Two invariants for adjointly equivalent graphs

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Abstract

Two graphs are defined to be adjointly equivalent if their complements are chromatically equivalent. We study the properties of two invariants under adjoint equivalence.

1 Introduction

In this paper, all graphs considered are simple graphs. For a graph G, let $\overline{G}$, $V(G)$, $E(G)$, $v(G)$, $e(G)$, $t(G)$, $c(G)$ and $P(G,\lambda)$, respectively, be the complement, vertex set, edge set, order, size, number of triangles, number of components and chromatic polynomial of G.

A partition $\{A_1, A_2, \cdots, A_k\}$ of $V(G)$, where k is a positive integer, is called a *k-independent partition* of a graph G if each A_i is a nonempty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions of G. Then

$$P(G,\lambda) = \sum_{k=1}^{v(G)} \alpha(G,k)(\lambda)_k, \tag{1}$$

where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$. (See [13].)

Two graphs G and H are said to be *chromatically equivalent* if they have the same chromatic polynomial. In this case we write $G \sim H$. The equivalence class determined by a graph G is denoted by $[G]$. A graph G is said to be *chromatically unique* if $[G] = \{G\}$.

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The determination of $[G]$ for a given graph G has received much attention in the literature (see [4, 5]). The adjoint polynomial of a graph is a useful tool for this study. We now proceed to define it.

Let G be a graph with order n. If H is a spanning subgraph of G and each component of H is complete, then H is called a *clique cover* [2] (or, by Liu [6], an *ideal subgraph*) of G. Two clique covers are considered to be different if they have different edge sets. For $k \geq 1$, let $N(G,k)$ be the number of clique covers H in G with $c(H) = k$. The number $N(G,k)$ is referred to as a *clique cover number*. It is clear that $N(G,n) = 1$ and $N(G,k) = 0$ for $k > n$. Define

$$h(G,\mu) = \begin{cases} \sum\limits_{k=1}^{n} N(G,k)\mu^k, & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases} \tag{2}$$

The polynomial $h(G,\mu)$ is called the *adjoint polynomial* of G. Observe that $h(G,\mu) = h(G',\mu)$ if $G \cong G'$. Hence $h(G,\mu)$ is a well-defined graph-function. The notion of the adjoint polynomial of a graph was introduced by Liu [6]. Note that the adjoint polynomial is a special case of an F-polynomial [2].

Two graphs G and H are said to be *adjointly equivalent* if they have the same adjoint polynomial. In this case we write $G \sim_h H$. The equivalence class determined by a graph G is denoted by $[G]_h$. A graph G is said to be *adjointly unique* if $[G]_h = \{G\}$. Note that

$$\alpha(G,k) = N(\overline{G},k), \qquad k = 1, 2, \cdots, n. \tag{3}$$

It follows that

Theorem 1.1 *(i)* $G \sim H$ *iff* $\overline{G} \sim_h \overline{H}$;
(ii) $[G] = \{H | \overline{H} \in [\overline{G}]_h\}$;
(iii) G *is chromatically unique if and only if* $\overline{G}$ *is adjointly unique.* □

Hence the goal of determining $[G]$ for a given graph G can be realised by determining $[\overline{G}]_h$. Thus, as has been observed in [6, 7, 8, 9, 10, 11, 12], if $e(G)$ is very large, it may be easier to study $[\overline{G}]_h$ rather than $[G]$.

Section 2 computes some clique cover numbers that are used to study two invariants for adjoint polynomials. These invariants, $R_1(G)$ and $R_2(G)$, are the subject matter of Sections 3 and 4 respectively. For a polynomial $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$, define

$$R_1(f) = \begin{cases} -\binom{b_1}{2} + b_1, & \text{if } n = 1, \\ b_2 - \binom{b_1}{2} + b_1, & \text{if } n \geq 2 \end{cases} \tag{4}$$

and

$$R_2(f) = b_3 - \binom{b_1}{3} - (b_1 - 2)\left(b_2 - \binom{b_1}{2}\right) - b_1,$$

where $b_k = 0$ for $k > n$. For any graph G, define

$$R_i(G) = R_i(h(G,\mu)) \tag{5}$$

for each $i \in \{1, 2\}$. It is clear that $R_i(G)$ is an invariant for adjointly equivalent graphs, since $N(G, k)$ is an invariant for each positive integer k. The invariant $R_1(G)$ was introduced by Liu [6] and used by him and others to study adjoint uniqueness of graphs. In particular in [12] Liu and Zhao showed that $R_1(G) \leq 1$ for any connected graph G, and characterised the connected graphs G with $R_1(G) \geq 0$. They also established the chromatic uniqueness of certain dense graphs. In Section 3 we obtain a recursive formula and a sharper upper bound for $R_1(G)$. We also show for which graphs this upper bound is met. In Section 4 we obtain alternative formulae for $R_2(G)$ which enable us to compute $R_2(G)$ for some specific graphs. In a subsequent paper we use both $R_1(G)$ and $R_2(G)$ to determine adjoint equivalence classes of certain graphs and confirm a conjecture of Liu [9] that P_n is adjointly unique for each even $n \neq 4$.

2 Computation of some clique cover numbers

In this section we calculate the clique cover numbers $N(G, n-k)$ for $k = 0, 1, 2, 3$ in order to obtain an expression for each $R_i(G)$, where $i = 1, 2$.

Theorem 2.1 [7] *For any graph G with order n,*
(i) $N(G, n) = 1$ *if* $n \geq 1$;
(ii) $N(G, n-1) = e(G)$ *if* $n \geq 2$;
(iii) $N(G, n-2) = t(G) + \binom{e(G)}{2} - \sum_{x \in V(G)} \binom{d_G(x)}{2}$ *if* $n \geq 3$. □

For $x \in V(G)$, let $\triangle_G(x)$ (or simply $\triangle(x)$) be the number of triangles in G which include x. For any graphs G and Q, let $n_G(Q)$ (or simply $n(Q)$) denote the number of subgraphs in G which are isomorphic to Q. Thus $n_G(K_2) = e(G)$ and $n_G(K_3) = t(G)$. In particular, let $p_k(G) = n_G(P_k)$, i.e., the number of paths of order k in G.

The next result gives an expression for $N(G, v(G) - 3)$.

Theorem 2.2 *For any graph G with order n, we have*

$$\begin{aligned} N(G, n-3) &= \binom{e(G)}{3} + p_4(G) + 5t(G) + n(K_4) - \sum_{x \in V(G)} d(x)\triangle(x) \\ &+ e(G)\left(t(G) - \sum_{x \in V(G)} \binom{d(x)}{2}\right) + 2\sum_{x \in V(G)} \binom{d(x)+1}{3}. \quad (6) \end{aligned}$$

Proof. By definition, $N(G, n-3)$ is the number of clique covers H in G with $c(H) = n-3$. Since $v(H) = n$, each component of H is of order at most 4, we find that H is one of the following types of graphs:

(i) $3K_2 \cup (n-6)K_1$,
(ii) $K_3 \cup K_2 \cup (n-5)K_1$,
(iii) $K_4 \cup (n-4)K_1$.

Thus

$$N(G, n-3) = n_G(3K_2) + n_G(K_3 \cup K_2) + n_G(K_4).$$

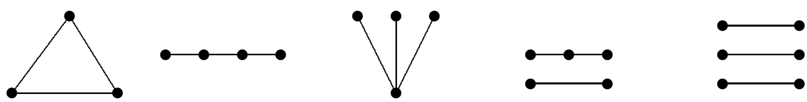
Observe that

$$n_G(K_3 \cup K_2) = \sum_{\triangle xyz \text{ in } G} (e(G) - d(x) - d(y) - d(z) + 3),$$

where the sum is taken over all triangles xyz in G. Hence

$$n_G(K_3 \cup K_2) = (e(G) + 3)t(G) - \sum_{x \in V(G)} d(x)\triangle(x).$$

Now consider the number $n_G(3K_2)$. The following figure shows all possible graphs with size 3 and no isolated vertices.



K_3 (b) $\quad$ P_4 (b) $\quad$ $K_{1,3}$ (c) $\quad$ $P_2 \cup P_3$ (d) $\quad$ $3P_2$ (e)

Figure 1

Observe that

$$n_G(K_{1,3}) = \sum_{x \in V(G)} \binom{d(x)}{3}$$

and

$$\sum_{x \in V(G)} \binom{d(x)}{2}(e(G) - d(x)) = 3n_G(K_3) + 2n_G(P_4) + n_G(P_2 \cup P_3).$$

Thus

$$\begin{aligned} n_G(3K_2) &= \binom{e(G)}{3} - n_G(K_3) - n_G(P_4) - n_G(K_{1,3}) - n_G(P_2 \cup P_3) \\ &= \binom{e(G)}{3} - \sum_{x \in V(G)} \binom{d(x)}{3} - \sum_{x \in V(G)} \binom{d(x)}{2}(e(G) - d(x)) \\ &\quad +2n_G(K_3) + n_G(P_4) \\ &= \binom{e(G)}{3} + 2 \sum_{x \in V(G)} \binom{d(x)+1}{3} - e(G) \sum_{x \in V(G)} \binom{d(x)}{2} \\ &\quad +2n_G(K_3) + n_G(P_4). \end{aligned}$$

The result is then obtained. □

3 The Invariant $R_1(G)$

By Theorem 2.1 and the definition of $R_1(G)$, we have

Lemma 3.1 *For any graph G,*

$$R_1(G) = t(G) + e(G) - \sum_{x \in V(G)} \binom{d_G(x)}{2}. \tag{7}$$

□

Corollary $R_1(G) = 0$ *if* $e(G) = 0$. □

By Lemma 3.1, the next result is obtained.

Lemma 3.2 *For any graph G with components $G_1, G_2, \cdots, G_k$,*

$$R_1(G) = \sum_{i=1}^{k} R_1(G_i). \tag{8}$$

□

If $e(G) = 0$, then $R_1(G) = 0$. We shall find a recursive expression for $R_1(G)$ when $e(G) > 0$. For $x, y \in V(G)$, let $N_G(x, y)$ (or simply $N(x, y)$) denote the set

$$(N(x) \cup N(y)) - \{x, y\}.$$

Observe that

$$|N_G(x,y)| = \begin{cases} d(x) + d(y) - |N(x) \cap N(y)|, & \text{if } xy \notin E(G); \\ d(x) + d(y) - |N(x) \cap N(y)| - 2, & \text{if } xy \in E(G). \end{cases}$$

Lemma 3.3 *For any graph G and $xy \in E(G)$, we have*

$$R_1(G) = R_1(G - xy) + 1 - |N_G(x, y)|. \tag{9}$$

Proof. By (7), we have

$$\begin{aligned} & R_1(G) - R_1(G - xy) \\ = \ & t(G) - t(G - xy) + (e(G) - e(G - xy)) \\ & - \left(\binom{d_G(x)}{2} - \binom{d_G(x) - 1}{2} \right) - \left(\binom{d_G(y)}{2} - \binom{d_G(y) - 1}{2} \right) \\ = \ & |N_G(x) \cap N_G(y)| + 1 - (d_G(x) - 1) - (d_G(y) - 1) \\ = \ & 1 - |N_G(x, y)|. \end{aligned}$$

□

By Lemma 3.3, we find a sufficient condition for two graphs G and G' to satisfy $R_1(G) = R_1(G')$.

Lemma 3.4 *Let xy be an edge in G with $N_G(x) \cap N_G(y) = \emptyset$. Let G' be any graph obtained from G by replacing the edge xy by a path containing no vertices of $V(G) - \{x, y\}$. Then*

$$R_1(G) = R_1(G'). \tag{10}$$

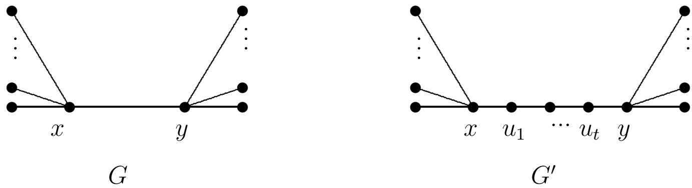


Figure 2

Proof. Let G' be the graph obtained from G by replacing the edge xy by the path with $t+2$ vertices, as shown in Figure 2. To prove the lemma, it suffices to show that $R_1(G') = R_1(G)$ for $t = 1$. Let $t = 1$. Assume that $d_G(x) = 1 + a$ and $d_G(y) = 1 + b$. By Lemma 3.3, we have

$$\begin{aligned} R_1(G') &= R_1(G' - xu_1) + 1 - (1 + a) \\ &= (R_1(G' - xu_1 - u_1y) + 1 - b) - a \\ &= R_1((G - xy) \cup K_1) + 1 - a - b \\ &= R_1(G - xy) + 1 - a - b \\ &= R_1(G). \end{aligned}$$

□

By using Lemmas 3.3 and 3.4, it is easy to compute $R_1(G)$ for some special graphs. Let $K_4 - e$ be the graph obtained from K_4 by deleting one edge.

Lemma 3.5 *(i)* $R_1(P_1) = 0$ *and* $R_1(P_t) = 1$ *for* $t \geq 2$.
(ii) $R_1(K_3) = 1$, $R_1(K_4) = -2$ *and* $R_1(K_4 - e) = -1$.
(iii) $R_1(C_k) = 0$ *for* $k \geq 4$. □

For positive integers k, s and t, let $T_{k,s,t}$ be the graph in Figure 3(a). Let

$$\mathcal{T}' = \{T_{k,s,t} | k \geq s \geq t \geq 1\}.$$

Let D_n and F_n be the graphs shown in Figure 3 (b) and (c).

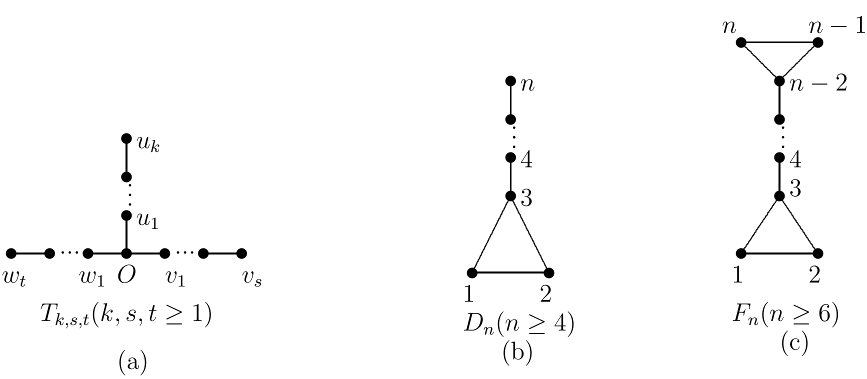


Figure 3

Theorem 3.1 [12] *Let G be a connected graph. Then $R_1(G) \leq 1$ and*

(i) $R_1(G) = 1$ if and only if $G \in \{K_3\} \cup \{P_n | n \geq 2\}$,

(ii) $R_1(G) = 0$ if and only if $G \in \{K_1\} \cup \mathcal{T}' \cup \{C_n, D_n | n \geq 4\}$, and

(iii) $R_1(G) = -1$ with $e(G) \geq v(G)+1$ if and only if $G \in \{K_4 - e\} \cup \{F_n | n \geq 6\}$.

□

From Theorem 3.1, we observe that for any connected graph G, if $G \ncong K_3$ and $R(G) \geq -1$, then $e(G)+R_1(G) \leq v(G)$. We shall show that for any connected graph G, if $G \ncong K_4$ and $R_1(G) \leq -2$, then $e(G) + R_1(G) \leq v(G) - 1$. First we establish the following result.

Theorem 3.2 *For any connected graph G, if $G \ncong K_4$, then*

$$R_1(G) \leq 2(v(G) - e(G)) + 1. \tag{11}$$

Proof. For any graph G, let

$$\phi(G) = R_1(G) - 2(v(G) - e(G)).$$

We have to show that for any connected graph G, if $G \ncong K_4$, then

$$\phi(G) \leq 1. \tag{12}$$

By Lemma 3.1, we have

$$\begin{aligned}
\phi(G) &= t(G) - 2v(G) + 3e(G) - \sum_{x \in V(G)} \binom{d(x)}{2} \\
&= t(G) + \sum_{x \in V(G)} (3d(x)/2 - 2) - \frac{1}{2} \sum_{x \in V(G)} d(x)(d(x) - 1) \\
&= t(G) - \frac{1}{2} \sum_{x \in V(G)} (d(x) - 2)^2. \qquad (13)
\end{aligned}$$

It follows that $\phi(G) \leq 1$ if $t(G) \leq 1$. Hence (12) holds for connected graphs G with $e(G) \leq 4$. Note that $\phi(K_4) = 4 - \frac{1}{2} \times 4 = 2$.

Suppose that H is a connected graph with minimum size such that $H \ncong K_4$ and $\phi(H) \geq 2$. We prove that such a graph H does not exist.

Claim 1: For any $x \in V(H)$, if $N_H(x) = \{y, z\}$, then $N_H(y) \cap N_H(z) \neq \{x\}$.

Suppose that $N_H(y) \cap N_H(z) = \{x\}$. Let H' be the graph $H \cdot xy$. Observe that H' has an edge which is not contained in any triangle, which implies that $H' \ncong K_4$. Since $e(H') < e(H)$, we have $\phi(H') \leq 1$. By Lemma 3.4, $R_1(H) = R_1(H')$. Since $v(H) = v(H')+1$ and $e(H) = e(H')+1$, we have $\phi(H) = \phi(H') \leq 1$, a contradiction. The claim holds.

Claim 2: $\delta(H) \geq 2$.

Suppose that $d_H(x) = 1$ and $N_H(x) = \{y\}$. Let $H' = H - x$. By (13),

$$\phi(H) - \phi(H') = -1/2 - 1/2(d_H(y) - 2)^2 + 1/2(d_H(y) - 3)^2 = 2 - d_H(y).$$

Since $H \neq K_2$ (as $\phi(K_2) = -1$) and $d_H(y) \neq 2$ by Claim 1, we have $d_H(y) \geq 3$. Hence $\phi(H) \leq \phi(H') - 1$. If $H' \ncong K_4$, then as H' is connected and $e(H') < e(H)$, we have $\phi(H') \leq 1$; thus $\phi(H) \leq 0$. If $H' \cong K_4$, we have $\phi(H) \leq 1$. Both cases lead to a contradiction.

Claim 3: H does not contain a bridge.

Suppose that xy is a bridge of H. Let H_1 and H_2 be the two components of $H - xy$. By (13),

$$\phi(H) - \phi(H_1) - \phi(H_2) = 5 - d_H(x) - d_H(y).$$

Observe that H_1, H_2 are connected. Thus $\phi(H_i) \leq 1$ if $H_i \ncong K_4$. By Claim 1 and 2, $d_H(x), d_H(y) \geq 3$. Let $x \in V(H_1)$ and $y \in V(H_2)$. Notice that $d_H(x) = 4$ if $H_1 \cong K_4$ and $d_H(y) = 4$ if $H_2 \cong K_4$. Hence $\phi(H) \leq 1$, a contradiction. The claim holds.

Claim 4: For each $xy \in E(H)$, $|N_H(x, y)| \leq 2$.

Suppose that $|N_H(x, y)| \geq 3$ for some $xy \in E(H)$. By Lemma 3.3,

$$R_1(H) \leq R_1(H - xy) - 2.$$

By Claim 3, $H - xy$ is connected. Since $H - xy$ is not complete and $e(H - xy) < e(H)$, we have $\phi(H - xy) \leq 1$. Hence by the definition of $\phi(H)$,

$$\phi(H) - \phi(H - xy) = R_1(H) - R_1(H - xy) + 2 \leq 0,$$

which implies that $\phi(H) \leq 1$, a contradiction. The claim follows.

Claim 5: $t(H) = 0$.

If H contains a subgraph isomorphic to $K_4 - e$, then $v(H) = 4$ by Claim 4, which implies that either $H \cong K_4$ or $H \cong K_4 - e$. But $\phi(K_4 - e) = 1$, a contradiction. Thus H does not contain any subgraph isomorphic to $K_4 - e$.

Suppose that xyz is a triangle in H. If $d(x) = d(y) = d(z) = 2$, then $H \cong K_3$ and $\phi(H) = 1$, a contradiction. By Claim 4, $d(x), d(y), d(z) \leq 3$. Now say $d(x) = 3$. Let $xw \in E(H)$, where $w \notin \{y, z\}$. Since $\delta(H) \geq 2$, we have $d(w) \geq 2$. Since H does not contain any subgraph isomorphic to $K_4 - e$, we have $|N_H(x, w)| \geq 3$, which contradicts Claim 4. Hence Claim 5 holds.

Since $t(H) = 0$ by Claim 5, we have $\phi(H) \leq 0$ by (13), a contradiction. Hence H does not exist. □

Recall from Theorem 3.1 that $R_1(G) \leq 1$. By Theorems 3.1 and 3.2, we have

Corollary 3.1 *For any connected graph G with $G \notin \{K_3, K_4\}$,*

(i) if $-1 \leq R_1(G) \leq 1$, then $R_1(G) \leq v(G) - e(G)$ with equality if and only if

$$G \in \{K_4 - e\} \cup \{P_n, C_{n+1}, D_{n+2}, F_{n+4} | n \geq 2\}.$$

(ii) if $R_1(G) \leq -2$, then $R_1(G) \leq v(G) - e(G) - 1$.

Proof. The result of (i) follows from Theorem 3.1.

(ii) If $R_1(G) \leq -2$, then by Theorem 3.2,

$$v(G) - e(G) \geq R_1(G)/2 - 1/2 = R_1(G) - R_1(G)/2 - 1/2 \geq R_1(G) + 1 - 1/2,$$

which implies $v(G) - e(G) \geq R_1(G) + 1$. Thus (ii) holds. □

4 The Invariant $R_2(G)$

Theorem 4.1 *For any graph G,*

$$\begin{aligned} R_2(G) &= 2\sum_{x\in V(G)}\binom{d(x)}{3} - \sum_{x\in V(G)} d(x)\triangle_G(x) \\ &\quad -e(G)+p_4(G)+7t(G)+n_G(K_4). \end{aligned} \tag{14}$$

Proof. Let $v(G)=n$. For $f(\mu)=h(G,\mu)$, we have $b_i=N(G,n-i)$, $i\geq 1$. Observe that

$$b_2-\binom{b_1}{2}=t(G)-\sum_{x\in V(G)}\binom{d(x)}{2},$$

and

$$\sum_{x\in V(G)}\binom{d(x)+1}{3}=\sum_{x\in V(G)}\binom{d(x)}{3}+\sum_{x\in V(G)}\binom{d(x)}{2}.$$

The result is then obtained from (5) and (6). □

The term $p_4(G)$ can be expressed in terms of $d_G(x)$ and $t(G)$. Thus there is another expression for $R_2(G)$.

Theorem 4.2 *For any graph G,*

$$\begin{aligned} R_2(G) &= 2\sum_{x\in V(G)}\binom{d(x)}{3} - \sum_{x\in V(G)} d(x)\triangle_G(x) - e(G)+4t(G)+n_G(K_4) \\ &\quad + \sum_{xy\in E(G)}(d_G(x)-1)(d_G(y)-1). \end{aligned} \tag{15}$$

Proof. For $xy\in E(G)$, let $p_4(xy)$ be the number of paths of the form $uxyv$ in G , where $u\neq v$. Observe that

$$p_4(xy)=(d(x)-1)(d(y)-1)-\triangle(xy),$$

where $\triangle(xy)$ is the number of triangles in G containing xy. Thus

$$\begin{aligned} p_4(G) &= \sum_{xy\in E(G)} p_4(xy) \\ &= \sum_{xy\in E(G)}((d(x)-1)(d(y)-1)-\triangle(xy)) \\ &= \sum_{xy\in E(G)}(d(x)-1)(d(y)-1)-3t(G). \end{aligned}$$

The result is then obtained. □

Corollary 4.1 *If G is K_3-free, then*

$$R_2(G) = 2\sum_{x\in V(G)} \binom{d(x)}{3} - e(G) + \sum_{xy\in E(G)} (d(x)-1)(d(y)-1). \qquad \square$$

Corollary 4.2 *If $G_1, G_2, \cdots, G_k$ are the components of G, then*

$$R_2(G) = \sum_{i=1}^{k} R_2(G_i).$$

Proof. It follows from Theorem 4.2. $\square$

Let Y_n denote the graph $T_{n-3,1,1}$, where $n \geq 4$. By applying Theorem 4.2, we have

Corollary 4.3 *(i) $R_2(P_1) = 0$, $R_2(P_2) = -1$ and $R_2(P_n) = -2$ for $n \geq 3$;*
(ii) $R_2(K_3) = -2$ and $R_2(C_n) = 0$ for $n \geq 4$;
(iii) $R_2(Y_4) = -1$ and $R_2(Y_n) = 0$ for $n \geq 5$;
(iv) $R_2(D_4) = 0$ and $R_2(D_n) = 1$ for $n \geq 5$;
(v) $R_2(F_6) = 5$ and $R_2(F_n) = 4$ for $n \geq 7$;
(vi) $R_2(K_4 - e) = 3$ and $R_2(K_4) = 7$. $\square$

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