The spectral radius of triangle-free graphs

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Abstract

In this note, we present two lower bounds for the spectral radius of the Laplacian matrices of triangle-free graphs. One is in terms of the numbers of edges and vertices of graphs, and the other is in terms of degrees and average 2-degrees of vertices. We also obtain some other related results.

1 Introduction

Let $G = (V, E)$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The value of a function $f : V(G) \longmapsto R$ at a vertex y is defined by $f(y)$. For $y \in V(G)$, we denote by $d(y)$ the degree of y. The Laplacian matrix $L(G)$ of G is defined by

$$L(x,y) = \begin{cases} d(y), & \text{if} \quad x = y, \\ -1, & \text{if} \ \ x \ \ \text{and} \ \ y \ \ \text{are adjacent}, \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to see that $L(G)$ is singular, positive semidefinite. Hence the eigenvalues of $L(G)$ can be denoted by $\lambda_1(L(G)) \geq \cdots \geq \lambda_n(L(G)) = 0$. The spectrum of $L(G)$ can be used to obtain much information about the graph; for example, estimates for the diameter of the graph (see the survey by Merris[8]). In particular, estimates or bounds for $\lambda_1(L(G))$ and $\lambda_{n-1}(L(G))$ are of great interest. Recently, some upper

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bounds for $\lambda_1(L(G))$ have been obtained in terms of degrees and average 2-degrees of vertices by Li and Zhang [7] and Merris [9]. As to the lower bounds for $\lambda_1(G)$, Fiedler in [4] proved the following result:

$$\lambda_1(L(G)) \geq \frac{n}{n-1} \max_{x \in V(G)} \{d(x)\}. \tag{1}$$

Recently, Grone and Merris in [5] improved the above result by showing that if G has at least one edge, then

$$\lambda_1(L(G)) \geq \max_{x \in V(G)} \{d(x)\} + 1. \tag{2}$$

In this note, we obtain two lower bounds for the spectral radius $\lambda_1(L(G))$ of triangle-free graphs; one is in terms of the numbers of edges and vertices of graphs, and the other is in terms of degrees and average 2-degrees of vertices. We also obtain some other related results. For triangle-free graphs, the second bound is better than (2) of Grone and Merris.

2 Lemmas

In this section, we present some lemmas which will be used to obtain our main results. We also give a new proof of inequality (2) and characterize the equality in (2).

Let G be a graph with the degree diagonal matrix $D(G)$ and the (0,1)-adjacency matrix $A(G)$. Let $Q(G) = D(G) + A(G)$.

Lemma 2.1 *Let G be a graph. Then*

$$\lambda_1(L(G)) \leq \lambda_1(Q(G)). \tag{3}$$

Moreover, if G is connected, then the equality in (3) *holds if and only if G is a bipartite graph.*

Proof. Since the absolute value of any (x, y)-th entry in $L(G)$ is no more than the corresponding (x, y)-th entry in $Q(G)$ and $Q(G)$ is nonnegative and positive semidefinite, the inequality in (3) follows from Wielandt's theorem (see [1], Theorem 2.2.14, for example). Moreover, if G is connected, then $L(G)$ and $Q(G)$ are irreducible. Hence it follows from Wielandt's theorem that the equality in (3) holds if and only if $L(G) = WQ(G)W^{-1}$, where W is a diagonal matrix whose diagonal entries have modulus one, say $W = \text{diag}(e^{i\theta_u},\ u \in V(G))$, where $i^2 = -1$ and θ_u is real. Let $L(G) = (l_{uv})$ and $Q(G) = (q_{uv})$. Then $l_{uv} = e^{i(\theta_u - \theta_v)} q_{uv}$ and therefore $e^{i(\theta_u - \theta_v)} = 1$ *or* -1 if $uv \in E(G)$. Since G is connected, for any two distinct vertices $u,\ v \in V(G)$, there exists a path $u = u_1 u_2 \cdots u_k = v$ in G. Thus, $e^{i(\theta_u - \theta_v)} = \prod_{j=1}^{k-1} e^{i(\theta_{u_j} - \theta_{u_{j+1}})}$ is 1 or -1. Therefore we may assume that $W = e^{i\theta} W_1$, where W_1 is a diagonal matrix whose diagonal entries are 1 or -1. Moreover, $L(G) = W_1 Q(G) W_1^{-1}$. By comparing with corresponding entries of

$L(G) = W_1 Q(G) W_1^{-1}$, it is easy to see that $L(G) = WQ(G)W^{-1}$ if and only if G is bipartite. ■

Remark: In fact, if G is bipartite and is not connected, the equality in (3) still holds.

The follow lemma is well-known (see [8], for example).

Lemma 2.2 *Let H be a bipartite subgraph of G. Then $\lambda_1(L(G)) \geq \lambda_1(Q(H))$.*

Now we are going to give a new proof of inequality (2).

Theorem 2.3 [5] *Let G be a graph with at least one edge. Then*

$$\lambda_1(L(G)) \geq \max_{x \in V(G)} \{d(x)\} + 1. \tag{4}$$

Moreover, if G is connected, then the equality in (4) *holds if and only if* $\max_{x \in V(G)} \{d(x)\} = |V(G)| - 1$*, where $|V(G)|$ is the cardinality of the vertex set $V(G)$.*

Proof. Let $d(z) = \max_{x \in V(G)} \{d(x)\}$ and H be the bipartite subgraph of G with edge set $E(H) = \{(z, x) \in E(G), x \in V(G)\}$. Then H is a star graph with $d(z) + 1$ vertices. Thus $\lambda_1(L(H)) = d(z) + 1$. Hence the inequality in (4) follows from Lemma 2.2.

Suppose that G is connected. If $\max_{x \in V(G)} \{d(x)\} = |V(G)| - 1$, then $\lambda_1(L(G)) \geq |V(G)|$. On the other hand, it is well known that $|V(G)| - \lambda_1(L(G))$ is an eigenvalue of $L(\overline{G})$, where $\overline{G}$ is the complement of G. So $|V(G)| - \lambda_1(L(G)) \geq 0$. Hence the equality in (4) holds.

Conversely, if $d(z) = \max_{x \in V(G)} \{d(x)\} < |V(G)| - 1$, then there exist vertices y_1 and y_2 such that $(z, y_1) \in E(G)$, $(z, y_2) \notin E(G)$ and $(y_1, y_2) \in E(G)$, since G is connected. Let H' be the bipartite subgraph of G with $E(H') = E(H) \cup \{(y_1, y_2)\}$. Define the function $f : V(H') \longmapsto R$ by $f(x) = 1$, if $x = z$; $f(x) = 1/d(z)$, if $(x, z) \in E(G)$; $f(x) = 0$, otherwise. Then

$$\begin{aligned} \lambda_1(Q(H')) &= \max_{f \neq 0} \frac{\langle f, Q(H')f \rangle}{\langle f, f \rangle} \\ &\geq \frac{d(z)(1 + 1/d(z))^2 + (1/d(z))^2}{1 + (1/d(z))^2 d(z)} \\ &> d(z) + 1. \end{aligned}$$

Hence, by Lemma 2.2, $\lambda_1(L(G)) > d(z) + 1$. This completes the proof. ■

Lemma 2.4 *Let G be a triangle-free graph on $|V(G)|$ vertices and $|E(G)|$ edges. Then there exists a bipartite subgraph H of G such that*

$$\begin{aligned} |E(H)| &\geq \max \left\{ \frac{4|E(G)|^2}{|V(G)|^2}, \quad \frac{|E(G)|}{2} + \frac{1}{8\sqrt{2}} \sum_{x \in V(G)} \sqrt{d(x)} \right\} \\ &\geq \max \left\{ \frac{4|E(G)|^2}{|V(G)|^2}, \quad \frac{|E(G)|}{2} + \frac{1}{8\sqrt{2}} |E(G)|^{3/4} \right\}. \end{aligned}$$

Proof. This follows from the results of Erdös et al. [3] and Shearer [10]. ■

3 Lower bounds for spectral radius of triangle-free graphs

Now we give the main results of this paper.

Theorem 3.1 *Let G be a triangle-free graph. Then*

$$\lambda_1(L(G)) \geq \max\left\{\frac{16|E(G)|^2}{|V(G)|^3}, \quad \frac{2|E(G)|}{|V(G)|} + \frac{|E(G)|^{3/4}}{2\sqrt{2}|V(G)|}\right\}. \tag{5}$$

Moreover, if G is the complete bipartite graph $K_{n,n}$ of order $2n$, then the equality in (5) *holds.*

Proof. Let H be a bipartite spanning subgraph of G with the largest number of edges. Hence by Lemmas 2.2 and 2.4, we have

$$\begin{aligned} \lambda_1(L(G)) &\geq \lambda_1(Q(H)) \\ &\geq \frac{\langle \mathbf{1}, Q(H)\mathbf{1}\rangle}{\langle \mathbf{1}, \mathbf{1}\rangle} \\ &= \frac{4|E(H)|}{|V(G)|} \\ &\geq \max\left\{\frac{16|E(G)|^2}{|V(G)|^3}, \quad \frac{2|E(G)|}{|V(G)|} + \frac{|E(G)|^{3/4}}{2\sqrt{2}|V(G)|}\right\}, \end{aligned}$$

where $\mathbf{1}$ is the vector with all coordinates 1. Moreover, if G is the complete bipartite graph $K_{n,n}$ of order $2n$, then by (5), we have $\lambda_1(L(G)) \geq 2n$. On the other hand, $\lambda_1(L(G)) \leq 2n$. Therefore, the equality in (5) holds. ■

From Theorem 3.1, it is easy to get a well known result, i.e., Turan's Theorem.

Corollary 3.2 (Turan's Theorem[2]) *Let G be a connected graph with $|E(G)| > \frac{1}{4}|V(G)|^2$. Then G contains at least one triangle.*

Proof. If G does not contain any triangle, by Theorem 3.1, we have

$$\frac{16|E(G)|^2}{|V(G)|^3} \leq \lambda_1(L(G)) \leq |V(G)|.$$

Hence $|E(G)| \leq \frac{1}{4}|V(G)|^2$, which contradicts the condition of Corollary 3.2. Therefore the result holds. ■

Corollary 3.3 *Let G be a triangle-free graph with maximum degree Δ. Then the smallest eigenvalue of the adjacency matrix $A(G)$ satisfies*

$$\lambda_n(A(G)) \leq \min\left\{\Delta - \frac{16|E(G)|^2}{|V(G)|^3}, \quad \Delta - \frac{2|E(G)|}{|V(G)|} - \frac{|E(G)|^{3/4}}{2\sqrt{2}|V(G)|}\right\}.$$

Proof. Let $D(G)$ be the degree diagonal matrix. Then

$$\lambda_1(L(G)) \leq \lambda_1(D(G)) - \lambda_n(A(G)).$$

Hence the result follows from Theorem 3.1. ■

Now we are going to give the second lower bound for $\lambda_1(L(G))$ in terms of degrees and average 2-degrees. The average 2-degree of a vertex u, denoted by m_u, is the average of the degrees of its neighbors.

Theorem 3.4 *Let $G = (V, E)$ be a triangle-free graph. If d_u and m_u are the degree and the average 2-degree of a vertex u, respectively, then*

$$\lambda_1(L(G)) \geq \max\left\{\frac{1}{2}(d_u + m_u + \sqrt{(d_u - m_u)^2 + 4d_u}, \quad u \in V\right\}. \tag{6}$$

Proof. Let $L(U)$ be the principal submatrix of $L(G)$ corresponding to U, where $U = \{u, v_1, \cdots, v_k\}$ is the closed neighborhood of a vertex u and $d_u = k$. Obviously, $\lambda_1(L(G)) \geq \lambda_1(L(U))$. Since G is triangle-free, we may assume that

$$L(U) = \begin{pmatrix} d_u & -1 & -1 & \cdots & -1 \\ -1 & d_{v_1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & \cdots & d_{v_k} \end{pmatrix}.$$

With elementary calculations, we see that the characteristic polynomial of $L(U)$ is

$$\det(\lambda I - L(U)) = (\lambda - d_u - \sum_{i=1}^{k} \frac{1}{\lambda - d_{v_i}}) \prod_{i=1}^{k} (\lambda - d_{v_i}).$$

Note that $\lambda_1(L(G)) \geq \lambda_1(L(U)) > d_{v_i}$ for each $i = 1, \cdots, k$. Hence $\lambda_1(L(G))$ satisfies

$$\lambda_1(L(G)) - d_u \geq \sum_{i=1}^{k} \frac{1}{\lambda_1(L(G)) - d_{v_i}}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{k} (\lambda_1(L(G)) - d_{v_i}) \sum_{i=1}^{k} \frac{1}{\lambda_1(L(G)) - d_{v_i}} \geq \left(\sum_{i=1}^{k} \frac{\sqrt{\lambda_1(L(G)) - d_{v_i}}}{\sqrt{\lambda_1(L(G)) - d_{v_i}}} \right)^2 = k^2.$$

Hence

$$\lambda_1(L(G)) - d_u \geq \frac{k^2}{\sum_{i=1}^{k} (\lambda_1(L(G)) - d_{v_i})} = \frac{d_u}{\lambda_1(L(G)) - m_u},$$

since $m_u = \frac{1}{k} \sum_{i=1}^{k} d_{v_i}$. This inequality yields the desired result. ■

For d-regular triangle-free graphs, we have the following result.

Corollary 3.5 *Let G be a d-regular triangle-free graph on n vertices . Then*

$$\lambda_1(L(G)) \quad \geq \max\left\{\frac{4d^2}{n}, \quad d+\sqrt{d}\right\}. \tag{7}$$

Proof. Since $\dfrac{16|E(G)|^2}{|V(G)|^3} = \dfrac{4d^2}{n}$ and $\frac{1}{2}(d_u + m_u + \sqrt{(d_u - m_u)^2 + 4d_u} = d + \sqrt{d}$, the inequality follows from Theorems 3.1 and 3.4. ■

Corollary 3.6 *Let G be a d-regular graph on n vertices. If the complement $\overline{G}$ of G is a triangle-free graph, then the algebraic connectivity of G satisfies*

$$\lambda_{n-1}(L(G)) \leq \min\left\{\frac{(3n-2d-2)(2d+2-n)}{n}, \quad d+1-\sqrt{n-1-d}\right\}.$$

Proof. Since $\lambda_{n-1}(L(G)) = n - \lambda_1(L(\overline{G}))$, the result follows from Corollary 3.5. ■

Remark. The bounds (2) and (5) are incomparable in general, as we will see in Example 3.7. However, for triangle-free graphs, (6) is better than (2) of Grone and Merris. In fact, if we denote by $f(m_u)$ the bracket of the right side in (6), then $f(m_u) \geq f(1) = d_u + 1$, since $f(m_u)' \geq 0$. Furthermore, in [6], the authors constructed, explicitly for every prime $p \equiv 1 \pmod 4$, and found for infinitely many values of n, a $d\,(= p+1)$-regular triangle-free graph G on n vertices whose smallest eigenvalue of the adjacency matrix exceeds $-2\sqrt{d-1}$. Therefore the spectral radius of the Laplacian matrix of G is no more than $d + 2\sqrt{d-1}$. Hence the result of Corollary 3.5 is good in some sense.

As the conclusion of this note, we give one example to illustrate our main results.
Example 3.7. Let G_1 and G_2 be graphs of order 6 and 7 respectively, as follows:

G_1 G_2

Fig. 1

The largest eigenvalues of the Laplacian matrices of graphs G_1 and G_2 and their lower bounds are as follows.

	$\lambda_1(L(G))$	bound in (5)	bound in (6)	bound in (2)
G_1	5.56	4.74	4.57	4
G_2	4.88	3.10	4.43	4

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