

# Minimally triangle-saturated graphs: adjoining a single vertex

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## Abstract

A graph  $G$  is triangle-saturated if every possible edge addition to  $G$  creates one or more new triangles (3-cycles). Such a graph is minimally triangle-saturated if removal of any edge from  $G$  leaves a graph that is not triangle-saturated. This paper investigates adding a single new vertex to a triangle-saturated graph so as to produce a new triangle-saturated graph, and determines the conditions under which the extended graph is minimally saturated. Particular attention is given to minimally saturated extensions which are *primitive* (no two vertices have the same neighbourhood). The results are applied to construct primitive maximal triangle-free graphs of every order  $n \geq 9$ .

## 1 Introduction

Given two finite simple graphs  $G$  and  $H$ , we say that  $G$  is  $H$ -saturated if every possible addition of an edge to  $G$  would create a new copy of the graph  $H$ . In particular,  $G$  is *triangle-saturated* (or just *saturated*) if addition of any edge  $e$  to  $G$  creates a triangle (3-cycle) in  $G + e$  which is not already present in  $G$ . One reason that triangle-saturated graphs are an important class to study is that they are precisely the graphs with diameter at most 2. A *minimally triangle-saturated* graph  $G$  is a triangle-saturated graph from which deletion of any edge  $e$  leaves a spanning subgraph  $G - e$  which is not triangle-saturated. The *maximal triangle-free graphs*

are precisely the minimally triangle-saturated graphs which contain no triangles. All triangle-saturated graphs can be generated from the minimally triangle-saturated graphs simply by adding edges. We call a graph *primitive* if it has no duplicate vertices, that is, if distinct vertices have distinct neighbourhoods. It turns out that all minimally triangle-saturated graphs can be generated by beginning with those which are primitive and successively duplicating appropriate vertices.

These notions were studied in [2] and various methods of constructing infinite families of primitive, minimally triangle-saturated graphs are described in [3] and [4]. The construction methods in those papers produce new triangle-saturated graphs from old, preserving minimality and primitivity. However, the graphs produced have considerably higher order than the starter graphs. In the present paper we consider the problem of constructing new triangle-saturated graphs from old by adjoining a single new vertex in an appropriate way. In Section 2, we describe a general construction and determine the conditions under which the extended graph is minimally saturated. Primitive extensions are discussed in Section 3. In Section 4, the construction is applied to construct one or more primitive maximal triangle-free graphs of each order  $n \geq 9$ .

## 2 The Construction

For any graph  $G$ , the graph produced by adjoining a new vertex  $x$  adjacent to every vertex of  $G$  is the *join* of  $G$  and  $x$ , denoted by  $G \vee x$ . Any additional edge inserted in  $G \vee x$  necessarily is incident with two vertices of  $G$ , both of which are adjacent to  $x$ , so the new edge forms a triangle with  $x$ . Hence the join  $G \vee x$  is always triangle-saturated. Indeed, when  $n \geq 2$  the graph  $\overline{K_n} \vee x$  is a star, and clearly is triangle-saturated, so

**Remark 1** *For any graph  $G$ , the join  $G \vee x$  is triangle-saturated, and is minimally triangle-saturated if and only if  $G$  has no edges.*

Consequently, to construct minimally triangle-saturated graphs by adjoining a new vertex  $x$  to a given nonempty graph  $G$ , we must make  $x$  adjacent to some proper subset of the vertices of  $G$ . For any subset  $D$  of the vertices of  $G$ , the *extension*  $[G, D, x]$  is the graph obtained by adjoining the vertex  $x$  to  $G$  so that it is adjacent to precisely the vertices in  $D$ . We are interested in the relationship between properties of  $D$  and properties of  $[G, D, x]$ .

For any vertex  $v$ , let  $N(v)$  and  $N[v]$  denote, respectively, the open and closed neighbourhoods of  $v$ . If  $D$  is any set of vertices, then

$$N(D) := \bigcup_{d \in D} N(d) \quad \text{and} \quad N[D] := \bigcup_{d \in D} N[d]$$

are the *open* and *closed neighbourhoods* of  $D$ . A vertex set  $D$  *dominates*  $G$ , or is a *dominating set* for  $G$ , if  $N[D]$  contains every vertex of  $G$ : equivalently,  $N(D)$

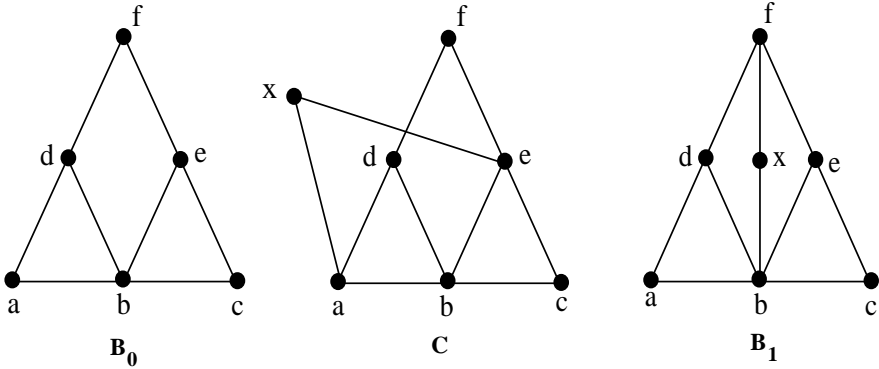


Figure 1:  $B_0$  and two extensions

contains every vertex of  $G - D$ . A *minimal* dominating set has no proper subset which dominates  $G$ .

For any graph  $G$ , it is easy to verify that if  $[G, D, x]$  is triangle-saturated, then  $D$  dominates  $G$ . Hence, we deduce

**Remark 2** *If  $G$  is any triangle-saturated graph, then the extension  $[G, D, x]$  is triangle-saturated if and only if  $D$  is a dominating set for  $G$ .*

Starting with a triangle-saturated graph  $G$ , what conditions on  $D$  will allow us to extend  $G$  to a minimally saturated graph  $[G, D, x]$ ? A necessary condition is easy to find:

**Lemma 1** *Let  $G$  be a triangle-saturated graph and suppose the extension  $[G, D, x]$  is minimally triangle-saturated. Then  $D$  is a minimal dominating set for  $G$ .*

**Proof.** By Remark 2,  $D$  must be a dominating set. If  $D$  is not minimal, then  $D - v$  is still a dominating set for some  $v$  in  $D$ . Consider  $[G, D - v, x] = [G, D, x] - vx$ . Because  $D - v$  dominates  $G$ , every vertex in  $G - (D - v)$  is adjacent to some neighbour of  $x$ , so adding any edge from  $x$  to  $G$  in  $[G, D, x] - vx$  will create a triangle. Also any edge inserted within  $G$  will create a triangle because  $G$  is saturated. Hence  $[G, D, x] - vx$  is saturated, contradicting the assumption that  $[G, D, x]$  was minimally saturated. Hence  $D$  is a minimal dominating set for  $G$ . ■

The condition that  $D$  be a minimal dominating set is generally not sufficient to guarantee that  $[G, D, x]$  is minimally triangle-saturated. Consider the graph  $B_0$  (Fig. 1), chosen because it is the graph of smallest order which is minimally triangle-saturated but not triangle-free [2]. The set  $\{a, e\}$  is a minimal dominating set for  $B_0$ . Let  $C := [B_0, \{a, e\}, x]$ , as shown in Fig. 1. Then  $C$  is triangle-saturated, by Remark 2, but is not minimally saturated because  $C - be$  is saturated.

To characterize those  $[G, D, x]$  which are minimally saturated we need several more definitions. A vertex set  $D$  is *irredundant* in  $G$  if its closed neighbourhood  $N[D]$  is strictly larger than the closed neighbourhoods of each of its proper subsets. If  $D$  dominates  $G$  and is irredundant, then it is easily seen to be a minimal dominating set and a maximal irredundant set. Indeed, a dominating set is minimal if and only if it is irredundant (see Prop. 4.1 of [1]).

An edge in a triangle-saturated graph is *essential* if its removal leaves an unsaturated graph. If a triangle-saturated graph is minimally saturated then every edge is essential. When an essential edge is removed, a corresponding *replacement edge* is any edge (possibly the same edge) which may be inserted without creating a triangle. For example, the essential edge  $ab$  in  $B_0$  has  $ac$  and  $ae$  as replacement edges.

**Remark 3** *If  $e$  is an essential edge of a triangle-saturated graph, any replacement edge is incident with at least one vertex of  $e$ .*

This is easily seen by noting that if  $e'$  is a replacement for  $e$ , then  $e'$  creates a triangle  $\Delta$  in  $G + e'$  but not in  $G + e' - e = G - e + e'$ . Hence  $e$  and  $e'$  must belong to  $\Delta$ , so  $e$  and  $e'$  have at least one vertex in common.

Let  $e$  be any essential edge in the triangle-saturated graph  $G$ , and let  $D$  be any set of vertices in  $G$ . Then  $D$  *blocks*  $e$  if every replacement edge for  $e$  has both its vertices in  $D$ . Note that an edge  $e$  could only be blocked by  $D$  if at least one vertex of  $e$  is in  $D$ , by Remark 3.

If  $D$  is any set of vertices in  $G$ , and  $e := ab$  is any edge of  $G$ , then  $e$  is a  *$D$ -spine* if  $a \in D$  and  $D \cap N[b] = \{a\}$ . It turns out that  $D$ -spines play a key role in determining whether  $[G, D, x]$  is minimally saturated.

**Theorem 1** *Let  $G$  be any minimally triangle-saturated graph and  $D$  any subset of vertices of  $G$ . Then the extension  $[G, D, x]$  is minimally triangle-saturated if and only if (1)  $D$  is an irredundant dominating set for  $G$  and (2) any edge of  $G$  which is blocked in  $G$  by  $D$  is a  $D$ -spine.*

**Proof.** To see the necessity of (1) and (2), assume that  $[G, D, x]$  is minimally saturated. Then  $D$  is a minimal dominating set for  $G$  by Lemma 1. Hence  $D$  is irredundant, and (1) follows. Now consider any edge  $e$  in  $G$ . There must be a replacement edge  $e'$  which can be added to  $[G, D, x] - e$  without creating a triangle. Evidently, at least one vertex  $b$  of  $e'$  must be in  $G - D$ . If there is such an  $e'$  with both vertices in  $G$ , then it is a replacement for  $e$  in  $G$ , so  $e$  is not blocked by  $D$ . On the other hand, if no such  $e'$  has both vertices in  $G$ , then  $e$  is blocked in  $G$  by  $D$  and necessarily  $e' = bx$ . Also  $b$  must be a vertex of  $e$ , by Remark 3. Because  $G$  is minimally saturated,  $e$  does have at least one replacement edge  $e''$  in  $G$ . Both vertices of  $e''$  must be in  $D$ , so at least one vertex  $a$  of  $e$  must be in  $D$  by Remark 3, and therefore  $e = ab$ . It now follows that  $D \cap N[b] = \{a\}$ , so  $e = ab$  is a  $D$ -spine. Thus every edge of  $G$  which is blocked by  $D$  is a  $D$ -spine, so (2) follows.

For the converse, assume that (1) and (2) hold. Remark 2 ensures that  $[G, D, x]$  is saturated, so it suffices to show that every edge of  $[G, D, x]$  has a replacement.

Suppose the edge  $dx$  is deleted, for some  $d \in D$ . Since  $D$  is irredundant, there is a vertex  $v$  in  $N[d]$  but not in  $N[D - d]$ , so insertion of  $vx$  in  $[G, D - d, x]$  does not create a triangle. Again, suppose an edge  $e$  of  $G$  is deleted. If  $e$  is not blocked in  $G$  by  $D$ , then there is a replacement edge for  $e$  in  $G$  which does not create a triangle with  $x$ . If  $e$  is blocked in  $G$  by  $D$ , then  $e$  is a  $D$ -spine, so  $e = ab$  with  $a \in D$  and  $D \cap N[b] = \{a\}$ . Then  $bx$  is a replacement for  $ab$ , since insertion of  $bx$  in  $[G, D, x]$  creates a unique triangle  $abxa$ , which is not present in  $[G - e, D, x]$ . Thus every edge of  $[G, D, x]$  is essential, so  $[G, D, x]$  is minimally saturated. ■

**Corollary 1** *Every maximal triangle-free graph has at least one maximal triangle-free extension.*

**Proof.** An easy induction shows that any finite graph has an independent dominating set. So suppose  $G$  is a maximal triangle-free graph and  $D$  is an independent dominating set for  $G$ . Then  $D$  automatically satisfies condition (1) of Theorem 1. Since  $G$  is triangle-free, each edge is a replacement for itself, and condition (2) is satisfied vacuously. Then  $[G, D, x]$  is minimally saturated, by Theorem 1. But  $[G, D, x]$  is triangle-free, so is a maximal triangle-free graph. ■

Theorem 1 can be extended to encompass all triangle-saturated graphs, not just those which are minimally saturated, as follows:

**Corollary 2** *Let  $G$  be any triangle-saturated graph and let  $D$  be any subset of vertices of  $G$ . Then  $[G, D, x]$  is minimally triangle-saturated if and only if (1)  $D$  is an irredundant dominating set for  $G$ , and (2) any edge of  $G$  which is inessential or blocked in  $G$  by  $D$  is a  $D$ -spine.*

**Proof.** Suppose  $[G, D, x]$  is minimally saturated. Then (1) follows as in the proof of Theorem 1. Also, in view of Theorem 1, (2) will follow from showing that every inessential edge  $e$  of  $G$  is a  $D$ -spine. Since there is no replacement edge for  $e$  in  $G$ , it must have a replacement edge  $bx$ , where  $b$  is a vertex of  $e$ , by Remark 3. Then  $b$  must be in  $G - D$ , since  $bx$  is not in  $[G, D, x]$ . But  $b \in N[D]$ , so  $b$  has a neighbour  $a \in D$ . If  $e \neq ab$ , adding  $bx$  to  $[G, D, x] - e$  would produce a triangle  $abxa$ , contradicting the fact that  $bx$  is a replacement edge for  $e$ . Hence  $e = ab$ . But adding  $bx$  to  $[G, D, x] - ab$  does not create a triangle, so  $D \cap N[b] = \{a\}$ . Thus any inessential edge of  $G$  must be a  $D$ -spine.

Checking the converse is straightforward, so is omitted. ■

For example, the 5-cycle  $C_5 := abcdea$  is a minimally saturated graph, so  $C_5 + ad$  is saturated but not minimally so. It can be verified that  $D = \{a, b\}$  is an irredundant dominating set and that the inessential edge  $ad$  and the essential edges  $bc$  and  $ae$  are all  $D$ -spines. The only remaining essential edge with at least one vertex in  $D$  is  $ab$ , and the replacement edge  $be$  shows that  $ab$  is not blocked by  $D$ . Hence  $[C_5 + ad, \{a, b\}, x]$  is minimally saturated, by Corollary 2; in fact, this is just the graph  $B_0$ . For convenience, any subset of the vertices of a triangle-saturated graph will be called a *support set* if it satisfies the conditions of Corollary 2. In particular, the proof of Corollary 1 shows that every maximal triangle-free graph has a support set. Perhaps this generalizes to all minimally triangle-saturated graphs.

| $D$           | Independent Set | Edges Blocked | Irredundant Domination | Support Set |
|---------------|-----------------|---------------|------------------------|-------------|
| $\{a, e\}$    | yes             | $be$          | yes                    | no          |
| $\{b, d\}$    | no              | none          | yes                    | yes         |
| $\{b, f\}$    | yes             | none          | yes                    | yes         |
| $\{d, e\}$    | yes             | none          | yes                    | yes         |
| $\{a, c, f\}$ | yes             | $ad, ce$      | yes                    | no          |

Table 1: Minimal dominating sets for  $B_0$

**Conjecture 1** *Every minimally triangle-saturated graph has a support set.*

This would imply that every minimally triangle-saturated graph has a minimally triangle-saturated extension.

The support sets for a minimally saturated graph must be among its minimal dominating sets. Let us examine systematically the non-isomorphic extensions of  $B_0$  (Fig. 1) based on minimal dominating sets. Up to automorphism, there are five minimal dominating sets (Table 1), three of which are support sets. By Theorem 1, there are just three minimally triangle-saturated extensions of  $B_0$ , namely  $B'_0 := [B_0, \{b, d\}, x]$ ,  $B_1 := [B_0, \{b, f\}, x]$  and  $B''_0 := [B_0, \{d, e\}, x]$ . An independent dominating set is automatically irredundant, so in a minimally saturated graph it is a support set precisely when it blocks no edge which belongs to a triangle.

Two of the three minimally triangle-saturated extensions of  $B_0$  are based on independent support sets. Moreover, every independent dominating set in a maximal triangle-free graph is a support set, as noted in the proof of Corollary 1. Perhaps this generalizes to all minimally triangle-saturated graphs.

**Conjecture 2** *Every minimally triangle-saturated graph has an independent support set.*

Further evidence supporting this conjecture is provided by the family of graphs produced by the *hanging planter* construction [4]. Indeed, if  $G$  is any minimally triangle-saturated hanging planter graph, duplication of its top vertex  $t$  yields another minimally saturated graph, so  $N(t)$  is an independent support set in  $G$ .

Note that the neighbourhood of  $a$  in  $B_0$  is  $N(a) = \{b, d\}$ , so  $x$  is a duplicate of  $a$  in  $B'_0$ . Similarly  $N(f) = \{d, e\}$ , so  $x$  is a duplicate of  $f$  in  $B''_0$ . Hence  $B_1$  (Fig. 1) is the only minimally saturated extension of  $B_0$  which is primitive. The next section takes up this subject.

### 3 Primitive Extensions

Suppose  $G$  is a maximal triangle-free graph and  $v$  is any vertex in  $G$ . Then the neighbourhood  $N(v)$  is a support set for  $G$ , since it is an independent set and all vertices of  $G - N[v]$  are at distance 2 from  $v$ , so are dominated by  $N(v)$ . Thus the extension  $[G, N(v), x]$  is minimally saturated by Theorem 1; but it is not primitive because  $x$  duplicates  $v$ .

**Remark 4** Let  $G$  be a minimally triangle-saturated graph. If every minimal dominating set in  $G$  is the neighbourhood of some vertex, then no minimally triangle-saturated extension of  $G$  is primitive.

For example, in the minimally saturated cycle  $C_5$  every minimal dominating set is the neighbourhood of some vertex, so  $C_5$  has no minimally saturated extension which is primitive. Again, no complete bipartite graph has a primitive minimally saturated extension because its partite sets are its only support sets and each is the neighbourhood of any vertex in the other set.

Which minimally triangle-saturated graphs have primitive extensions that are also minimally saturated? We do not yet know the full answer. However, a theorem of Pach [6] provides the answer for those graphs which are triangle-free, as we shall now show.

Let  $G_0 := K_2$  and for any integer  $k \geq 1$  let  $G_k$  be the graph formed from  $C_{3k+2}$  by adding all edges (chords) between vertices at distance 1 (mod 3) on the cycle. Equivalently, if  $k \geq 0$  then  $G_k$  is the graph with vertex set  $\mathbb{Z}_{3k+2}$  (the residue classes of integers modulo  $3k + 2$ ), and edges  $ij$  precisely when  $|i - j| \equiv 1 \pmod{3}$ . The graphs  $G_k$  are triangle-free circulant graphs [6], and in fact it is easy to see that they are maximal triangle-free graphs. Let  $\{G_k\}^*$  denote the family of all graphs which can be derived from  $G_k$  by duplicating any vertices a finite number of times. For example,  $\{G_0\}^*$  is the family of all complete bipartite graphs and  $\{G_1\}^*$  is the family of graphs derived from  $C_5$  by vertex duplication. All members of  $\{G_k\}^*$  are maximal triangle-free graphs, by Corollary 2.3 of [2].

**Theorem 2 (Pach)** Let  $G$  be any triangle-free graph. Then every independent set of vertices in  $G$  has a common neighbour if and only if  $G \in \{G_k\}^*$  for some  $k \geq 0$ .

If  $[G, D, x]$  is triangle-free, then  $G$  must be triangle-free and  $D$  must be an independent set. By the results of Section 2, Pach's Theorem then implies

**Corollary 3** A maximal triangle-free graph  $G$  has a primitive maximal triangle-free extension if and only if  $G \notin \cup_{k \geq 0} \{G_k\}^*$ .

Next let us explore the existence of primitive minimally triangle-saturated extensions of minimally triangle-saturated graphs which are not triangle-free. The smallest such graph is  $B_0$  (Fig. 1), of order 6. We noted in Section 2 that  $B_1 = [B_0, \{b, f\}, x]$  is the unique minimally triangle-saturated extension of  $B_0$  which is primitive. There are four minimally triangle-saturated graphs of order 7 which are not triangle-free [2]: three extensions of  $B_0$ , namely  $B'_0$ ,  $B''_0$  and  $B_1$  (introduced prior to Table 1), together with the graph  $M_0$  (Fig. 2). Of these,  $B_1$  and  $M_0$  are primitive, and  $B'_0$  and  $B''_0$  each have one pair of duplicate vertices. No minimal dominating set in  $B'_0$  or  $B''_0$  contains exactly one of the duplicate vertices, so neither has a primitive minimally triangle-saturated extension.

Let us call a support set  $D$  *primitive* if  $D$  is not the neighbourhood of any vertex of  $G$ . Note that if  $D$  is a primitive support set for the minimally saturated graph  $G$ , the extension  $[G, D, x]$  is certainly primitive if  $G$  is primitive, and can still be

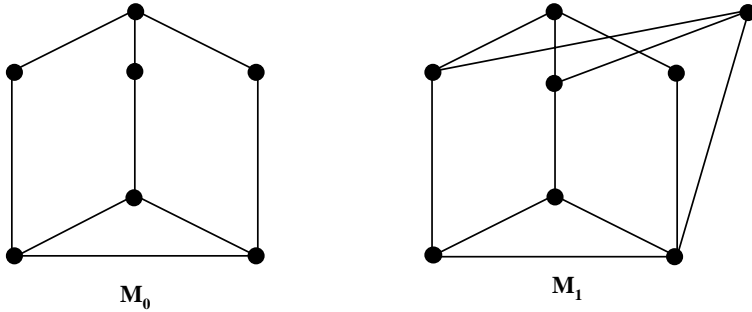


Figure 2:  $M_0$  and its unique primitive minimally saturated extension

primitive even if  $G$  is not primitive (provided exactly one of each pair of duplicate vertices in  $G$  belongs to  $D$ ). Note in particular that if some vertex of  $G$  has more than one duplicate, no extension of  $G$  is primitive. Up to automorphism, there are three primitive support sets in  $B_1$ , which yield the order 8 primitive minimally saturated graphs  $B_2 := [B_1, \{b, x\}, y]$ ,  $B_3 := [B_1, \{a, c, x\}, y]$ ,  $B_4 := [B_1, \{d, e, f\}, y]$ . Also, up to automorphism there is a unique primitive support set in  $M_0$ , so  $M_0$  has a unique primitive minimally triangle-saturated extension  $M_1$  (Fig. 2).

We are indebted to Brendan MacKay for carrying out a computer search which established that there are 11 primitive minimally triangle-saturated graphs of order 8, including  $B_2$ ,  $B_3$ ,  $B_4$ , and  $M_1$  (Fig. 3). With the exception of Pach's graph  $G_2$ , they all contain one or more triangles. Examination shows that all except  $G_2$  have at least one primitive minimally saturated extension, suggesting

**Conjecture 3** *Every primitive minimally triangle-saturated graph which is not triangle-free has a primitive minimally triangle-saturated extension.*

If this conjecture is true, it would follow that the Pach graphs  $G_k$  must be the only primitive minimally triangle-saturated graphs without a primitive minimally triangle-saturated extension.

Note that  $\{b, f\}$  is an independent primitive support set for  $B_0$ . Note further that  $\{a, c, x\}$  is an independent primitive support set for  $B_1$ , and any primitive support set for  $M_0$  is also independent. Of course, Pach's Theorem implies that  $G_2$  has no primitive support set which is independent, but one is tempted to conjecture that all minimally triangle-saturated graphs which are not triangle-free have an independent primitive support set. However, it turns out that this fails for  $M_1$ , though it holds for the other 9 graphs of order 8 (Fig. 3). Up to automorphism, the primitive support sets of  $M_1$  are  $\{b, f, g\}$  and  $\{c, f, g\}$ , neither of which is independent. The independent support sets of  $M_1$  are  $N(x) = \{a, e, f\}$ ,  $N(d) = \{a, g\}$ ,  $N(f) = \{c, g, x\}$  and  $N(g) = \{d, e, f\}$ , none of which is primitive. Hence

**Remark 5**  $M_1$  is a primitive minimally triangle-saturated graph which is not triangle-free and has no independent primitive support set.



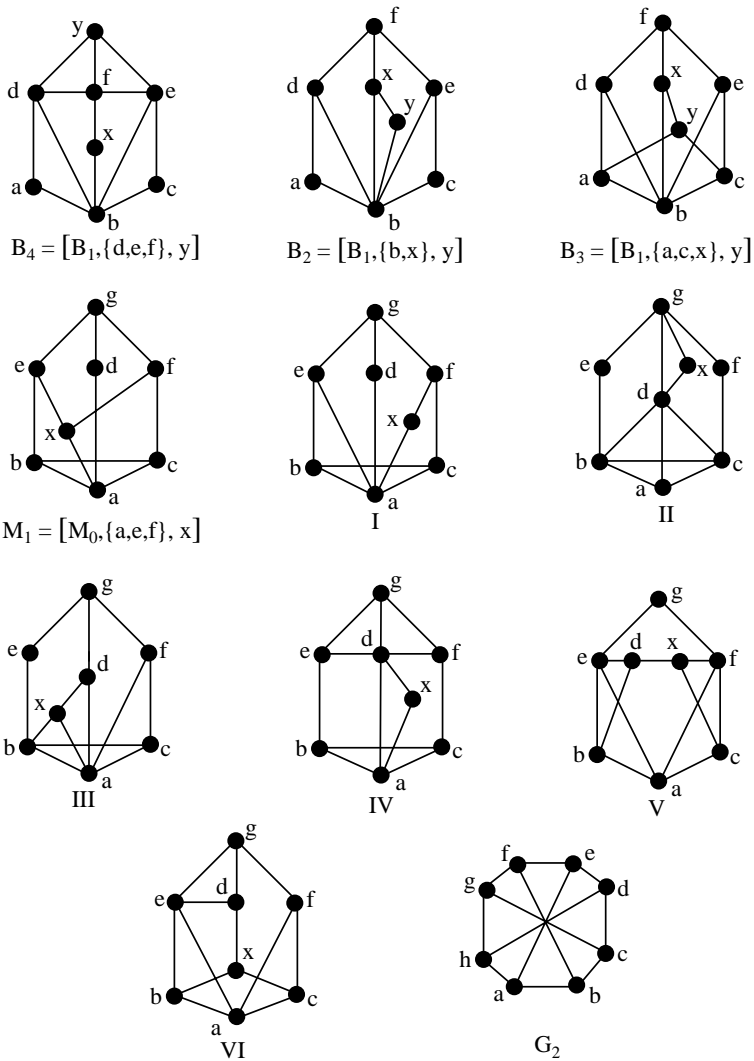


Figure 3: All order 8 primitive minimally triangle-saturated graphs

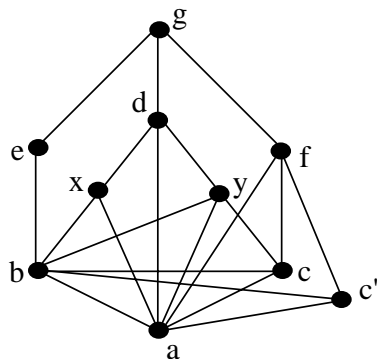


Figure 4: Primitive minimally saturated extension of an imprimitive minimally saturated graph

It was remarked earlier that an extension  $[G, D, x]$  of a minimally saturated graph might be primitive and minimally saturated even though  $G$  itself is not primitive. An example based on graph III (Fig. 3) can be obtained by duplicating the selective vertex  $c$  (see [2]) to produce a minimally triangle-saturated graph  $A$  of order 9. Then  $\{a, b, c, d\}$  is a primitive support set for  $A$ , so  $A' = [A, \{a, b, c, d\}, y]$  is primitive (Fig. 4). Hence

**Remark 6** *The minimally triangle-saturated graph  $A$  is not primitive, but it has a primitive minimally triangle-saturated extension.*

Indeed, let  $B^*$  be the graph obtained from  $G_1 = C_5 = 012340$  by duplicating vertex 1 and adding the edge 24. This new graph is triangle-saturated, but is neither minimal (because of the edge 24) nor primitive (because of the duplicate of vertex 1). However  $\{1, 2\}$  is a primitive support set and  $[B^*, \{1, 2\}, x]$  is in fact  $B_1$ .

**Remark 7** *The triangle-saturated graph  $B^*$  is neither minimally saturated nor primitive, but it has a triangle-saturated extension which is both minimally saturated and primitive.*

## 4 Maximal Triangle-free Graphs

As shown in [3] and [4], for sufficiently large  $n$  one can construct large numbers of triangle-saturated graphs of order  $n$  that are both minimally saturated and primitive. However, those constructions typically produce graphs that contain triangles. It is more difficult to find primitive maximal triangle-free graphs. At least for small orders, they form a tiny proportion of the total. Table 2 compares the number  $f_n$  of primitive maximal triangle-free graphs of order  $n$  with the number  $g_n$  of minimally

| $n$ | $g_n$ | $f_n$ |
|-----|-------|-------|
| 4   | 0     | 0     |
| 5   | 1     | 1     |
| 6   | 1     | 0     |
| 7   | 2     | 0     |
| 8   | 11    | 1     |
| 9   | 36    | 1     |
| 10  | 240   | 2     |
| 11  | 1986  | 4     |
| 12  | ...   | 8     |
| 13  | ...   | 24    |
| 14  | ...   | 91    |

Table 2: Primitive minimally saturated and maximal triangle-free graphs

triangle-saturated graphs of order  $n$ , for all orders that are known. We are again indebted to Brendan McKay [5] for implementing a computer search to determine these numbers.

In this section we describe a construction which yields primitive maximal triangle-free graphs of every order  $n \geq 9$ . We start by finding a family of such graphs, one for each order  $n \equiv 0 \pmod{3}$  and then show how to extend each of them repeatedly. By Corollary 1, there always exists an independent support set in a maximal triangle-free graph  $G$ . The difficulty for guaranteeing primitivity is to make sure that the newly added vertex does not duplicate a vertex of  $G$ . In fact, this is impossible if  $G \in \{G_k\}^*$  for any  $k \geq 0$ , by Corollary 3.

Our construction begins with any of the circulant graphs  $G_k$  (defined in Sec. 3), provided  $k \geq 1$ . Add three new vertices  $s_0, s_1, s_2$  so that the  $G_k$ -neighbourhood of  $s_0$  is

$$N_0 = \{i \equiv 0 \pmod{3}, 0 \leq i \leq 3k + 1\}$$

and the  $G_k$ -neighbourhood of  $s_r$  is

$$N_r = \{i + r \pmod{3k + 2} : i \in N_0\}$$

with  $r = 1, 2$ . Written compactly, these  $G_k$ -neighbourhoods are

$$N_r = N_0 + r \pmod{3k + 2}, \quad r = 1, 2.$$

We say that  $N_r$  is obtained from  $N_0$  by *rotation through  $r$  modulo  $3k + 2$* . These neighbourhoods are intimately related to the structure of  $G_k$  as is shown by the following observation.

**Remark 8**  $N_1 = N_0 + 1$  is precisely the neighbourhood of 0 in  $G_k$ .

Finally add a fourth vertex  $t$  adjacent only to  $s_0, s_1$ , and  $s_2$ . The resulting graph  $H_n$  has order  $n = 3k + 6$ . It is actually an example of the *hanging planter* construction described in [4]. In the terminology of that paper the circulant graph

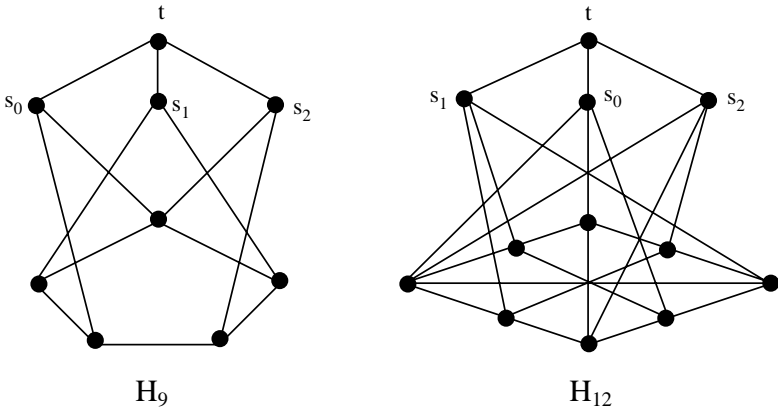


Figure 5: The starter graphs  $H_9$  and  $H_{12}$

$G_k$  is the *base*, the three  $s_i$  are the *support vertices*, and  $t$  is the *top vertex*. Note that  $H_n$  is a maximal triangle-free graph, by Theorem 3 of [4], because  $G_k$  is a maximal triangle-free graph and  $H_n$  is a hanging planter with base  $G_k$  and each of the sets  $N_0$ ,  $N_1$  and  $N_2$  is independent. Moreover  $H_n$  is primitive, by Theorem 2 of [4], because  $G_k$  is primitive. For the graphs  $H_9$  and  $H_{12}$  obtained from  $G_1$  and  $G_2$  respectively, see Fig. 5.

Beginning with  $H_n$  for  $n = 3k + 6$  and  $k \geq 1$ , we now define a sequence of extensions  $\{H_n(r) : 2 \leq r \leq 3k + 1\}$ . Let  $H_n(2) := H_n$ . Successively extend, at each step adding a new vertex  $s_r$  adjacent to the top vertex  $t$  and to the set  $N_r := N_0 + r \pmod{3k + 2}$  obtained from  $N_0$  by rotation through  $r \pmod{3k + 2}$ . Thus

$$H_n(r) = [H_n(r - 1), N_r \cup \{t\}, s_r] \quad \text{for } 3 \leq r \leq 3k + 1.$$

**Theorem 3** For  $2 \leq r \leq 3k + 1$ ,  $k \geq 1$  and  $n = 3k + 6$ , each  $H_n(r)$  is a primitive maximal triangle-free graph.

**Proof.** The set  $N_0$  is an independent dominating set for  $G_k$ . Hence each of the sets  $N_r := N_0 + r \pmod{3k + 2}$  with  $0 \leq r \leq 3k + 1$  is an independent dominating set for  $G_k$  because  $G_k$  is a circulant graph. Moreover no two of these  $3k + 2$  sets are identical.

The hanging planter construction ensures that  $H_n(2) = H_n$  is a primitive maximal triangle-free graph. Now suppose that  $H_n(r - 1)$  is primitive and maximal triangle-free, for some  $r$  satisfying  $3 \leq r \leq 3k + 1$ , and the neighbourhood of  $t$  in  $H_n(r - 1)$  is  $\{s_0, s_1, \dots, s_{r-1}\}$ . Because  $N_r$  is an independent dominating set for  $G_k$ , it follows that  $N_r \cup \{t\}$  is an independent dominating set for  $H_n(r - 1)$ . Then, trivially,  $N_r \cup \{t\}$  dominates  $H_n(r - 1)$  irredundantly, and since  $H_n(r - 1)$  is triangle-free,  $N_r \cup \{t\}$  blocks no edge of  $H_n(r - 1)$ . Hence  $N_r \cup \{t\}$  is a support set for  $H_n(r - 1)$ , so Theorem 1 guarantees that  $H_n(r) = [H_n(r - 1), N_r \cup \{t\}, s_r]$  is minimally saturated, and hence

is a maximal triangle-free graph. Further, because the sets  $N_i$  with  $0 \leq i \leq r$  are all different, and  $t$  is not adjacent to  $G_k$  in  $H_n(r-1)$ , it follows that  $N_r \cup \{t\}$  is not the neighbourhood of any vertex in  $H_n(r-1)$ , so  $H_n(r)$  is primitive. Hence by finite induction each  $H_n(r)$  with  $2 \leq r \leq 3k+1$  is a primitive maximal triangle-free graph. ■

**Lemma 2** For  $2 \leq r \leq 3k+1$ ,  $k \geq 1$  and  $n = 3k+6$ , the graph  $H_n(r)$  has order  $3k+r+4$  and size  $\frac{1}{2}(3k+2)(k+1) + (r+1)(k+2)$ .

**Proof.** The base of the hanging planter  $H_n(2) = H_n$  is the circulant graph  $G_k$ . But  $N_1 = N_0 + 1$  is the neighbourhood of 0 in  $G_k$ , by Remark 8, and  $|N_0| = k+1$ , so  $G_k$  is a regular graph of order  $3k+2$  and degree  $k+1$ . Hence  $G_k$  has size  $\frac{1}{2}(3k+2)(k+1)$  and  $H_n$  has order  $3k+6$ .

Also  $H_n(r)$  is formed from  $H_n(2)$  by  $r-2$  extensions, so the order of  $H_n(r)$  is  $3k+r+4$ . Each edge of  $H_n(r)$  is either in  $G_k$  or is incident with a unique vertex  $s_i$ , where  $0 \leq i \leq r$ , and each  $s_i$  has degree  $k+2$  in  $H_n(r)$ , so the size of  $H_n(r)$  is  $\frac{1}{2}(3k+2)(k+1) + (r+1)(k+2)$ . ■

**Lemma 3** No two graphs in the family  $\{H_n(r) : 2 \leq r \leq 3k+1, k \geq 1, n = 3k+6\}$  are isomorphic.

**Proof.** Suppose  $H_n(r)$  and  $H_{n'}(r')$  have the same order, and  $n' > n$ . Let  $n = 3k+6$  and  $n' = 3k'+6$ , and put  $s := k' - k$ . Because the orders are equal,  $r' = r - 3s$  by Lemma 2. By a routine computation, Lemma 2 now implies

$$\text{size } H_{n'}(r') - \text{size } H_n(r) = \frac{1}{2}s(r' + r - 5).$$

This difference is always positive because  $r' \geq 2$  and  $r = r' + 3s \geq 5$ . Hence  $H_{n'}(r')$  is larger than  $H_n(r)$  and, consequently, no two graphs in this family can be isomorphic. ■

**Theorem 4** For  $n \geq 9$  there are at least  $\lfloor \frac{n-3}{6} \rfloor$  primitive maximal triangle-free graphs of order  $n$ .

**Proof.** By Theorem 3 and Lemma 2, the graph  $H_s(r)$  is a primitive maximal triangle-free graph of order  $n := r + s - 2$ . With  $2 \leq r \leq 3k+1$ ,  $k \geq 1$  and  $s = 3k+6$ , it follows that there is an  $H_s(r)$  of each order from  $3k+6$  to  $6k+5$  inclusive. Every integer  $n \geq 9$  lies in at least one such interval. Indeed,  $3k+6 \leq n \leq 6k+5$  holds if and only if

$$\left\lfloor \frac{n-5}{6} \right\rfloor \leq k \leq \left\lfloor \frac{n-6}{3} \right\rfloor.$$

By considering  $n$  modulo 6, it follows that the number of solutions for  $k$  is

$$\left\lfloor \frac{n-6}{3} \right\rfloor - \left\lfloor \frac{n-5}{6} \right\rfloor + 1 = \left\lfloor \frac{n-3}{6} \right\rfloor.$$

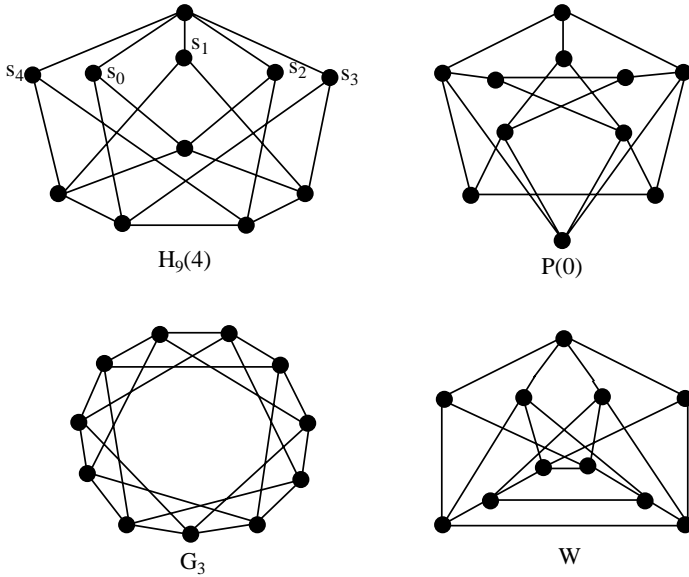


Figure 6: The 4 primitive maximal triangle-free graphs of order 11

The theorem now follows because Lemma 3 guarantees that no two of these graphs are isomorphic. ■

In fact, other non-isomorphic primitive maximal triangle-free graphs can be produced by “mixing” the order of extensions to the graph  $H_n$ . The first case where this possibility occurs is  $[H_{12}, N_4, s_4]$ , which turns out not to be isomorphic to  $H_{12}(3) = [H_{12}, N_3, s_3]$ .

The primitive maximal triangle-free graphs formed by extending the circulant graphs in the fashion described above make up a family whose cardinality we have shown to be a linear function of  $n$ . Even for the relatively small values of  $n$  that have been enumerated, there are many primitive maximal triangle-free graphs not produced by our construction, so the question remains as to the cardinality of the whole set. It is tempting to conjecture that this is exponential in  $n$ , but as yet we have no strong support for this conjecture.

As indicated in Table 2, there are two primitive maximal triangle-free graphs of order 10 and four of order 11. The order 10 graphs are the Petersen graph and the extension  $H_9(3)$  described above. The four order 11 graphs are shown in Fig. 6; these include the circulant graph  $G_3$ , the extension  $H_9(4)$ , and an extension  $P(0)$  of the Petersen graph which will be discussed in the next section. The graph  $W$  is also formed by extending  $H_9(3)$ , but using a different support set from that described in the general construction above.

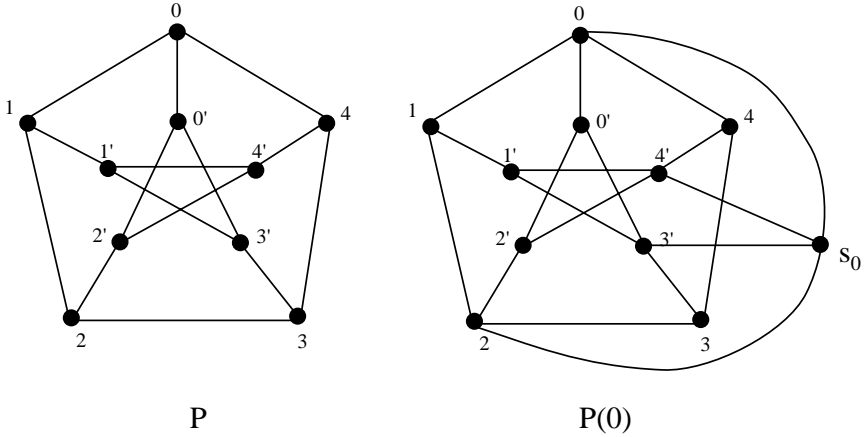


Figure 7: The Petersen graph and its unique primitive extension

## 5 Other Triangle-free Extensions

We constructed the graphs  $H_n(r)$  in the previous section by labelling a circulant base graph  $G_k$  with  $\mathbb{Z}_{3k+2}$  and rotating an independent support set modulo  $3k+2$  to define neighbourhoods for support vertices in a sequence of hanging planter constructions. This construction method can be adapted to other “starter” (base) graphs.

To illustrate, we shall outline such a construction based on the Petersen graph,  $P$ . Label the vertices of  $P$  with  $\mathbb{Z}_5 \times \mathbb{Z}_2$  as shown in Fig. 7. For brevity, we write  $i$  for  $(i, 0) \in \mathbb{Z}_5 \times \mathbb{Z}_2$  and  $i'$  for  $(i, 1) \in \mathbb{Z}_5 \times \mathbb{Z}_2$ . Up to automorphism,  $P$  has only one primitive independent support set, namely  $\{0, 2, 3', 4'\}$ . Add a new vertex  $s_0$  with neighbourhood  $N_0^* := \{0, 2, 3', 4'\}$ , forming the extension  $P(0) := [P, N_0^*, s_0]$ , also shown in Fig. 7.

For  $1 \leq r \leq 4$ , form a sequence of extensions  $P(r)$  by adjoining at each step a new vertex with neighbourhood

$$N_r^* := N_0^* + (r, 0) \subseteq \mathbb{Z}_5 \times \mathbb{Z}_2$$

so that

$$P(r) = [P(r-1), N_r^*, s_r], \quad 1 \leq r \leq 4.$$

With our abbreviated notation, we can write  $N_r^* = N_0^* + r \pmod{5}$ , and say that  $N_r^*$  is obtained from  $N_0^*$  by *rotation through  $r \pmod{5}$* . Then,  $N_1^* := \{1, 3, 4', 0'\}$ , and so on. Each set  $N_r^*$  is an independent dominating set for  $P$ , and it is easy to verify

**Remark 9** Any two of the sets  $N_r^*, 0 \leq r \leq 4$ , have a common vertex.

Hence each set  $N_r^*$  is a primitive independent support set for  $P(r-1), 1 \leq r \leq 4$ . It follows that  $\{P(r) : 0 \leq r \leq 4\}$  is a family of primitive maximal triangle-free graphs.

Now extend the order 15 graph  $P(4)$  by adjoining a new vertex  $t$  with neighbourhood  $N(t) := \{s_r : 0 \leq r \leq 4\}$ , forming the extension

$$P(5) := [P(4), N(t), t].$$

This is a hanging planter with base  $P(4)$ , support set  $N(t)$  and top vertex  $t$ . It is again a primitive maximal triangle-free graph.

For  $6 \leq r \leq 10$ , form a sequence of further extensions  $P(r)$  by adjoining at each step a new vertex  $s_r$  with neighbourhood  $N_r^* \cup \{t\}$  where

$$N_6^* := \{0, 2, 1'\}, \quad \text{and} \\ N_r^* := N_6^* + (r - 6, 0) \subseteq \mathbb{Z}_5 \times \mathbb{Z}_2.$$

Thus

$$P(r) = [P(r - 1), N_r^* \cup \{t\}, s_r], \quad 6 \leq r \leq 10.$$

For  $6 \leq r \leq 10$ , each set  $N_r^* \cup \{t\}$  is a primitive independent support set for  $P(r)$ . Hence

**Remark 10**  $\{P(r) : 0 \leq r \leq 10\}$  is a family of primitive maximal triangle-free graphs, where each  $P(r)$  has order  $r + 11$ .

A further sequence of five extensions can be defined similarly, but it is not clear whether the sequence of extensions can be continued indefinitely.

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