Note on k-contractible edges in k-connected graphs

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Abstract

It is proved that if G is a k-connected graph which does not contain K_4^- with k being odd, then G has an edge e such that the graph obtained from G by contracting e is still k-connected. The same conclusion does not hold when k is even. This result is a generalization of the famous theorem of Thomassen [J. Graph Theory **5** (1981), 351–354] when k is odd.

1 Introduction

In this paper, all graphs considered are finite, undirected, and without loops or multiple edges. For a graph G, $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices and the set of edges and the minimum degree of G, respectively. For a given graph G and $v \in V(G)$, we denote by $N_G(x)$ the neighbourhood of $V(G)$ and $d_G(x) = |N_G(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $\langle S \rangle$. We often use $|H|$ instead of $|V(H)|$.

Let $k \geq 2$ be an integer. An edge e of a k-connected graph is said to be *k-contractible* if the graph obtained from G by contracting e (and replacing each of the resulting pairs of double vertices by a single vertex) is still k-connected. A *k-cutset* is a cutset consisting of k vertices.

It is well known that every 3-connected graph of order 5 or more contains a 3-contractible edge. But, Thomassen [5] stated that there exist infinitely many k-connected k-regular graphs which do not have a k-contractible edge for $k \geq 4$.

Egawa [1] studied the minimum degree condition for a k-connected graph to have a contractible edge and proved the following theorem.

Theorem 1 *Let $k \geq 2$ be an integer, and let G be a k-connected graph with $\delta(G) \geq [\frac{5k}{4}]$. Then G has a k-contractible edge, unless $2 \leq k \leq 3$ and G is isomorphic to K_{k+1}.*

Thomassen [5] proved the following theorem.

Theorem 2 *Let G be a k-connected triangle-free graph. Then G contains an edge e such that the contraction of e results in a k-connected graph.*

Egawa et al. [2] proved the following theorem.

Theorem 3 *Let G be a k-connected triangle-free graph. Then G contains $\min\{|V(G)| + \frac{3}{2}k^2 - 3k, |E(G)|\}$ k-contractible edges.*

To see Theorem 3, a k-connected triangle-free graph has a lot of k-contractible edges. Hence the condition " triangle-free" may be too strong. In fact, we prove the following theorem.

Theorem 4 *Let K_4^- be the graph obtained from K_4 by removing just one edge. Let $k \geq 3$ be an odd integer, and let G be a k-connected graph which does not contain K_4^-. Then G has a k-contractible edge.*

The same conclusion does not hold when k is even. Let G be a graph $G = K_3 \times K_3 \times \cdots \times K_3 = K_3^{k/2}$ with k even. G is k-regular, k-connected and each edge is contained in only one triangle. Clearly, G does not contain K_4^- and G does not have a k-contractible edge.

Actually, we will prove a stronger result. The graph is said to be *minimally k-connected* if it is k-connected but on omitting any of the edges the resulting graph is no longer k-connected. We will prove the following theorem.

Theorem 5 *Let $k \geq 3$ be an odd integer, and let G be a minimally k-connected graph which does not contain K_4^-. Then G has a k-contractible edge.*

Theorem 5 implies Theorem 4. If G is not minimally k-connected, we can delete edges until G is minimally k-connected. The graph G' obtained from G by such edge deleting operation keeps the property that G' does not contain K_4^-. By Theorem 5, G' has a k-contractible edge e. Then clearly e is also a k-contractible edge in G.

2 Proof of Theorem 5

Before we prove Theorem 5, we need the following two lemmas due to Halin [3] and Mader [4].

Lemma 1 (Halin [3]) *Every minimally k-connected graph has a vertex whose degree is k.*

Lemma 2 (Mader [4]) *Let G be a minimally k-connected graph and let T be the set of vertices of degree k. Then $G - T$ is a (possibly empty) forest.*

Now, we turn back to the proof of Theorem 5. We prove the following lemmas.

Lemma 3 *Let $k \geq 3$ be an odd integer, and let G be a k-connected graph which does not contain K_4^-. Suppose $v \in V(G)$ is a vertex of degree k. Then there exists an edge e such that e is not contained in any triangle and v is in $V(e)$.*

Proof. Let W be the subgraph of G induced by $N_G(v)$. Since G does not contain K_4^-, every vertex in W has degree at most 1. Since W consists of odd vertices, there exists a vertex $w \in W$ of degree 0 in W. Then $e = uv$ is not contained in any triangle. ∎

Lemma 4 *Let $k \geq 3$ be an odd integer, and let G be a k-connected graph which does not contain K_4^-. Let A be a k-cutset such that A contains x and y where $xy \in E(G)$, and xy is not contained in any triangle. Let H be a component in $G - A$. Then $|H| \geq k - 1$.*

Proof. Let u be a vertex in H. Since $ux \notin E(G)$ or $uy \notin E(G)$, there exists an edge uv in H. Since G does not contain K_4^-, $|N_G(u) \cap N_G(v)| \leq 1$. So, $|N_G(u) \cup N_G(v)| \geq 2k - 1$. Hence, $|H| \geq 2k - 1 - |A| = k - 1$. ∎

Lemma 5 *Let $k \geq 3$ be an odd integer, and let G be a k-connected graph which does not contain K_4^-. Let A and A' be k-cutsets such that A contains $e = xy$ and A' contains $e' = x'y'$, where both e and e' are not contained in any triangle. Let H be a component in $G - A$. Then $H \not\subseteq A'$.*

Proof. Assume, not. Let $W = G - A - H$. Let H' be a component in $G - A'$ and also, let W' denote $G - A' - H'$. By Lemma 4, $|H|, |H'|, |W|, |W'| \geq k - 1$. Let H_1, H_2 and H_3 denote $H \cap H'$, $H \cap A'$ and $H \cap W'$, respectively. Also, let W_1, W_2 and W_3 denote $W \cap H'$, $W \cap A'$ and $W \cap W'$, respectively. Let Q_1, Q_2 and Q_3 denote $A \cap H'$, $A \cap A'$ and $A \cap W'$, respectively. Since $|A| = |A'| = k$, $|A| + |A'| = \sum_{i=1}^{3} |Q_i| + |W_2| + |Q_2| + |H_2| = 2k$. Also, by the assumption, $H_1 = H_3 = \emptyset$. Since $|H| \geq k - 1$, $|H_2| \geq k - 1$. Hence $|W_2 \cup Q_2| \leq 1$. Since $|W| \geq k - 1$ and $|W_2| \leq 1$, $W_1 \neq \emptyset$ or $W_3 \neq \emptyset$. First, assume $W_1 \neq \emptyset$. Since G is k-connected, $|W_2 \cup Q_1 \cup Q_2| \geq k$. Since $|W_2 \cup Q_2| \leq 1$, this implies $|Q_1| \geq k - 1$, and hence $|Q_3| \leq 1$. Since $|W'| = |W_3| + |Q_3| \geq k - 1$, we have $W_3 \neq \emptyset$. On the other hand, $|W_2 \cup Q_2 \cup Q_3| = |W_2 \cup Q_2| + |Q_3| \leq 2 < k$. This contradicts the connectivity of G.

Finally, assume $W_3 \neq \emptyset$. This case follows by using the same argument as in the proof of the previous paragraph. ∎

Now, we can finish the proof of Theorem 5. By Lemma 1, there exists a vertex x whose degree is k. By Lemma 3, there exists an edge xy which is not contained in any triangle. Since G does not have a k-contractible edge, any two adjacent vertices are contained in a k-cutset. Let A be a k-cutset such that A contains x and y. Let H be a component in $G - A$. And also, let $W = G - A - H$. We choose $\{x, y\}$, A and H such that $|H|$ is least possible. By Lemma 4, $|H| \geq k - 1$ and $|W| \geq k - 1$.

If, for each vertex $h \in H$, $d_G(h) \geq k + 1$, then, by Lemma 2, H must be a forest. Hence, there must exist a vertex s such that $d_H(s) = 1$. Therefore, s is adjacent to all the vertices in A. But, $\langle s, x, y \rangle$ is a triangle, a contradiction. So, we may assume that there exists a vertex $x' \in V(H)$ whose degree is k. By Lemma 3, there exists an edge $x'y'$ which is not contained in any triangle. Let A' be a k-cutset such that A' contains

x' and y'. Let H' be a component in $G - A'$. And also, let $W' = G - A' - H'$. By Lemma 4, $|H|, |H'|, |W|, |W'| \geq k-1$. Let H_1, H_2 and H_3 denote $H \cap H'$, $H \cap A'$ and $H \cap W'$, respectively. Also, let W_1, W_2 and W_3 denote $W \cap H'$, $W \cap A'$ and $W \cap W'$, respectively. Let Q_1, Q_2 and Q_3 denote $A \cap H'$, $A \cap A'$ and $A \cap W'$, respectively. By Lemma 5, $H_1 \neq \emptyset$ or $H_3 \neq \emptyset$. Without loss of generality, we may assume $H_1 \neq \emptyset$. Since $|A| = |A'| = k$, $|A| + |A'| = \sum_{i=1}^{3} |Q_i| + |W_2| + |Q_2| + |H_2| = 2k$.

We claim $W_3 = \emptyset$. Assume, not. Then, by the connectivity of G, $|W_2 \cup Q_2 \cup Q_3| \geq k$. By the minimality of H, $|Q_1 \cup Q_2 \cup H_2| \geq k+1$. Notice that the edge $x'y'$ is contained in $\langle Q_2 \cup H_2 \rangle$. But $2k = \sum_{i=1}^{3} |Q_i| + |W_2| + |Q_2| + |H_2| = |W_2 \cup Q_2 \cup Q_3| + |Q_1 \cup Q_2 \cup H_2| \geq k + k + 1 = 2k+1$, a contradiction. So, $W_3 = \emptyset$. By Lemma 5, we have $W \not\subseteq A'$ and hence, $W_1 \neq \emptyset$. Since G is k-connected, $|W_2 \cup Q_1 \cup Q_2| \geq k$.

If $H_3 \neq \emptyset$, then by the same argument as used for $H_1 \neq \emptyset$, we can conclude $W_1 = \emptyset$. This implies that $H_3 = \emptyset$. However, we have $W' \subseteq A$, which contradicts Lemma 5. ∎

References

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