Unique minimum domination in trees

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Abstract

A set D of vertices in a graph G is a distance-k dominating set if every vertex of G either is in D or is within distance k of at least one vertex in D. A distance-k dominating set of G of minimum cardinality is called a minimum distance-k dominating set of G. For any graph G and for a subset F of the edge set of G the set F is an edge dominating set of G if every edge of G either is in D or is adjacent to at least one edge in D. An edge dominating set of G of minimum cardinality is called a minimum edge dominating set of G. We characterize trees with unique minimum distance-k dominating sets, which is a generalization of a result of Gunther, Hartnell, Markus, and Rall. Further, we give a characterization of trees with unique minimum edge dominating sets, which contains some results of Topp.

1 Terminology and Introduction

For any graph G the vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, and $n(G) = |V(G)|$ and $m(G) = |E(G)|$. The number of components of G is denoted by $\kappa(G)$. For any subset $A \subseteq V(G)$ we define the induced subgraph $G[A]$ as the graph with vertex set A and edge set $\{ab \in E(G) \mid a, b \in A\}$. For any set $A \subseteq V(G)$ and any vertex $x \in V(G)$ we define $G - A = G[V(G) \setminus A]$ and $G - x = G - \{x\}$. For two vertices x and y in a connected graph G the *distance* $d(x, y)$ between x and y is the minimum number of edges of a path in G from x to y. If we define $e(v) = \max_{w \in V(G)} d(v, w)$, then the *diameter* of G is $\operatorname{diam}(G) = \max_{v \in V(G)} e(v)$ and the *radius* of G is $\operatorname{rad}(G) = \min_{v \in V(G)} e(v)$. For any vertex $x \in V(G)$ the open k-neighborhood of x, denoted $N_k(x)$, is the set $N_k(x) = \{y \in V(G) \mid y \neq x \text{ and } d(x, y) \leq k\}$ and the set $N_k[x] = N_k(x) \cup \{x\}$ is called the closed k-neighborhood of x. If $A \subseteq V(G)$, then $N_k(A) = \bigcup_{x \in A} N_k(x)$ and $N_k[A] = N_k(A) \cup A$. For a subset D of $V(G)$ and a vertex $x \in D$ the set $P_k(x, D) = N_k[x] \setminus N_k[D \setminus \{x\}]$ is called the *private k-neighborhood* of x with regard to D and a vertex $y \in P_k(x, D)$ is called a *private k-neighbor* of x with regard to D.

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A set $D \subseteq V(G)$ is a *distance-k dominating set* of G if $N_k[D] = V(G)$. The minimum cardinality of a distance-k dominating set is called the *distance-k domination number* denoted by $\gamma_{\leq k}(G)$. A distance-k dominating set D of G with cardinality $\gamma_{\leq k}(G)$ is called a *$\gamma_{\leq k}$-set* or a *minimum distance-k dominating set*. Note that the case $k = 1$ leads to the ordinary domination. There are several publications on distance domination as e.g. [1], [2], [13], [14] and the chapter 'Distance domination in graphs' by M.A. Henning in [11].
For any subset $B \subseteq E(G)$ we define the subgraph $G(B)$ as the graph with edge set B and vertex set $\{v, w \in V(G) \mid vw \in B\}$. For any set $B \subseteq E(G)$ and any edge $e \in E(G)$ we define $G - B = G(E(G) \setminus B)$ and $G - e = G - \{e\}$. Notice that the subgraphs $G - B$ and $G - e$ contain no isolated vertices. A subset F of the edge set $E(G)$ is an *edge dominating set* of G if every edge in G either is in F or is adjacent to at least one edge in F. The *edge domination number* $\gamma'(G)$ is the smallest cardinality of all edge dominating sets and an edge dominating set of cardinality $\gamma'(G)$ is called a *minimum edge dominating set* of G. The edge domination is studied in numerous publications as e.g. in [3], [4], [12], [15], and in [18].
For other graph theory terminology we follow [10].

2 Unique minimum distance domination in trees

Theorem 2.1 *Let T be a tree of order at least 3, let D be a subset of $V(T)$, and let k be a positive integer. Then the following conditions are equivalent:*

(i) D is the unique $\gamma_{\leq k}$-set of T.

(ii) D is a distance-k dominating set of T such that every vertex in D has at least two private k-neighbors v and w with $d(v, w) = 2k$.

(iii) D is a $\gamma_{\leq k}$-set of T such that $\gamma_{\leq k}(T - x) > \gamma_{\leq k}(T)$ for every vertex $x \in D$.

Proof.
(i) $\Rightarrow$ (ii): Let D be the unique $\gamma_{\leq k}$-set of T. Then, we have $|P_k(x, D)| \geq 2$ for every vertex $x \in D$. Suppose there is a vertex $x \in D$ such that $d(a, b) < 2k$ for every pair of vertices in $P_k(x, D)$. If $d(a, x) < k$ for every vertex a in $P_k(x, D)$, then for some arbitrary, fixed $z \in N_1(x)$ we have $d(a, z) \leq k$ for every vertex a in $P_k(x, D)$, and $(D \setminus \{x\}) \cup \{z\}$ is a $\gamma_{\leq k}$-set of T different from D, which is a contradiction. Hence, there is a vertex $a \in P_k(x, D)$ with $d(x, a) = k$. Let $z \in N_1(x)$ with $d(z, a) = k - 1$. Suppose there is a vertex $b \in P_k(x, D)$ with $d(z, b) > k$. Then $d(x, b) = k$ and the vertex x lies on the unique path from a to b. This yields the contradiction $d(a, b) = d(a, x) + d(x, b) = 2k$. Therefore $d(z, b) \leq k$ for every $b \in P_k(x, D)$ and $(D \setminus \{x\}) \cup \{z\}$ is a $\gamma_{\leq k}$-set of T different from D, which again is a contradiction.
(ii) $\Rightarrow$ (i): We prove this by induction on the order $n(T)$. If a tree T has a distance-k dominating set D as in (ii), then the diameter of T is greater or equal $2k$ and $n(T) \geq 2k + 1$. First, let T be a tree of order $n(T) = 2k + 1$, that has a distance-k dominating set D as in (ii). Since the diameter of T is greater or equal $2k$, the tree T

is isomorphic to the path $x_1x_2\dots x_{2k+1}$ and $D = \{x_{k+1}\}$. Obviously, D is the unique $\gamma_{\le k}$-set of T. Assume the claim holds for every tree T' of order $2k+1 \le n(T') < n$. Now, let T be a tree of order $n(T) = n$, and let D be a distance-k dominating set of T as in (ii). Suppose there exists a $\gamma_{\le k}$-set of T different from D. Let $D' \ne D$ be a $\gamma_{\le k}$-set of T such that $|D \cap D'|$ is maximal. There is at least one vertex $x \in D \setminus D'$ and there are two vertices $y_1, y_2 \in P_k(x, D)$ with $d(y_1, y_2) = 2k$. Hence, we have $d(x, y_1) = d(x, y_2) = k$ and x lies on the unique path from y_1 to y_2 in T. Let $T_1, T_2, \dots, T_\kappa$ be the components of $T - x$ such that $y_i \in V(T_i)$ for $i = 1, 2$. Further, let $D_i = D \cap V(T_i)$ and $D'_i = D' \cap V(T_i)$ for $i = 1, 2$. Since D_i does not distance-k dominate the vertex y_i but D'_i dominates T_i, there is a vertex $z_i \in D'_i \setminus D_i$ with $d(z_i, y_i) \le k$ for $i = 1, 2$. The set $D'' = (D' \setminus (D'_1 \cup D'_2)) \cup (D_1 \cup D_2 \cup \{x\})$ is a distance-k dominating set of T, which implies $|D''| \ge |D'|$ and $|D_1| + |D_2| + 1 \ge |D'_1| + |D'_2|$. If $|D'_1| > |D_1|$ and $|D'_2| > |D_2|$, then we obtain a contradiction. Hence, without loss of generality, we have $|D'_1| \le |D_1|$. Let P be the unique path in T from x to y_2 and let $T' = T[V(T_1) \cup V(P)]$. It is easy to see that $D_1 \cup \{x\}$ is a distance-k dominating set of T' that fulfils (ii). If $n(T') = n(T)$, then $T_2 = P - x$ and $N_k[z_2] \subseteq N_k[x]$. Hence, $D'' = (D' \setminus \{z_2\}) \cup \{x\}$ is a $\gamma_{\le k}$-set of T with $|D'' \cap D| > |D' \cap D|$. Since $z_1 \in D'' \setminus D$, we have $D'' \ne D$, and this is a contradiction to the maximality of $|D' \cap D|$. Hence, let $n(T') < n(T)$. Then, by the induction hypothesis, the set $D_1 \cup \{x\}$ is the unique $\gamma_{\le k}$-set of T'. But $D'_1 \cup \{x\}$ is also a distance-k dominating set of T' with $|D'_1 \cup \{x\}| \le |D_1 \cup \{x\}|$ and $z_1 \in D'_1 \setminus D_1$, which is a contradiction to the uniqueness of $D_1 \cup \{x\}$.

(i) $\Rightarrow$ (iii): Let D be the unique $\gamma_{\le k}$-set of T, let $x \in D$ arbitrary, let $\kappa = \kappa(T - x)$ and let $T_1, T_2, \dots, T_\kappa$ be the components of $T - x$. Further, let D' be a $\gamma_{\le k}$-set of $T - x$ and for every $1 \le i \le \kappa$ let $D_i = D \cap V(T_i)$, and $D'_i = D' \cap V(T_i)$. For every $1 \le i \le \kappa$ the set $D''_i = (D \setminus D_i) \cup D'_i$ is a distance-k dominating set of T, which implies that either $D_i = D'_i$ or $|D_i| < |D'_i|$. By (i) $\Rightarrow$ (ii), the vertex x has at least two private k-neighbors x_1, x_2 in T with $d(x_1, x_2) = 2k$. Without loss of generality, let $x_1 \in V(T_1)$ and $x_2 \in V(T_2)$. Then, for $i = 1, 2$, the set D_i is not a distance-k dominating set of T_i, in contrary to the set D'_i. Hence, we have $D_i \ne D'_i$ and $|D_i| < |D'_i|$ for $i = 1, 2$, which implies $\gamma_{\le k}(T - x) = |D'| = \sum_{i=1}^{\kappa} |D'_i| \ge 2 + \sum_{i=1}^{\kappa} |D_i| = 1 + |D| > \gamma_{\le k}(T)$.

(iii) $\Rightarrow$ (i): Let D be a $\gamma_{\le k}$-set of T such that $\gamma_{\le k}(T - x) > \gamma_{\le k}(T)$ for every vertex $x \in D$. Suppose that there is a $\gamma_{\le k}$-set $D' \ne D$ of T. Since there exists at least one vertex $x \in D \setminus D'$, the set D' is distance-k dominating set of $T - x$. Hence, $\gamma_{\le k}(T - x) \le |D'| = \gamma_{\le k}(T)$ for some $x \in D$, which is a contradiction. $\square$

For $k = 1$, Theorem 2.1 yields immediately the next corollary.

Corollary 2.2 [Gunther, Hartnell, Markus, Rall [9]] *Let T be a tree of order at least 3. Then the following conditions are equivalent:*

(i) T has the unique γ-set D.

(ii) T has a γ-set D for which every vertex in D has at least two private neighbors other than itself.

(iii) T has a γ-set D for which $\gamma(T - x) > \gamma(T)$ for every vertex $x \in D$.

Remark 2.3 *Since the problem of finding a minimum dominating set (i.e. a minimum distance-1 dominating set) in an arbitrary graph is NP-complete, this is also NP-complete for minimum distance-k dominating sets.*

Let G^k denote the k-th *power graph* of G with the vertex set $V(G)$ and the edge set $\{uv \mid u,v \in V(G), d_G(u,v) \leq k\}$. It is easy to see that $\gamma_{\leq k}(G)$ equals $\gamma_{\leq 1}(G^k) = \gamma(G^k)$. Further, G.J. Chang and G.L. Nemhauser [2] have proved that $\gamma_{\leq k}(T) = \alpha(T^{2k}) = \theta(T^{2k})$ for any tree T, where $\alpha(G)$ denotes the cardinality of a maximum independent set of G and $\theta(G)$ denotes the minimum number of cliques in G that cover G. Hence, in view of [2], the following problems are equivalent:

a) The problem of finding a minimum distance-k dominating set of any tree.

b) The problem of finding a minimum (distance-1) dominating set of a graph G which is the k-th power graph of some tree.

c) The problem of finding a maximum independent set of a graph G which is the $2k$-th power graph of some tree.

d) The problem of finding a minimum clique covering of a graph G which is the $2k$-th power graph of some tree.

Lubiw [17] has shown that powers of strongly chordal graphs are also strongly chordal. Trees are strongly chordal, and G.J. Chang and G.L Nemhauser have noticed in [2] that we can construct the strongly chordal graph T^k in $O(n^3)$ time for any tree T of order n. They have also mentioned that therefore we can use every algorithm for finding the cardinality of a minimum dominating set, a maximum independent set, or a minimum clique covering on strongly chordal graphs to determine the distance-k domination number of a tree in polynomial time. There are efficient such algorithms by M. Farber [5], A.W.J. Kolen [16], A. Lubiw [17], A. Frank [6], and F. Gavril [8]. For example, the algorithm of M. Farber [5] solves the domination problem for strongly chordal graphs, and the algorithm of A. Frank [6] solves the independent problem for chordal graphs, both in linear time.

Remark 2.4 *We are able to check in polynomial time whether a given tree T has a unique minimum distance-k dominating set or not by constructing T^k (or T^{2k}, respectively) and by using one of the mentioned algorithms and Theorem 2.1.*

3 Unique minimum edge domination in trees

First, we need some further definitions. For any graph G and any edge $e \in E(G)$ we define $N'(e) = \{f \in E(G) \mid f \text{ adjacent to } e\}$ and the set $N'[e] = N'(e) \cup \{e\}$. If $B \subseteq E(G)$, then $N'(B) = \bigcup_{e \in B} N'(e)$ and $N'[B] = N'(B) \cup B$. For a subset F of $E(G)$ and an edge $e \in F$ we define the set $P'(e,F) = N'[e] \setminus N'[F \setminus \{e\}]$, and we call an edge $f \in P'(e,F)$ a *private adjacent edge* of e with regard to F.

The first lemma contains a simple necessary condition for unique minimum edge dominating sets in graphs. It is a generalization of a result of Topp (Proposition 2.8 in [18]).

Lemma 3.1 *Let G be a connected graph of order at least 3 and let F be a unique minimum edge dominating set of G. Then the set F is independent, and every edge $e \in F$ contains at least two non adjacent edges in $P'(e, F)$.*

Proof. Let $e \in F$ be arbitrary. Since F is minimal, we have $P'(e, F) \neq \emptyset$. If $P'(e, F) = \{e\}$, then we can take any edge f adjacent to e and $(F \setminus \{e\}) \cup \{f\}$ is a minimum edge dominating set of G different from F, which is a contradiction. If $f \in P'(e, F) \setminus \{e\} \neq \emptyset$ and every edge in $P'(e, F) \setminus \{f\}$ is adjacent to f, then again $(F \setminus \{e\}) \cup \{f\}$ is a minimum edge dominating set of G different from F, which is a contradiction. Hence, for every edge $e \in F$ the set $P'(e, F)$ contains two non adjacent edges. This also implies that no two edges in F are adjacent. $\square$

The next theorem is a characterization of trees with unique minimum edge dominating sets similar to the characterization in Corollary 2.2. One part of this theorem says that, for trees, the necessary condition in Lemma 3.1 is also sufficient. The converse does not hold in general, as we can see with the simple graph G with vertex set $V(G) = \{u, v, w, x\}$, edge set $E(G) = \{uv, uw, ux, vw\}$ and with the two minimum edge dominating sets $\{uv\}$ and $\{uw\}$.

Theorem 3.2 *Let T be a tree of order at least 3 and let F be a subset of $E(T)$. Then the following conditions are equivalent:*

(i) F is the unique minimum edge dominating set of T.

(ii) F is an edge dominating set of T such that every edge e in F has at least two non adjacent edges in $P'(e, F)$.

(iii) F is an independent edge dominating set of T such that every edge e in F has at least two non adjacent edges in $P'(e, F)$.

(iv) F is a minimum edge dominating set of T such that $\gamma'(T-e) > \gamma'(T)$ for every edge $e \in F$.

Proof.
(i) $\Rightarrow$ (iii): Follows immediately from Lemma 3.1.
(iii) $\Rightarrow$ (ii): Obviously.
(ii) $\Rightarrow$ (i): Let F be an edge dominating set of T as in (ii). For any subset B of the edge set of T we define $V(B) = \{u, u' \in V(T) \mid uu' \in B\}$. Thus, for every edge $e = vw \in F$ there are two edges vv' and ww' with $v' \neq w'$ and $v, v', w, w' \notin V(F \setminus \{e\})$. Hence, no two edges in F are adjacent. Suppose there is a minimum edge dominating set $F' \neq F$ of T. Then, $|F \setminus F'| \geq |F' \setminus F|$. Let the set $B = (F \setminus F') \cup (F' \setminus F)$ and $H = T[V(B)]$. Let $F'_1 = \{vw \in F' \setminus F \mid v, w \in V(F \setminus F')\}$, $F'_2 = \{vw \in F' \setminus F \mid |\{v, w\} \cap V(F \setminus F')| = 1\}$, and $F'_3 = \{vw \in F' \setminus F \mid v, w \notin V(F \setminus F')\}$. The set $F' \setminus F$ is the disjoint union of F'_1, F'_2 and F'_3. We get for the vertex set of H

$$|V(H)| = |V(B)| \leq 2|F \setminus F'| + |F'_2| + 2|F'_3|.$$

By (ii), for every vertex $v \in V(F \setminus F')$ there is an edge $vw \in F \setminus F'$ and an edge $vv' \neq vw$ such that $v, v' \notin V(F \setminus \{e\})$. Since F' is an edge dominating set of T, we get that v or v' is in $V(F')$. If $v \in (V(F) \setminus V(F')) \subseteq V(F \setminus F')$, then $v' \in (V(F') \setminus V(F)) \subseteq V(F' \setminus F)$ and $vv' \in E(H) \setminus B$. This implies that

$$|E(H) \setminus B| \geq |V(F) \setminus V(F')| \geq 2|F \setminus F'| - 2|F_1'| - |F_2'|.$$

Hence, we obtain for the cardinality of $E(H)$

$$\begin{aligned}
|E(H)| &= |F \setminus F'| + |F' \setminus F| + |E(H) \setminus B| \\
&\geq 2|F' \setminus F| + (2|F \setminus F'| - 2|F_1'| - |F_2'|) \\
&= 2(|F_1'| + |F_2'| + |F_3'|) + (2|F \setminus F'| - 2|F_1'| - |F_2'|) \\
&= |F_2'| + 2|F_3'| + 2|F \setminus F'| \\
&\geq |V(H)|.
\end{aligned}$$

But, since H is a forest, we have $m(H) = n(H) - \kappa(H) < n(H)$, which is a contradiction.

(i) $\Rightarrow$ (iv): Let F be the unique minimum edge dominating set of T, let $e = v_1v_2 \in F$ be arbitrary, and let T_1 and T_2 be the components of $T - e$ where $v_i \in V(T_i)$ for $i = 1, 2$.

Further, let F' be a minimum edge dominating set of $T - e$ and for $i = 1, 2$ let $F_i = F \cap E(T_i)$ and $F_i' = F' \cap E(T_i)$. By (i) $\Rightarrow$ (ii), the edge e is adjacent to at least two edges $v_1w_1 \in E(T_1)$ and $v_2w_2 \in E(T_2)$ that are not adjacent to any other edge in F. Hence, the set F_i is not an edge dominating set of T_i, contrary to F_i' for $i \in \{1, 2\}$. Thus, we have $F_i \neq F_i'$ for $i = 1, 2$. Since the set $F_i'' = (F \setminus F_i) \cup F_i' \neq F$ is an edge dominating set of T, we get $|F_i| < |F_i'|$. This yields $\gamma'(T - e) = |F'| = |F_1'| + |F_2'| \geq |F_1| + |F_2| + 2 = |F| + 1 > \gamma'(T)$.

(iv) $\Rightarrow$ (i): Let F be a minimum edge dominating set of T such that $\gamma'(T-e) > \gamma'(T)$ for every edge $e \in F$. Suppose that there is a minimum edge dominating set $F' \neq F$ of T. There exists at least one edge $e \in F \setminus F'$ and the set F' is an edge dominating set of $T - e$. Hence, $\gamma'(T-e) \leq |F'| = \gamma'(T)$ for some $e \in F$, which is a contradiction. □

As a corollary of Theorem 3.2 we obtain a characterization of caterpillars with unique minimum edge dominating sets by Topp (Corollary 3.1 in [18]). Further, we get the following corollary, that also contains a result of Topp (Theorem 2.11 in [18]).

Corollary 3.3 *Let T be a tree of diameter at least 3, let F be a minimum edge dominating set of T, and let $e \in F$ arbitrary. Then F is the unique minimum edge dominating set of T if and only if every component of the forest $H = T - N'[e]$ is of order at least 4 and H has the unique minimum edge dominating set $F \setminus \{e\}$.*

Proof. Let F be a minimum edge dominating set of T and let $e \in F$ be arbitrary. First, let F be unique. Hence F fulfils (ii) in Theorem 3.2, and this implies that $F \setminus \{e\}$ fulfils (ii) for the forest H. Thus each component of H is of order at least 4. If we use Theorem 3.2 on these components, then we get that H has the unique minimum edge dominating set $F \setminus \{e\}$. Now, let $F \setminus \{e\}$ be the unique edge dominating set of H and every component of H be of order at least 4. By Theorem 3.2, the set $F\backslash\{e\}$ fulfils (ii). This implies that F fulfils (ii) for T. Thus F is unique, by Theorem 3.2. □

Remark 3.4 *There are some algorithms known to determine minimum edge dominating sets in special classes of graphs ([3],[4],[7],[12],[15]). For trees a linear time algorithm to determine minimum edge dominating sets is given by S. Hedetniemi and S. Mitchell [12], and a linear time algorithm to determine minimum independent edge dominating sets is given by F. Gavril and M. Yannakakis [7]. Further, G.J. Chang and S.-F. Hwang [3] found a linear time algorithm to determine minimum edge dominating sets in block graphs. Hence, we can inspect in linear time whether a given tree has a unique minimum edge dominating set or not, by using one of these algorithms and Theorem 3.2.*

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