{(4)}$ has property Q(n,k).

Proof: Let A be subsets of n vertices of $D_q^{(4)}$. Then, there are at least k other vertices each of which dominates A if and only if

$$h=\sum_{\substack{x\in F_q\\ x\notin A}}\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}\geq k4^n.$$

To show that $h \geq k4^n$, it is clearly sufficient to establish that $h > (k-1)4^n$.

Let

$$g=\sum_{x\in F_q}\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}.$$

Then, by Lemma 2.1, we have

$$g \geq q-[1+(3n-4)4^{n-1}]\sqrt{q}.$$

Consider

$$g-h=\sum_{x\in A}\prod_{i=1}^{n}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}.$$

If $g-h\neq 0$, then for some a_k the product

$$\prod_{i=1}^{n}\{1+\beta(a_k-a_i)+\beta^2(a_k-a_i)+\beta^3(a_k-a_i)\}\neq 0. \tag{3.2}$$

For (3.2) to hold we must have $\beta(a_k - a_i) + \beta^2(a_k - a_i) + \beta^3(a_k - a_i) \neq -1$ for all i. This means that for $i \neq k$, $\beta(a_k - a_i) + \beta^2(a_k - a_i) + \beta^3(a_k - a_i) = 3$. Hence a_k dominates all other vertices in A. Therefore a_k is unique and $g - h = 4^{n-1}$. Then, since $g - h$ could be 0 we conclude that $g - h \leq 4^{n-1}$ and so

$$\begin{aligned} h &\geq g - 4^{n-1} \\ &\geq q - [1 + (3n-4)4^{n-1}]\sqrt{q} - 4^{n-1}. \end{aligned}$$

Now, if inequality (3.1) holds, then $h > (k - 1)4^n$ as required. As A is arbitrary, this completes the proof. ❑

For the property Q(m,n,k), we have the following result.

Theorem 3.2. Let $q \equiv 5 \pmod 8$ be a prime power and k a positive integer. If

$$q > (t2^{t-1} - 2^t + 1)3^m\sqrt{q} + (t + 4k - 4)3^{-n}4^{t-1}, \tag{3.3}$$

then $D_q^{(4)}$ has property Q(m,n,k) for all m, n with $t = m + n$.

Proof: Let A and B be disjoint subsets of vertices of $D_q^{(4)}$ with $|A| = m$ and $|B| = n$. Then, there are at least k vertices, each of which is dominates every vertex of A but is dominated by every vertex of B if and only if

$$\begin{aligned} h = &\sum_{\substack{x\in F_q\\ x\notin A\cup B}}\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}\prod_{b\in B}\{3-\beta(x-b)-\beta^2(x-b)-\beta^3(x-b)\} \\ &> (k-1)4^t. \end{aligned}$$

Let

$$g = \sum_{x\in F_q}\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}\prod_{b\in B}\{3-\beta(x-b)-\beta^2(x-b)-\beta^3(x-b)\}.$$

Using Lemma 2.2 we have

$$g \geq 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q}.$$

Consider

$$g - h = \sum_{x\in A\cup B}\prod_{a\in A}\{1+\beta(x-a)+\beta^2(x-a)+\beta^3(x-a)\}\prod_{b\in B}\{3-\beta(x-b)-\beta^2(x-b)-\beta^3(x-b)\}.$$

Since, in each product, each factor is at most 4 and one factor is 1, so each of these terms is at most 4^{t-1} we have

$$g - h \le t4^{t-1}.$$

Consequently,

$$h \ge 3^n q - (t2^{t-1} - 2^t + 1)3^t\sqrt{q} - t4^{t-1}.$$

Now, if inequality (3.3) holds, then $h > (k-1)4^t$ as required. Since A and B are arbitrary, this completes the proof of the theorem. ❑

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