Equivalence classes of n-dimensional proper Hadamard matrices

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Abstract

Equivalence operations for n-dimensional proper Hadamard matrices are defined. Their equivalence classes are investigated and bounds on the number of such equivalence classes are found to be associated with the number of equivalence classes of 2-dimensional Hadamard matrices. Properties of planes (2-dimensional sections) in an n-dimensional proper Hadamard matrix are investigated. It is shown that all planes of an n-dimensional proper Hadamard matrix are Hadamard equivalent to the 2-dimensional Hadamard matrix from which it is constructed, regardless of whether the construction technique applied is Product Construction, Group Development or Relative Difference Set Construction. The relationship of the three constructions is demonstrated.

1 Introduction

A $v \times v$ matrix H with entries $\{\pm 1\}$ is a Hadamard matrix if $HH^\top = vI$, where v has to be 1, 2, or multiple of 4. A Hadamard matrix H is *normalized* if the entries of the first row and the first column are all 1's. Two dimensional Hadamard matrices were discovered over 100 years ago and have been found extremely useful in areas such as error-correcting codes. For general information on Hadamard matrices, see [6, 9].

In 1971, Shlichta [7] discovered the existence of higher dimensional Hadamard matrices and constructed a 3-dimensional proper Hadamard matrix. Since then a relatively

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small amount of research has been done on higher dimensional Hadamard matrices. Even the elementary equivalence operations for equivalence classes of such matrices have not been defined. Our fundamental intention here is to define elementary equivalence operations and investigate equivalence classes of n-dimensional proper Hadamard matrices. In the process we demonstrate links between all three known constructions of n-dimensional proper Hadamard matrices.

An n-dimensional order v proper Hadamard matrix [3] is an n-dimensional $\{\pm 1\}$ array $[A(i_1, \ldots, i_n) | 1 \leq i_j \leq v, 1 \leq j \leq n]$ which has the following property: for each distinct pair k and l of coordinates, letting the k^{th} coordinate take the values x and y, $1 \leq x, y \leq v$, letting the l^{th} coordinate run from 1 to v,

$$\sum_{1 \leq i_l \leq v} A(i_1, \ldots, i_{k-1}, x, \ldots, i_l, \ldots, i_n) A(i_1, \ldots, i_{k-1}, y, \ldots, i_l, \ldots, i_n) = v\delta_{xy} \tag{1}$$

for every choice of indices in the other coordinates. Another way to say this is that every 2-dimensional submatrix, obtained by fixing all but 2 coordinates, is a Hadamard matrix. Given a 2-dimensional Hadamard matrix $H = [h(i,j)]$, Yang [10] developed a simple technique to construct an n-dimensional proper Hadamard matrix $[A(i_1, i_2, \ldots, i_n)]$, setting

$$A(i_1, i_2, \ldots, i_n) = \prod_{1 \leq s < t \leq n} h(i_s, i_t). \tag{2}$$

This technique is termed *product construction* [5]. Moreover, if H is developed over an abelian group G, that is, $H = [h(i,j)] = [\phi(g_1 g_2)]_{g_i \in G}$, where $\phi : G \to \{\pm 1\}$ is a set mapping, then the n-dimensional array $[A(g_1, g_2, \ldots, g_n)]_{g_i \in G}$, where $A(g_1, g_2, \ldots, g_n) = \phi(g_1 g_2 \cdots g_n)$, is an n-dimensional proper Hadamard matrix [2]. It was noted later by Horadam and Lin [5] that G does not need to be abelian. We call this technique *group development*. On the other hand if H is cocyclic, that is, $H = [h(i,j)] = [\psi(g_1, g_2)]_{g_i \in G}$, where $\psi : G \times G \to \{\pm 1\}$ is a 2-dimensional cocycle, that is, ψ is a set mapping satisfying

$$\psi(g_1, g_2)\psi(g_1 g_2, g_3) = \psi(g_2, g_3)\psi(g_1, g_2 g_3) \ \ \forall g_1, g_2, g_3 \in G, \tag{3}$$

then the n-dimensional array, $[A(g_1, g_2, \ldots, g_n)]_{g_i \in G}$, where

$$A(g_1, g_2, \ldots, g_n) = \prod_{i=2}^{n} \psi(\prod_{j=1}^{i-1} g_j, g_i), \tag{4}$$

is an n-dimensional proper Hadamard matrix [1, 5]. We call this technique *relative difference set construction*. Given a 2-dimensional Hadamard matrix H, different construction techniques applied to H generally produce different n-dimensional proper Hadamard matrices.

We will employ the following notations.

$\mathcal{P}(n, v, H)$ = n-dimensional order v proper Hadamard matrix constructed from a $v \times v$ Hadamard matrix H by product construction.

$\mathcal{G}(n, v, G, \phi)$ = n-dimensional order v proper Hadamard matrix constructed by group development over a finite group G of order v with mapping $\phi : G \to \{\pm 1\}$.

$\mathcal{R}(n, v, G, \psi)$= n-dimensional order v proper Hadamard matrix constructed from a cocyclic Hadamard matrix over a finite group G of order v with cocycle $\psi : G \times G \to \{\pm 1\}$ by relative difference set construction.

In Section 2, planes are defined and some of their fundamental properties are uncovered. Section 3 defines equivalence operations for n-dimensional proper Hadamard matrices and properties of planes in equivalent n-dimensional proper Hadamard matrices are investigated. Section 4 gives bounds on the number of such equivalence classes and investigates relationships among n-dimensional proper Hadamard matrices constructed by the above standard constructions.

2 Planes in n-Dimensional Proper Hadamard Matrices

Definition 2.1 *Let $M = [A(i_1, i_2, \ldots, i_n) \mid 1 \leq i_j \leq v, 1 \leq j \leq n]$ be an n-dimensional order v proper Hadamard matrix. The set*

$$P = \{A(i_1, i_2, \ldots, i_{k-1}, x, i_{k+1}, \ldots, i_{l-1}, y, i_{l+1}, \ldots, i_n) \mid 1 \leq x, y \leq v\},$$

where $i_1, i_2, \ldots, i_{k-1}, i_{k+1}, \ldots, i_{l-1}, i_{l+1}, \ldots, i_n$ are fixed indices, is called a plane. Two planes P and $Q = \{A(j_1, j_2, \ldots, j_{k-1}, a, j_{k+1}, \ldots, j_{l-1}, b, j_{l+1}, \ldots, j_n) \mid 1 \leq a, b \leq v\}$ are parallel if they have their fixed indices in the same coordinates and there exists $h \in \{1, 2, \ldots, k-1, k+1, \ldots, l-1, l+1, \ldots, n\}$ such that $i_h \neq j_h$.

From now on, we denote the entry $h(i, j)$ in a 2-dimensional matrix $H = [h(i, j)]$ by (i, j). To further investigate planes in n-dimensional proper Hadamard matrices, the following lemma is useful.

Lemma 2.2 *Let H be a normalized $v \times v$ Hadamard matrix. If we take any v-tuple of 1's and -1's, and multiply coordinatewise each row (or, alternatively, each column) of H by this v-tuple to form another $v \times v$ matrix H', then H' is Hadamard and is equivalent to H.*

Proof. H' is equivalent to H because H' is formed by multiplying each column (or row) of H by a fixed value (either 1 or -1). □

We begin by investigating the relationship between any plane in $\mathcal{P}(n, v, H)$ and the Hadamard matrix H from which it is constructed.

Lemma 2.3 *Any plane P in $\mathcal{P}(3, v, H)$ is equivalent to H.*

Proof. One can easily show that any plane P in $\mathcal{P}(3, v, H)$ is formed by first multiplying coordinatewise each row of H by the i^{th} row of H, where i is the fixed index of P, and then multiplying each row x by the value of (i, x), which is ± 1. Therefore by Lemma 2.2 P is equivalent to H. □

Lemma 2.4 *Every plane in $\mathcal{P}(n + 1, v, H)$ is equivalent to some plane in a $\mathcal{P}(n, v, H)$.*

Proof. Let $1 \leq k < l \leq n + 1$ and

$$P_{n+1} = \{A(i_1, i_2, \ldots, i_{k-1}, c, i_{k+1}, \ldots, i_{l-1}, d, i_{l+1}, \ldots, i_n, i_{n+1}) | 1 \leq c, d \leq v\}$$

be a plane in $\mathcal{P}(n + 1, v, H)$. Consider the case $l < n + 1$. Let

$$P_n = \{A(i_1, i_2, \ldots, i_{k-1}, c, i_{k+1}, \ldots, i_{l-1}, d, i_{l+1}, \ldots, i_n) | 1 \leq c, d \leq v\},$$

$$c_x = A(i_1, i_2, \ldots, i_{k-1}, c, i_{k+1}, \ldots, i_{l-1}, x, i_{l+1}, \ldots, i_n), 1 \leq c, x \leq v$$

and

$$K = (i_1, i_{n+1}) \cdots (i_{k-1}, i_{n+1})(i_{k+1}, i_{n+1}) \cdots (i_{l-1}, i_{n+1})(i_{l+1}, i_{n+1}) \cdots (i_n, i_{n+1}).$$

Then for any two "rows" a and b of P_{n+1}, we have

row a: $a_1 K(a, i_{n+1})(1, i_{n+1}) \quad a_2 K(a, i_{n+1})(2, i_{n+1}) \cdots\cdots\cdots \quad a_v K(a, i_{n+1})(v, i_{n+1})$

row b: $b_1 K(b, i_{n+1})(1, i_{n+1}) \quad b_2 K(b, i_{n+1})(2, i_{n+1}) \cdots\cdots\cdots \quad b_v K(b, i_{n+1})(v, i_{n+1})$

Thus P_{n+1} is formed by multiplying each entry of P_n by the value of K, which is ± 1, and then multiply each "row" x by the value of (x, i_{n+1}), which is ± 1, and each "column" y by value of (y, i_{n+1}), which is ± 1. P_{n+1} is therefore equivalent to P_n. The proof is similar for the case $l = n + 1$. □

Theorem 2.5 *Any plane in $\mathcal{P}(n, v, H)$ is equivalent to H.*

Proof. Suppose that any plane P_n in $\mathcal{P}(n, v, H)$ is equivalent to H, then by Lemma 2.4, any plane P_{n+1} in $\mathcal{P}(n + 1, v, H)$ is also equivalent to H. The result follows by use of Lemma 2.3 and mathematical induction. □

We now turn to investigate the relationship between any plane in $\mathcal{G}(n, v, G, \phi)$ and the Hadamard matrix $H = [(i, j)] = [\phi(g_1 g_2)]_{g_i \in G}$ from which $\mathcal{G}(n, v, G, \phi)$ is constructed. Consider a plane

$$P = \{A(g_{i_1}, g_{i_2}, \ldots, g_{i_{k-1}}, g_a, g_{i_{k+1}}, \ldots, g_{i_{l-1}}, g_b, g_{i_{l+1}}, \ldots, g_{i_n}) | 1 \leq a, b \leq v\}$$

in $\mathcal{G}(n, v, G, \phi)$ constructed from H. Let g_1 denote the identity of G. Consider a typical entry $A(g_{i_1}, \ldots, g_x, g_{i_{k+1}}, \ldots, g_y, g_{i_{l+1}}, \ldots, g_{i_n})$ in any x^{th} "row" and y^{th} "column" in P.

$$\begin{aligned} A(g_{i_1}, \ldots, g_x, g_{i_{k+1}}, \ldots, g_y, g_{i_{l+1}}, \ldots, g_{i_n}) &= \phi(g_{i_1} \cdots g_x g_{i_{k+1}} \cdots g_y g_{i_{l+1}} \cdots g_{i_n}) \\ &= \phi(K g_x L g_y M) \end{aligned}$$

where $K = g_{i_1}g_{i_2}\cdots g_{i_{k-1}}$, $L = g_{i_{k+1}}\cdots g_{i_{l-1}}$ and $M = g_{i_{l+1}}\cdots g_{i_n}$. Since K, L and M are constants, there exist permutations θ and σ on $\{1, 2, \ldots, v\}$ such that $Kg_xL = g_{\theta(x)}$ and $g_yM = g_{\sigma(y)}$. Thus $\phi(Kg_xLg_yM) = \phi(g_{\theta(x)}g_{\sigma(y)})$. P is therefore equivalent to H as a result of permutations of rows and columns of H.

Theorem 2.6 *Any plane in $\mathcal{G}(n, v, G, \phi)$ is equivalent to the group developed Hadamard matrix from which it is constructed.* □

Before we turn our attention to the relationship between any plane in $\mathcal{R}(n, v, G, \psi)$ and the cocyclic Hadamard matrix $H = [(i, j)] = [\psi(g_1, g_2)]_{g_i\in G}$ from which $\mathcal{R}(n, v, G, \psi)$ is constructed, we first uncover a fundamental property of the 2-dimensional cocycle.

If $\psi : G \times G \to \{\pm 1\}$ is a cocycle, then ψ is a set mapping satisfying Equation (3). By induction,

$$\begin{aligned}\psi(g_1, g_2)\psi(g_1g_2, g_3)\cdots\psi(g_1g_2\cdots g_{n-1}, g_n) &= \\ \psi(g_{n-1}, g_n)\psi(g_{n-2}, g_{n-1}g_n)\cdots\psi(g_1, g_2g_3\cdots g_n). \quad (5)\end{aligned}$$

We now make use of Equation (5) in our investigation of planes in $\mathcal{R}(n, v, G, \psi)$.

Theorem 2.7 *Any plane in $\mathcal{R}(n, v, G, \psi)$ is equivalent to the cocyclic Hadamard matrix from which $\mathcal{R}(n, v, G, \psi)$ is constructed.*

Proof. Consider any plane
$P = \{A(g_{i_1}, g_{i_2}, \ldots, g_{i_{k-1}}, g_a, g_{i_{k+1}}, \ldots, g_{i_{l-1}}, g_b, g_{i_{l+1}}, \ldots, g_{i_n}) \mid 1 \le a, b \le v\}$.
For any entry $A(g_{i_1}, g_{i_2}, \ldots, g_{i_{k-1}}, g_x, g_{i_{k+1}}, \ldots, g_{i_{l-1}}, g_y, g_{i_{l+1}}, \ldots, g_{i_n})$ in any x^{th} "row" and y^{th} "column" in P, by use of Equation (5), we have

$$A(g_{i_1}, g_{i_2}, \ldots, g_{i_{k-1}}, g_x, g_{i_{k+1}}, \ldots, g_{i_{l-1}}, g_y, g_{i_{l+1}}, \ldots, g_{i_n}) = XZY\psi(L, g_yg_{i_{l+1}}\cdots g_{i_n})$$

where

$$X = \psi(g_{i_1}, g_{i_2})\psi(g_{i_1}g_{i_2}, g_{i_3})\cdots\psi(g_{i_1}g_{i_2}\cdots g_xg_{i_{k+1}}\cdots g_{i_{l-2}}, g_{i_{l-1}}),$$
$$Y = \psi(g_y, g_{i_{l+1}}\cdots g_{i_n}),$$
$$Z = \psi(g_{i_{n-1}}, g_{i_n})\psi(g_{i_{n-2}}, g_{i_{n-1}}g_{i_n})\cdots\psi(g_{i_{l+1}}, g_{i_{l+2}}g_{i_{l+3}}\cdots g_{i_n}),$$
$$L = g_{i_1}g_{i_2}\cdots g_{i_{l-1}}.$$

Thus by multiplying each "row" x by the corresponding inverse of X, each "column" y by the corresponding inverse of Y, and all entries by the inverse of the constant Z, the plane P is Hadamard equivalent to a matrix H_1 where each entry in the (x, y) position of H_1 is in the form of $\psi(g_1g_xg_2, g_yg_3)$, where $g_1 = g_{i_1}g_{i_2}\cdots g_{i_{k-1}}$, $g_2 = g_{i_{k+1}}\cdots g_{i_{l-1}}$ and $g_3 = g_{i_{l+1}}\cdots g_{i_n}$ are fixed elements of G. Since g_3 is fixed, there exists a permutation σ on $\{1,2,\ldots,v\}$ such that $g_yg_3 = g_{\sigma(y)}$. Also since g_1, g_2 are fixed elements of G, there exists a permutation θ on $\{1,2,\ldots,v\}$ such that $g_1g_xg_2$

$= g_{\theta(x)}$. Thus by suitable row and column operations performed on H_1, it can be shown that H_1 is Hadamard equivalent to H whose entry in the (x, y) position is $\psi(g_x, g_y)$. $\square$

The following corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8 *If $\mathcal{R}(n, v, G, \psi)$ is constructed from a cocyclic Hadamard matrix H and $\mathcal{R}'(n, v, G', \psi')$ from H' such that H is equivalent to H', then all planes in $\mathcal{R}(n, v, G, \psi)$ and $\mathcal{R}'(n, v, G', \psi')$ are equivalent.* $\square$

Corollary 2.9 *If H is a $v \times v$ Hadamard matrix from which $\mathcal{P}(n, v, H)$, $\mathcal{G}(n, v, G, \phi)$, and $\mathcal{R}(n, v, G', \psi)$ are constructed, then all planes in the above n-dimensional order v proper Hadamard matrices are equivalent.*

Proof. All planes in $\mathcal{P}(n, v, H)$, $\mathcal{G}(n, v, G, \phi)$ and $\mathcal{R}(n, v, G', \psi)$ are equivalent to H by previous results. $\square$

3 Equivalence of n-Dimensional Proper Hadamard Matrices

Since the appearance of [7] and [8], relatively little research has been done on higher dimensional proper Hadamard matrices, principally by Hammer and Seberry (e.g. [2, 3]), de Launey (e.g. [1]) and Yang (e.g. [10]). The fundamental question of equivalence relations between such matrices remains untouched until now. Our basic aim is to define a set of elementary equivalence operations for higher dimensional proper Hadamard matrices and subsequently to investigate their equivalence classes.

Definition 3.1 *Elementary Equivalence Operations for n-dimensional Proper Hadamard Matrices. Let M be a n-dimensional proper Hadamard matrix. An n-dimensional matrix M' is equivalent to M if M' can be obtained from M by performing a finite sequence of the following operations:*
1. Permute parallel $(n-1)$-dimensional planes, that is, given a coordinate k, exchange any corresponding entries indexed by x and y in the k^{th} coordinate.
2. Negate any $(n-1)$-dimensional plane, that is, given a coordinate k and index x, multiply all entries by -1 whose k^{th} coordinate takes the value x.

Notice that the above operations preserve propriety, and when $n = 2$, they define Hadamard equivalence.

Lemma 3.2 *If M and M' are two n-dimensional proper Hadamard matrices and M' is equivalent to M by performing Operation 2 to M, then every plane P in M is equivalent to the corresponding plane P' in M'.*

Proof. Consider a plane P in M where

$$P = \{A(i_1, i_2, \ldots, i_{l-1}, a, i_{l+1}, \ldots, i_{m-1}, b, i_{m+1}, \ldots, i_n) | 1 \leq a, b \leq v\}$$

and a corresponding plane

$$P' = \{A'(i_1, i_2, \ldots, i_{l-1}, a, i_{l+1}, \ldots, i_{m-1}, b, i_{m+1}, \ldots, i_n) | 1 \leq a, b \leq v\}$$

in M', and suppose that M' is equivalent to M by performing Operation 2 to M, that is,

$$A'(j_1, \ldots, j_{k-1}, x, j_{k+1}, \ldots, j_n) = -A(j_1, \ldots, j_{k-1}, x, j_{k+1}, \ldots, j_n)$$

and

$$A'(j_1, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_n) = A(j_1, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_n) \quad \forall j_k \neq x,$$

where $j_1, j_2, \ldots, j_n$ are any indices. If $k \neq l$ and $k \neq m$, then if i_k, which is a fixed coordinate in all entries of P and P', equals x, then all entries of P are multiplied by -1 to obtain P', P is thus equivalent to P'. If $i_k \neq x$, then P and P' are identical. If $k = l$ (or m), then P' is obtained from P by multiplying the x^{th} "row" of P by -1, P is thus equivalent to P'. $\square$

Although this result may not hold in general for Operation 1, we can prove it holds for Operation 1 for the standard constructions.

Theorem 3.3 *If M and M' are two n-dimensional proper Hadamard matrices and M' is equivalent to M, where M is constructed by one of the three constructions, then every plane P in M is equivalent to the corresponding plane P' in M'.*

Proof. Let P and P' be planes in M and M' respectively as defined in Lemma 3.2. The result is immediate if M' is equivalent to M by performing Operation 2 to M. Suppose that M' is equivalent to M by performing Operation 1 to M. Let P and P' be planes in M and M' respectively as defined in Lemma 3.2. If $k = l$ (or m), then "row" x and "row" y of P are interchanged to obtain P', so P is equivalent to P'. If $k \neq l$ and $k \neq m$, we then consider different cases of constructions by which M is constructed. Typical entries of P and P' in this case are

$$A'(j_1, \ldots, j_{k-1}, x, j_{k+1}, \ldots, j_n) = A(j_1, \ldots, j_{k-1}, y, j_{k+1}, \ldots, j_n)$$

and

$$A'(j_1, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_n) = A(j_1, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_n) \quad \forall j_k \neq x, y,$$

where $j_1, j_2, \ldots, j_n$ are any indices.

M is constructed by product construction Define

$$\begin{aligned} I_{ab} = \quad & (i_1,i_2)(i_1,i_3)\cdots(i_1,i_{k-1})(i_1,i_{k+1})\cdots(i_1,a)\cdots(i_1,b)(i_1,i_{m+1})\cdots(i_1,i_n) \\ & (i_2,i_3)\cdots(i_2,i_{k-1})(i_2,i_{k+1})\cdots(i_2,a)\cdots(i_2,b)(i_2,i_{m+1})\cdots(i_2,i_n) \\ & \vdots \\ & (i_{k-2},i_{k-1})(i_{k-2},i_{k+1})\cdots(i_{k-2},a)\cdots(i_{k-2},b)(i_{k-2},i_{m+1})\cdots(i_{k-2},i_n) \\ & (i_{k+1},i_{k+2})(i_{k+1},i_{k+3})\cdots(i_{k+1},a)\cdots(i_{k+1},b)(i_{k+1},i_{m+1})\cdots(i_{k+1},i_n) \\ & \vdots \\ & (i_{n-1},i_n), \end{aligned}$$

$$\begin{aligned} I_x = (i_1,x)(i_2,x)\cdots(i_{k-1},x)(x,i_{k+1})(x,i_{k+2}) & \cdots \quad (x,i_{l-1})(x,i_{l+1}) \\ & \cdots \quad (x,i_{m-1})(x,i_{m+1})\cdots(x,i_n). \end{aligned}$$

Then the plane P can be written as

$$P = \begin{bmatrix} I_{11}I_x(x,1)(x,1) & I_{12}I_x(x,1)(x,2) & \cdots & I_{1v}I_x(x,1)(x,v) \\ I_{21}I_x(x,2)(x,1) & I_{22}I_x(x,2)(x,2) & \cdots & I_{2v}I_x(x,2)(x,v) \\ \vdots & & & \\ I_{v1}I_x(x,v)(x,1) & I_{v2}I_x(x,v)(x,2) & \cdots & I_{vv}I_x(x,v)(x,v) \end{bmatrix}.$$

Multiply every row (or column) by I_x, then multiply each row i of P by (x,i) and each column j by (x,j). Now multiply every row (or column) by I_y, then multiply each row i of P by (y,i) and each column j by (y,j). P is then transformed into

$$P' = \begin{bmatrix} I_{11}I_y(y,1)(y,1) & I_{12}I_y(y,1)(y,2) & \cdots & I_{1v}I_y(y,1)(y,v) \\ I_{21}I_y(y,2)(y,1) & I_{22}I_y(y,2)(y,2) & \cdots & I_{2v}I_y(y,2)(y,v) \\ \vdots & & & \\ I_{v1}I_y(y,v)(y,1) & I_{v2}I_y(y,v)(y,2) & \cdots & I_{vv}I_y(y,v)(y,v) \end{bmatrix}.$$

M is constructed by group development Define

$$G_1 = g_{i_1}g_{i_2}\cdots g_{i_{k-1}}, G_{ab} = g_{i_{k+1}}\cdots g_{i_{l-1}}g_a\cdots g_{i_{m-1}}g_b\cdots g_{i_n}$$

By Theorem 2.6, P is equivalent to $H = [\phi(g_1g_2)]_{g_i\in G}$, from which M is constructed. Consider a typical entry in any a^{th} "row" and b^{th} "column" of P',

$$\begin{aligned} & A'(g_{i_1},g_{i_2},\ldots,g_x,g_{i_{k+1}},\ldots,g_a,g_{i_{l+1}},\ldots,g_b,g_{i_{m+1}},\ldots g_{i_n}) \\ = \quad & \phi(g_{i_1}g_{i_2}\cdots g_y g_{i_{k+1}}\cdots g_a g_{i_{l+1}}\cdots g_b g_{i_{m+1}}\cdots g_{i_n}) \\ = \quad & \phi(K_y g_a L g_b M) \end{aligned}$$

where $K_y = g_{i_1}\cdots g_{i_{k-1}}g_y\cdots g_{i_{l-1}}$, $L = g_{i_{l+1}}\cdots g_{i_{m-1}}$ and $M = g_{i_{m+1}}\cdots g_{i_n}$. Since K_y, L and M are constants, there exist permutations θ and σ on $\{1,2,\ldots,v\}$ such that

$K_y g_a L = g_{\theta(a)}$ and $g_b M = g_{\sigma(b)}$. Thus $\phi(K_y g_a L g_b M) = \phi(g_{\theta(a)} g_{\sigma(b)})$. P' is therefore equivalent to H as a result of permutations of rows and columns of H. P and P' are therefore equivalent.

M is constructed by relative difference set construction Define

$$G_1 = \psi(g_{i_1}, g_{i_2})\psi(g_{i_1}g_{i_2}, g_{i_3})\cdots\psi(g_{i_1}g_{i_2}\cdots g_{i_{k-2}}, g_{i_{k-1}})$$

$$G_x = \psi(g_{i_1}\cdots g_{i_{k-1}}, g_x)\psi(g_{i_1}\cdots g_{i_{k-1}}g_x, g_{i_{k+1}})\cdots\psi(g_{i_1}\cdots g_x\cdots g_{i_{l-2}}, g_{i_{l-1}})$$

$$\begin{aligned} G_{ax} = \psi(g_{i_1}\cdots g_x\cdots g_{i_{l-1}}, g_a) \quad & \psi \quad (g_{i_1}\cdots g_x\cdots g_{i_{l-1}}g_a, g_{i_{l+1}})\cdots \\ & \psi \quad (g_{i_1}\cdots g_x\cdots g_a\cdots g_{i_{m-2}}, g_{i_{m-1}}) \end{aligned}$$

$$\begin{aligned} G_{bx} = \quad & \psi(g_{i_1}\cdots g_x\cdots g_a\cdots g_{i_{m-1}}, g_b)\psi(g_{i-1}\cdots g_x\cdots g_a\cdots g_{i_{m-1}}g_b, g_{i_{m+1}})\cdots \\ & \psi(g_{i_1}\cdots g_x\cdots g_a\cdots g_b\cdots g_{i_{n-1}}, g_{i_n}). \end{aligned}$$

Likewise define G_y, G_{ay} and G_{by}.
Then the plane P can be written as

$$P = \begin{bmatrix} G_1G_xG_{1x}G_{1x} & G_1G_xG_{1x}G_{2x} & \cdots & G_1G_xG_{1x}G_{vx} \\ G_1G_xG_{2x}G_{1x} & G_1G_xG_{2x}G_{2x} & \cdots & G_1G_xG_{1x}G_{vx} \\ \vdots & & & \\ G_1G_xG_{vx}G_{1x} & G_1G_xG_{vx}G_{2x} & \cdots & G_1G_xG_{vx}g_{vx} \end{bmatrix}.$$

Multiply each row a of P by G_{ax} and G_x, then multiply each column b of P by G_{bx}, P is then equivalent to P' after further multiplying each row a by G_y and G_{ay}, and each column b by G_{by}. $\square$

4 Relationships Between The Standard Classes

We now start our investigation into equivalence classes of n-dimensional proper Hadamard matrices by firstly looking into the relationship between the number of equivalence classes of $\mathcal{P}(n, v, H)$ and that of 2-dimensional Hadamard matrices. Let H and H' be two equivalent $v\times v$ Hadamard matrices such that H' is obtained by multiplying some row x of H by -1, $1 \leq x \leq v$. Construct $\mathcal{P}(n, v, H)$ and $\mathcal{P}(n, v, H')$. Any entry $A(i_1, i_2, \ldots, i_n)$ in $\mathcal{P}(n, v, H)$ and $A'(i_1, i_2, \ldots, i_n)$ in $\mathcal{P}(n, v, H')$ are identical if $i_k \neq x$ $\forall k = 1, 2, \ldots, n-1$. For any $i_k = x$, $k = 1, 2, \ldots, n-1$, the product $(i_k = x, i_{k+1})(x, i_{k+2})\cdots(x, i_n)$ in $A(i_1, i_2, \ldots, i_n)$ becomes

$$\begin{aligned} & -(i_k = x, i_{k+1})(-(x, i_{k+2}))\cdots(-(x, i_n)) \\ & = (-1)^{n-k}(i_k = x, i_{k+1})(x, i_{k+2})\cdots(x, i_n) \end{aligned}$$

in $A'(i_1, i_2, \ldots, i_n)$; i.e., $\mathcal{P}(n, v, H')$ can be obtained from $\mathcal{P}(n, v, H)$ by applying Operation 2 to $\mathcal{P}(n, v, H)$ $n-k$ times whenever the k^{th} coordinate takes the value of x, $\forall k = 1, 2, \ldots, n-1$. Notice that $A(i_1, i_2, \ldots, i_{n-1}, x) = A'(i_1, i_2, \ldots, i_{n-1}, x)$ if $i_l \neq x$ $\forall l = 1, 2, \ldots, n-1$ since such factors (i_l, x) are in the column x of H and H'.

A similar argument applies when column y of H is multiplied by -1 instead.

If H and H' are equivalent by exchanging row x with row y, then $\mathcal{P}(n, v, H')$ can be obtained from $\mathcal{P}(n, v, H)$ by applying Operation 1 to $\mathcal{P}(n, v, H)$ successively in each coordinate $k = 1, 2, \ldots, n-1$. Similarly if H and H' are equivalent by exchanging column x with column y, then $\mathcal{P}(n, v, H')$ can be obtained from $\mathcal{P}(n, v, H)$ by applying Operation 1 to $\mathcal{P}(n, v, H)$ successively in each coordinate $k = 2, 3, \ldots, n$. The following lemma has therefore been proved.

Lemma 4.1 *If H and H' are two $v \times v$ equivalent Hadamard matrices, then $\mathcal{P}(n, v, H)$ and $\mathcal{P}(n, v, H')$ are equivalent.* □

Lemma 4.2 *If $\mathcal{P}(n, v, H)$ and $\mathcal{P}(n, v, H')$ are equivalent, then H and H' are equivalent.*

Proof. If $\mathcal{P}(n, v, H)$ is equivalent to $\mathcal{P}(n, v, H')$, then by Theorem 3.3, every pair of corresponding planes P in $\mathcal{P}(n, v, H)$ and P' in $\mathcal{P}(n, v, H')$ are equivalent. Also by Theorem 2.5, P is equivalent to H and P' is equivalent to H'. Thus H is equivalent to H'. □

Theorem 4.3 *$\mathcal{P}(n, v, H)$ is equivalent to $\mathcal{P}(n, v, H')$ if and only if H is equivalent to H'.*

Proof. This follows immediately from the results of Lemma 4.1 and Lemma 4.2. □

From Theorem 4.3, we then have the following corollary.

Corollary 4.4 *The number of equivalence classes of $\mathcal{P}(n, v, H)$ is equal to the number of equivalence classes of $v \times v$ Hadamard matrices.* □

Next we investigate the relationship between the number of equivalence classes of group developed n-dimensional proper Hadamard matrices and that of 2-dimensional Hadamard matrices. Let $H = \mathcal{G}(2, v, G, \phi)$ and H' be obtained from H by multiplying some row x of H by -1. It is clear that $H' \neq \mathcal{G}(2, v, G, \phi')$, i.e., H' cannot be developed over the same group G. Suppose that $H' = \mathcal{G}(2, v, G', \psi)$. Without loss of generality, we name the first row of elements of G' in its multiplication table as $g'_1, g'_2, \ldots, g'_v$. Then $\phi(g_i) = \psi(g'_i)$ $\forall i = 1, 2, \ldots, v$. Compare row x of H with row x of H' and let $\phi(g_{xy})$ and $\psi(g'_{xy})$ denote $\phi(g_x g_y)$ and $\psi(g'_x g'_y)$ respectively, we have $\phi(g_{x1}) = -\psi(g'_{x1}) = -\phi(g_{\theta(x1)})$, $\phi(g_{x2}) = -\psi(g'_{x2}) = -\phi(g_{\theta(x2)})$,$\ldots\ldots$,

$\phi(g_{xv}) = -\psi(g'_{xv}) = -\phi(g_{\theta(xv)})$. Notice that θ has to be a fixed point free permutation on $\{1, 2, \ldots, v\}$, i.e., $i \neq \theta(i)$ $\forall i = 1, 2, \ldots, v$ since $\phi(g_i) = \psi(g'_i)$ $\forall i = 1, 2, \ldots, v$. Thus, there exists exactly $n/2$ g_i's such that $\phi(g_i) = 1$ and exactly $n/2$ g_j's such that $\phi(g_j) = -1$. But then H cannot be developed over G with such a definition of ϕ. Indeed if H' is obtained from H by multiplying rows only of H by -1, then H' cannot be group developed unless it is the exact negation of H. A similar argument applies if only columns of H are multiplied by -1.

Lemma 4.5 *If $H = \mathcal{G}(2, v, G, \phi)$ and $H' \neq -H$ is obtained from H by multiplying rows only (or columns only) of H by -1, then H' cannot be group developed.* $\square$

However it is possible that a combination of row and column permutations and multiplication of rows and columns of H by -1 leads to another H' which is group developed. We will restrict discussion of equivalence to $\mathcal{G}(2, v, G, \phi)$'s which are equivalent as a result of row and column permutations only. In light of this the following lemma then becomes trivial.

Lemma 4.6 *If $H = \mathcal{G}(2, v, G, \phi)$ and $H' = \mathcal{G}'(2, v, G', \phi')$ are equivalent by row and column permutations alone, then $\mathcal{G}(n, v, G, \phi)$ and $\mathcal{G}'(n, v, G', \phi')$ are equivalent.* $\square$

Lemma 4.7 *If $\mathcal{G}(n, v, G, \phi)$ is equivalent to $\mathcal{G}(n, v, G', \phi')$, then $\mathcal{G}(2, v, G, \phi)$ is equivalent to $\mathcal{G}(2, v, G', \phi')$.* $\square$

Proof. This follows immediately from Theorem 3.3 and Theorem 2.6. $\square$

From Lemma 4.7, we have the following corollary.

Corollary 4.8 *Given G and v, for each $n \geq 2$, the number of equivalence classes of $\mathcal{G}(n, v, G, \phi)$ is at least equal to the number of equivalence classes of $\mathcal{G}(2, v, G, \phi)$.* $\square$

We now turn our attention to the equivalence classes of $\mathcal{R}(n, v, G, \psi)$. Let $H = \mathcal{G}(2, v, G, \varphi)$ be a $v \times v$ group developed Hadamard matrix. Every such φ defines a special type of cocycle called a *coboundary* $\partial\varphi : G \times G \to \{\pm 1\}$ where $\partial\varphi(g_1, g_2) = \varphi(g_1)^{-1}\varphi(g_2)^{-1}\varphi(g_1 g_2)$, $g_1, g_2 \in G$. Consider two n-dimensional proper Hadamard matrices constructed from H, $\mathcal{G}(n, v, G, \varphi)$ and $\mathcal{R}(n, v, G, \partial\varphi)$.

Let $A_1(g_{i_1}, g_{i_2}, \ldots, g_{i_n}) = \varphi(g_{i_1} g_{i_2} \cdots g_{i_n})$ be an entry in $\mathcal{G}(n, v, G, \varphi)$ and

$$\begin{aligned}
&A_2(g_{i_1}, g_{i_2}, \ldots, g_{i_n}) \\
&= \partial\varphi(g_{i_1}, g_{i_2})\partial\varphi(g_{i_1} g_{i_2}, g_{i_3}) \cdots \partial\varphi(g_{i_1} g_{i_2} \cdots g_{i_{n-1}}, g_{i_n}) \\
&= \varphi(g_{i_1})^{-1}\varphi(g_{i_2})^{-1}\varphi(g_{i_1} g_{i_2})\varphi(g_{i_1} g_{i_2})^{-1}\varphi(g_{i_3})^{-1}\varphi(g_{i_1} g_{i_2} g_{i_3}) \cdots \\
&\qquad \cdots \varphi(g_{i_1} g_{i_2} \cdots g_{i_{n-1}})^{-1}\varphi(g_{i_n})^{-1}\varphi(g_{i_1} g_{i_2} \cdots g_{i_n}) \\
&= \varphi(g_{i_1})^{-1}\varphi(g_{i_2})^{-1} \cdots \varphi(g_{i_n})^{-1}\varphi(g_{i_1} g_{i_2} \cdots g_{i_n})
\end{aligned}$$

be an entry in $\mathcal{R}(n, v, G, \partial\varphi)$. Since each entry in $\mathcal{R}(n, v, G, \partial\varphi)$ with the first coordinate indexed by $g_{i_1} \in G$ has a factor $\varphi(g_{i_1})^{-1}$ and with the second coordinate indexed by $g_{i_2} \in G$ has a factor $\varphi(g_{i_2})^{-1}$ and so on, $\mathcal{G}(n, v, G, \varphi)$ is equivalent to $\mathcal{R}(n, v, G, \partial\varphi)$ as each entry in $\mathcal{R}(n, v, G, \partial\varphi)$ can be obtained from $\mathcal{G}(n, v, G, \varphi)$ by repeatedly applying the second equivalence operation. Conversely, every $H = \mathcal{R}(2, v, G, \partial\varphi)$ induces an equivalent $H' = \mathcal{G}(2, v, G, \varphi)$ and each entry in $\mathcal{G}(n, v, G, \varphi)$ can be obtained from $\mathcal{R}(n, v, G, \partial\varphi)$ by repeatedly applying the second equivalence operation, reversing the process described above.

Theorem 4.9 *$\mathcal{G}(n, v, G, \varphi)$ and $\mathcal{R}(n, v, G, \partial\varphi)$ are equivalent.* $\square$

If H is a binary cocyclic matrix over a finite group G with cocycle ψ, then ψ is unique in the sense that if there exists another cocycle ψ' such that for some $g_x, g_y \in G$, $\psi(g_x, g_y) \neq \psi'(g_x, g_y)$, then $H(g_x, g_y) \neq H(g_x, g_y)$.

Corollary 4.10 *The number of equivalence classes of $\mathcal{R}(n, v, G, \psi)$ is at least equal to the number of equivalence classes of $\mathcal{G}(n, v, G, \phi)$.* $\square$

Lemma 4.11 *If $\mathcal{R}(n, v, G, \psi)$ exists, then the identity g_1 of G is the only element of G with the property $\psi(g_1, g) = \psi(g, g_1) = \psi(g_1, g_1)\ \forall g \in G$.*

Proof. Consider $H = \mathcal{R}(2, v, G, \psi)$ and let g_1 denote the identity of G. Suppose that there exists $g_x \in G$, $g_x \neq g_1$, such that $\psi(g_x, g) = \psi(g, g_x) = \psi(g_x, g_x)\ \forall g \in G$, then the inner product

$$
\begin{aligned}
&\psi(g_1, g_1)\psi(g_x, g_1) + \psi(g_1, g_2)\psi(g_x, g_2) + \cdots + \psi(g_1, g_v)\psi(g_x, g_v) \\
&= [\psi(g_x, g_1) + \psi(g_x, g_2) + \cdots + \psi(g_x, g_v)]\psi(g_1, g_1) \\
&= v\psi(g_x, g_1)\psi(g_1, g_1) \\
&= \pm v \neq 0
\end{aligned}
$$

contradicting the definition of Hadamard matrix. $\square$

Lemma 4.12 *If H' is obtained from $H = \mathcal{R}(2, v, G, \psi)$ only by multiplying some non-initial rows of H by -1, or only by multiplying some non-initial columns only of H by -1, then H' is not cocyclic over any group with any cocycle.*

Proof. Let $H' = \mathcal{R}(2, v, G', \chi)$ be obtained from $H = \mathcal{R}(2, v, G, \psi)$ by multiplying some row g_x of H by -1, $x \neq 1$, where g_1 denotes the identity of G. Then in particular, $\chi(g'_x, g'_1) = -\phi(g_x, g_1)$. Since the first rows of H and H' are identical, by Lemma 4.11, it is necessary that g'_1 is the identity of G' and $\phi(g_1, g_x) = \chi(g'_1, g'_x) = \phi(g_x, g_1) = \chi(g'_x, g'_1)$, contradicting $\chi(g'_x, g'_1) = -\phi(g_x, g_1)$. Indeed apart from the first row, for any number of row multiplications by -1, the same argument applies. Similarly if some column g_y, $y \neq 1$, of H is multiplied by -1 to obtain H', then in particular, $\chi(g'_1, g'_y) = -\phi(g_1, g_y)$. Since the first columns of H and H' are identical, by Lemma 4.11, g_1' is the identity of G' and $\phi(g_y, g_1) = \chi(g_y', g_1') = \phi(g_1, g_y) = \chi(g_1', g_y')$, contradicting $\chi(g_1', g_y') = -\phi(g_1, g_y)$. Apart from the first column, for any number of column multiplications by -1, the same argument applies. □

However it is possible that a combination of row and column permutations and multiplication of rows and columns of H by -1 leads to another H' which is cocyclic. We will restrict discussion of equivalence to $\mathcal{R}(2, v, G, \phi)$'s which are equivalent as a result of row and column permutations. In light of this the following lemma then becomes trivial.

Lemma 4.13 *If $H' = \mathcal{R}(2, v, G', \psi')$ is obtained from $H = \mathcal{R}(2, v, G, \psi)$ by row and column permutations alone, then $\mathcal{R}(n, v, G, \psi)$ is equivalent to $\mathcal{R}(n, v, G', \psi')$* □

Lemma 4.14 *If $\mathcal{R}(n, v, G, \psi)$ and $\mathcal{R}(n, v, G', \psi')$ are equivalent, then $\mathcal{R}(2, v, G, \psi)$ and $\mathcal{R}(2, v, G', \psi')$ are equivalent.*

Proof. This follows immediately from Theorem 3.3 and Theorem 2.7. □

From Lemma 4.14, we then have the following corollary.

Corollary 4.15 *The number of equivalence classes of $\mathcal{R}(n, v, G, \psi)$ is at least equal to the number of equivalence classes of $\mathcal{R}(2, v, G, \psi)$.* □

A cocycle $\psi : G \times G \to C$ is *multiplicative in the first coordinate* if $\forall g_x, g_y, g_z \in G$, $\psi(g_x g_y, g_z) = \psi(g_x, g_z)\psi(g_y, g_z)$. Similarly ψ is *multiplicative in the second coordinate* if $\forall g_x, g_y, g_z \in G$, $\psi(g_x, g_y g_z) = \psi(g_x, g_y)\psi(g_x, g_z)$. An example of a multiplicative cocycle is Example 4.3 in [4], for which $\mathcal{R}(2, 2^m, (\mathbb{Z}_2)^m, \psi)$ is the Sylvester-Hadamard matrix of order 2^m. Notice that

$$\begin{aligned}
& \psi(g_x g_y, g_z) = \psi(g_x, g_z)\psi(g_y, g_z) \\
\iff \quad & \psi(g_x, g_y)\psi(g_x g_y, g_z) = \psi(g_x, g_y)\psi(g_x, g_z)\psi(g_y, g_z) \\
\iff \quad & \psi(g_x, g_y g_z)\psi(g_y, g_z) = \psi(g_x, g_y)\psi(g_x, g_z)\psi(g_y, g_z) \\
\iff \quad & \psi(g_x, g_y g_z) = \psi(g_x, g_y)\psi(g_x, g_z)
\end{aligned}$$

Hence a 2-dimensional cocycle is multiplicative in the first coordinate if and only if it is multiplicative in the second coordinate.

Let $H = \mathcal{R}(2, v, G, \psi)$. Consider any entry $A(i_1, i_2, \ldots, i_n)$ in $\mathcal{P}(n, v, H)$ and $A'(g_{i_1}, g_{i_1}, \ldots, g_{i_n})$ in $\mathcal{R}(n, v, G, \psi)$.

$$A(i_1, i_2, \ldots, i_n) = (i_1, i_2)(i_1, i_3) \cdots (i_1, i_n)(i_2, i_3) \cdots (i_2, i_n) \cdots (i_{n-1}, i_n)$$

$$A'(g_{i_1}, g_{i_2}, \ldots, g_{i_n}) = \psi(g_{i_1}, g_{i_2})\psi(g_{i_1} g_{i_2}, g_{i_3}) \cdots \psi(g_{i_1} g_{i_2} \cdots g_{i_{n-1}}, g_{i_n})$$

Notice that if ψ is multiplicative in the first coordinate,

$$\psi(g_{i_1}, g_{i_2}) = (i_1, i_2)$$

$$\psi(g_{i_1} g_{i_2}, g_{i_3}) = \psi(g_{i_1}, g_{i_3})\psi(g_{i_2}, g_{i_3}) = (i_1, i_3)(i_2, i_3)$$

and in general, for $1 < k \leq n$,

$$\psi(\prod_{j=1}^{k-1} g_{i_j}, g_{i_k}) = \prod_{j=1}^{k-1}(i_j, i_k)$$

Any two corresponding entries of $\mathcal{P}(n, v, H)$ and $\mathcal{R}(n, v, G, \psi)$ are therefore identical.

Theorem 4.16 *If ψ in $H = \mathcal{R}(2, v, G, \psi)$ is multiplicative in either coordinate, then $\mathcal{R}(n, v, G, \psi) = \mathcal{P}(n, v, H)$.* $\square$

It has been well known that $\mathcal{G}(n, v, G, \phi)$ and $\mathcal{R}(n, v, G, \psi)$ are associated with each other. Now that we have defined equivalence of n-dimensional proper Hadamard matrices, Theorem 4.9 makes this more precise: each of the former is equivalent to one of the latter. Moreover, Theorem 4.16 identifies a condition under which $\mathcal{R}(n, v, G, \psi)$ and $\mathcal{P}(n, v, H)$ are not only equivalent but identical, hence identifies some linkages among the n-dimensional proper Hadamard matrices constructed by those three techniques. Since it is not known if every $v \times v$ Hadamard matrix H is equivalent to a $\mathcal{R}(2, v, G, \psi)$, the relationship between the two constructions for a general cocycle ψ is still open to be investigated.

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