On the number of cut edges in a regular graph

Nirmala Achuthan

Department of Mathematics and Statistics
Curtin University of Technology
GPO Box U1987
Perth, WA 6845
AUSTRALIA

A. Ramachandra Rao

Statistics and Mathematics Unit
Indian Statistical Institute
203 BT Road, Kolkata 700108
INDIA

Abstract

A *cut edge* in a graph G is an edge whose removal increases the number of connected components of G. In this paper we determine the maximum number of cut edges in a connected d-regular graph G of order p.

1 Introduction and definitions

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. Let $\Delta(G)$ and $\delta(G)$ denote the largest degree and the smallest degree of G respectively. A *cut vertex* of a graph G is a vertex whose removal increases the number of connected components of the graph. A *cut edge* of a graph is defined in a similar manner as an edge whose removal increases the number of connected components of the graph. It is easy to see that a connected graph of order p has at most $p-2$ cut vertices and at most $p-1$ cut edges. The problem of determining the largest number of cut vertices or cut edges becomes non-trivial if one places additional restrictions on the graph G. Many such problems have been considered in the literature. Rao [5,6] determined the ranges of the number of cut vertices and the number of cut edges in a graph of order p and size q. These problems with additional constraints on the degrees such as $\Delta(G) \leq d$ and $\delta(G) \geq d$ are also considered in Rao [6] and Rao [7]. The problem of determining the maximum number of cut vertices in a connected graph G of order n where $\delta(G) \geq d$ is considered in Clark and Entringer [3] for $d \geq 5$, and in Albertson and Berman [1] for $d \geq 2$. Nirmala and Rao [4] have shown that the maximum number of cut vertices in a connected d-regular graph of order n is either $\left\lfloor \frac{2n-d-5}{d+1} \right\rfloor - 1$ or $\left\lfloor \frac{2n-d-5}{d+1} \right\rfloor - 2$ for odd $d \geq 5$, and have obtained an upper bound for even $d \geq 6$.

For integers $p \geq 1$ and $d \geq 0$, let $g(p, d)$ denote the maximum number of cut edges in a connected d-regular graph G of order p. We define $g(p, d)$ to be zero if there exists no connected d-regular graph of order p. The exact value of $g(p, 3)$ has been determined in Rao [6] and an upper bound for $g(p, d)$ is obtained in Rao [7]. In this paper we determine the exact value of $g(p, d)$.

We define a *pendant block* of a graph G to be a block that contains exactly one cut vertex of G. Let x be a cut vertex of a connected graph G and $C_1, C_2, \ldots$ be the connected components of $G - x$. We define by a *piece* of G at x, a subgraph $C_i \cup \{x\}$ from which the vertex x, but not its incident edges, is deleted (note that some edges may be 'hanging'). A *neutral vertex* of a graph G is a vertex which is not a cut vertex. A *nearly pendant* block of G is a block which contains two cut vertices of G. For general definitions and notation we refer to Chartrand and Lesniak [2].

2 The number of cut edges in a regular graph

Let G be a connected d-regular graph of order p. Clearly at least one of d and p is even. It is easy to see that G has no cut edges if d is even. Thus $g(p, d) = 0$ if d is even. Again $g(p, 1) = 1$ or 0 according as $p = 2$ or $p > 2$. Henceforth we will assume that $d \geq 3$ is an odd integer, p is an even integer and $p \geq d + 1$.

Theorem 1. *If $p \geq d^2+2d+1$, let $p = d^2+2d+1+m(d^2-3)+\delta$, where $0 \leq \delta < d^2-3$ and $\delta = l(d + 1) + \epsilon$ where $0 \leq \epsilon < d + 1$. Then $g(p, d) = d + m(d - 1) + l$.*
If $2(d + 2) \leq p < d^2 + 2d + 1$, let $p = 2(d + 2) + l(d + 1) + \theta$, where $0 \leq \theta < d + 1$. Then $g(p, d) = l + 1$. Finally, if $d + 1 \leq p < 2(d + 2)$ then $g(p, d) = 0$.

Proof: Let us first assume that $p \geq d^2 + 2d + 1$. Note that m and l are non-negative integers and ϵ is an even integer. We will prove that the number of cut edges in an arbitrary connected d-regular graph G of order p is at most $d + m(d - 1) + l$. This is obviously true if G does not have any cut edges. If G has at least one cut edge then we will apply a reduction procedure to the graph G and reduce it, in a finite number of steps, to a connected graph H with exactly $d + m(d - 1) + l$ cut edges. We ensure that the number of cut edges of the graph does not decrease at each step of the reduction process. This will prove the theorem in this case.

The graph H is described in the following. Take a graph H_1 isomorphic to G_1 of Figure 1. We attach to H_1 a graph H_2 isomorphic to G$_2$ (of the same figure) by identifying the vertex β of H_1 with the vertex α of H_2. Next we attach to H_2 a graph H_3 isomorphic to G_2 by identifying the vertex β of H_2 with the vertex α of H_3. In this way we add graphs $H_2, H_3, \ldots, H_{m+1}$ all isomorphic to G_2.

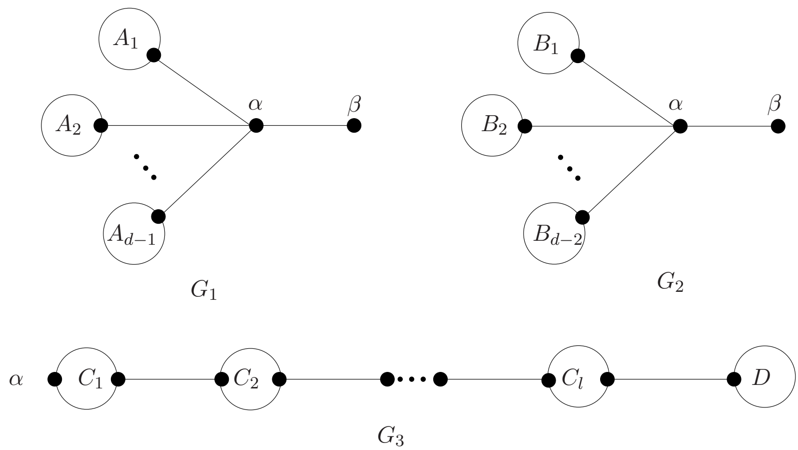

Figure 1

Now attach to H_{m+1} a graph H_{m+2} isomorphic to G_3 of Figure 1 by identifying β of H_{m+1} with the vertex α of H_{m+2}. Let H be the resulting graph. Note that in Figure 1, the blocks $A_1, A_2, \ldots, A_{d-1}$, $B_1, B_2, \ldots, B_{d-2}$ which are represented by circles are pendant blocks on $d+2$ vertices. Also $C_1, C_2, \ldots, C_l$ are nearly pendant blocks on $d+1$ vertices and D is a pendant block on $d+2+\epsilon$ vertices.

Let G be a connected d-regular graph, with at least one cut edge, of order p. We will now apply the reduction procedure to G and reduce it to the graph H described above.

It is easy to see that the size of a pendant block P of G is even or odd according as the degree in P of the cut vertex belonging to P is odd or even.

Let P be a pendant block of G with size n. Clearly $n \geq d+2$. Let x be the cut vertex in P and d_1 the degree of x in P. Note that $2 \leq d_1 \leq d-1$. The first step of the reduction process is to replace the pendant block P by the graph G' described below.

Suppose n is odd. Note that d_1 is even. In this case we take a d-regular graph on the vertices of $P - x$ which is the edge-disjoint union of $(d-1)/2 \geq 1$ Hamiltonian cycles $C_1, C_2, \ldots, C_{\frac{d-1}{2}}$ and a perfect matching. This is possible since $n-3 \geq d-1 \geq 2$. Now remove a matching M_1 of size $d_1/2$ from the perfect matching and join x to the end vertices of the edges of M_1. This is possible since $n-1 \geq d_1+2 \geq 4$. Let G' be the resulting graph. Clearly G' is 2-connected.

Suppose n is even. Note that d_1 is odd, $d-2 \geq d_1 \geq 3$ and $n \geq d+3 \geq 8$. Choose a vertex $y \neq x$ of P. Take a d-regular graph on the vertices of $P - \{x, y\}$ which is the edge disjoint union of $(d-1)/2 (\geq 2)$ Hamiltonian cycles $C_1, C_2, \ldots, C_{\frac{d-1}{2}}$ and a perfect matching. Join x and y. Now from the edges of $C_{\frac{d-1}{2}}$ choose edge-disjoint matchings M_1 and M_2 of sizes $(d_1-1)/2(\geq 1)$ and $(d-1)/2(\geq 2)$ respectively. This is possible since $n-2 \geq d+1 \geq d_1+3 \geq 6$. Remove the edges of M_1 and M_2 and join the end vertices of the edges of $M_1(M_2)$ to $x(y)$. Let G' be the resulting graph. Now using the fact that there is a cycle passing through all the vertices of $P-\{x,y\}$ it is easy to check that G' is 2-connected.

Thus it is easy to see in both the cases that there are $(d-3)/2$ edge disjoint cycles $C_1, C_2, \ldots, C_{(d-3)/2}$ in G' not containing the cut vertex x. If the order of G' is odd these cycles pass through all the neutral vertices of G'. Otherwise they pass through all the neutral vertices of G' except possibly one.

We now reduce the graph G until all pendant blocks, except possibly one, have size $d+2$ or $d+3$. If P is a pendant block of size $n > d+3$ then we replace P by the graph G' (described above) of order $d+2$ or $d+3$ according as n is odd or even. Since the size of every pendant block has been reduced by an even number, this procedure generates an even number of spare (unused) vertices which are now transferred to one of the pendant blocks. Thus all pendant blocks of G contain $d+2$ or $d+3$ vertices except possibly one, whose size may be larger than $d+3$. This exceptional pendant block is referred to as P' in the rest of the paper. Henceforth we refer to the above step of the reduction process as Step 2.

Next we prove that G can be reduced to a graph in which all blocks adjacent to any pendant block are cut edges.

Let P be a pendant block of G. Let $B_1, B_2, \ldots, B_t$ be blocks of G which are not cut edges, adjacent to P. Note that $t \leq (d-3)/2$. Let a_1b_1 and a_2b_2 be edges of the cycle C_1 of P and the block B_1 respectively. Remove them from G and introduce the edges a_1a_2 and b_1b_2. Using the cycle C_1 it is easy to check that this operation combines P and B_1 into a single block. Similarly the blocks B_i, $i = 2, 3, \ldots, t$ are combined with P using the cycles C_i, $i = 2, 3, \ldots, t$. This is Step 3 of the reduction process.

The fourth step in the reduction process involves getting rid of pendant blocks which are adjacent to more than one cut edge. This is done by replacing every pendant block which is adjacent to $k \geq 2$ cut edges by a non-pendant block of the same size. Take a pendant block A which is adjacent to $k \geq 2$ cut edges $e_1, e_2, \ldots, e_k$. Clearly $k \leq d-2$. Let x be the cut vertex in A and P_i the piece at x containing e_i. Then the degree of x in A is $d-k$. We can assume that the size of A is $d+2$ or $d+3$ according as k is odd or even.

Case i: *k is odd.* Note that $k \geq 3$ and $d+2-k(\geq 4)$ is even. Replace A by a block A' which is constructed as follows: Take the complete graph of order $d+2$. Choose k vertices $x_1, x_2, \ldots, x_k$ and remove a cycle of length k passing through them and a matching on the remaining $d+2-k$ vertices. Let A' be the resulting graph. Clearly the degree of x_i in A' is $d-1$ and the degree in A' of every other vertex is d.

Case ii: *k is even.* Note that $d+3-k(\geq 6)$ is even. Replace A by a block A' on $d+3$ vertices which has k vertices of degree $d-1$ in A' and $d+3-k$ vertices with degree d in A'. The block A' is constructed as follows: Take a d-regular graph on $d+3$ vertices which contains a perfect matching and remove a matching of size $k/2$ from it. Let A' be the resulting graph and let $x_1, x_2, \ldots, x_k$ be the vertices with degree $d-1$ in A'.

After replacing the block A by A' as explained above, we attach the piece P_i to A' at the vertex x_i for $i = 1, 2, \ldots, k$.

Thus every pendant block in the graph is adjacent to exactly one block which is a cut edge. Also all pendant blocks except possibly one have size $d+2$.

Next we reduce the graph G until at any cut vertex of G there is at most one block of G which is not a cut edge. Suppose A and B are blocks which are not cut edges and with a common cut vertex x.

Case i: One of the blocks A and B, A say, contains a cycle which does not pass through x. Take an edge uv on that cycle and an edge wz in $B - x$. Remove the edges uv and wz from G and introduce the edges uw and vz into G. Thus A and B are combined into one block.

Case ii: There does not exist any cycle in $A \cup B - \{x\}$. Let $x_1, x_2, \ldots, x_k$ be all the vertices in $A \cup B$ which have degree 2 in $A \cup B$. Note that $k \geq 4$. (Notice that in this case A and B cannot be combined into a single block with the same degree sequence.) Now replace $A \cup B$ by a cycle passing through $x_1, x_2, \ldots, x_k$. Let y be a vertex which originally had an even degree $d_1 > 2$ in $A \cup B$. Then the pieces at y not containing A and B together with y are attached to some pendant block as explained below. Take a matching of $d_1/2$ edges in that pendant block, remove them and join the d_1 end vertices to the vertex y.

Next let z be a vertex which originally had degree d_2 in $A \cup B$ where d_2 is odd. Take a pendant block P of size $d+2$. Note that the cut vertex in P has degree $d-1$ in P. Replace P by a graph G'' which has one vertex of degree $d-1$, one vertex, say z', of degree d_2 and d vertices of degree d each. Now attach the pieces at z not containing A and B at the vertex z' of G''. Note that in this procedure we have not used the vertex z. Since the number of vertices in $A \cup B$ which have odd degree in $A \cup B$ is even, the above procedure leaves an even number of unused (free) vertices which can easily be put in a pendant block. Repeating this procedure as many times as necessary, we finally get a graph with the property that at any cut vertex there is at most one block which is not a cut edge.

Before proceeding further let us make an observation. The size n of a block B which is adjacent to exactly k cut edges is odd or even according as k is odd or even. This follows from the identity $\sum_{i=1}^{n} d_i + k = nd$, where the d_i are the degrees in B of the vertices of B.

In the next step of the reduction process we get rid of blocks, with more than 2 vertices, which are adjacent to $k \geq 3$ cut edges. Firstly let A be a block of size $n \geq 3$ which is adjacent to $k \geq d$ cut edges. Note that these are the only blocks that are adjacent to A. We now shrink the block A to a single vertex a. This produces $n-1$ free vertices (unused vertices of A) which will be placed in some pendant block as explained below.

Case i: $k = d$. Note that n is odd and hence the $n-1$ free vertices can easily be placed into a pendant block.

Case ii: $k > d$. In this case the procedure of shrinking A to a single vertex a, makes the degree of a equal to $k > d$. The degree of a can be made d by removing $k-d$ of the pieces at a. Each of these $k-d$ pieces is now attached to some pendant block successively in the following way.

Let P be a pendant block of size $d+2$. Recall that the block adjacent to P is a cut edge, and hence the unique cut vertex (say x) in P has degree $d-1$ in P. Now replace P by a block of size $d+1$ which has two vertices of degree $d-1$ and $d-1$ vertices of degree d each. Let y be the other vertex of degree $d-1$ in P. Now attach a piece (removed from a) at the vertex y. In this process we get, in all, $n-1+k-d$ free (unused) vertices. Since $n-1+k-d$ is even, these free vertices can easily be put in a pendant block.

Next take a block A which is adjacent to k cut edges where $3 \le k \le d-1$. We first show that the size of A is at least $d+1$. Suppose the size of A is $n \le d$. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of A. Then $\sum_{i=1}^{n} d_i = nd - k \ge (n-1)d + 1$. But $d_i \le n-1$, so we get $n(n-1) \ge (n-1)d+1$, a contradiction which proves that $n \ge d+1$. Now let $e_1, e_2, \ldots, e_k$ be the cut edges adjacent to A. Replace the block A by a block B on $d+1$ vertices such that two vertices x_1, x_2 of B have degree $d-1$ in B and the rest have degree d. Attach the pieces starting with e_1 and e_2 to B at the vertices x_1 and x_2 respectively. Attach the remaining pieces to some pendant blocks successively as explained in the preceding paragraph. By this process we get $n-(d+1)+k-2$ free (unused) vertices. Since $n+k-d-3$ is even these vertices can easily be put in some pendant block.

Thus the graph is reduced until every block in the graph is either pendant or nearly pendant, every nearly pendant block has size 2 or $d+1$, every pendant block is adjacent to exactly one block which is a cut edge, and every pendant block, except possibly one, has size $d+2$.

Now we bring together all the cut vertices with d pieces so that the subgraph generated by them is a tree. For this, if a block C on more than two vertices separates two vertices x and y each of which is a cut vertex with d pieces, remove the block C together with a cut edge and introduce C along with the cut edge at a pendant block as shown in Figure 2.

Thus the graph can be reduced until it consists of a tree T such that all its nonpendant vertices have degree d in T with a chain of nearly pendant blocks and one pendant block attached at each pendant vertex of T. Call each such chain a 'terminal chain'. Now the tree T can be chosen to be a path. We can also assume that all the terminal chains except possibly one consist of a pendant block. Let l' be the number of nearly pendant blocks different from K_2 contained in the exceptional terminal chain. We may assume that all the pendant blocks except the one in the

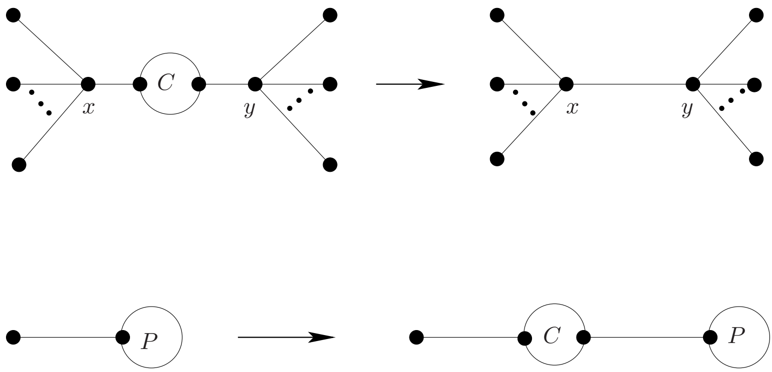

Figure 2

exceptional terminal chain contain $d+2$ vertices. Let α be the vertex common to T and the exceptional terminal chain. If l' is greater than $d-2$, the first $2d-2$ blocks in the exceptional terminal chain can be replaced by a graph isomorphic to G_2 of Figure 1 thereby gaining two free vertices. When $l' = d-2$ and the size of the pendant block in the exceptional terminal chain is at least $2d+1$ we replace the exceptional terminal chain by a graph isomorphic to G_2 of Figure 1 with a pendant block at the vertex β, thereby increasing the number of cut edges. Thus when $l' = d-2$ we may take that the size of the pendant block in the exceptional terminal chain is at most $2d$.

We can now assume that the pendant block in the exceptional terminal chain has at most $2d+2$ vertices since otherwise we can get an extra cut edge. Let $m'+1$ be the number of cut vertices with d pieces of the resulting graph. Now counting the number of vertices in the exceptional terminal chain we see that l' and m' coincide with l and m respectively, defined in the statement of the theorem. Thus the final graph coincides with H described at the beginning of the proof.

Next let $2d+4 \leq p < d^2+2d+1$. By a procedure similar to that used when $p \geq d^2+2d+1$, we reduce any connected d-regular graph on p vertices with at least one cut edge, to the graph shown in Figure 3 without decreasing the number of cut edges. Note that $A_2, A_3, \ldots, A_{l+1}$ are blocks on $d+1$ vertices, A_1 is a block on $d+2$ vertices and A_{l+2} is a block on $d+2+\theta$ vertices.

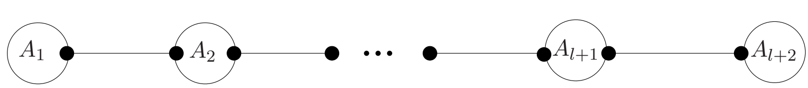

Figure 3

Finally it is easy to see that when $d+1 \leq p < 2(d+2)$, $g(p,d) = 0$. This completes the proof of the theorem.

Acknowledgement

The authors would like to thank the referee for his/her valuable suggestions which improved the presentation of the paper.

References

[1] M.O. Albertson and D.M. Berman, *The number of cut vertices in a graph of given minimum degree*, Discrete Math. **89** (1991), 97–100.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, 2nd Edition, Wadsworth and Brooks/Cole, Monterey, California (1986).

[3] L.H. Clark and R.C. Entringer, *The number of cut vertices in graphs with given minimum degree*, Discrete Math. **18** (1990), 137–145.

[4] K. Nirmala and A.R. Rao, *The number of cut vertices in a regular graph*, Cahiers Centre Etudes Reserche Oper. **17** (1975), 295–299.

[5] A. Ramachandra Rao, *An extremal problem in graph theory*, Israel J. Math. **6** (1968), 261–266.

[6] A. Ramachandra Rao, *Some extremal problems and characterizations in the theory of graphs*, Ph.D. Thesis, Indian Statistical Institute (1969).

[7] S.B. Rao, *Contributions to the theory of directed and undirected graphs*, Ph.D. Thesis, Indian Statistical Institute, (1970).

(Received 28/7/2000; revised 26/6/2002)