Optimal holey packing $\mathrm{OHP}_4(2,4,v,2)$ for $v \equiv 2 \pmod 3$

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Abstract

Maximum distance holey packing $\mathrm{MDHP}(2,k,v,g)$ was first introduced by Yin and used to construct an optimal $(g+1)$-ary constant weight code $(v,k,2k-3)$ CWC. In this paper, an *optimal holey packing* $\mathrm{OHP}_d(2,k,v,g)$ is introduced to construct an optimal $(g+1)$-ary constant weight code (v,k,d) CWC. For $k=4$, $d=4$ and $g=2$, it is proved that there exists an $\mathrm{OHP}_d(2,k,v,g)$ for any integer $v \equiv 2 \pmod 3$ and $v \geq 5$.

1 Introduction

The concept of an H-design $H(v,g,k,t)$ was first introduced by Hanani [3] as a generalization of Steiner systems (the notion of H-design is due to Mills [4]). As stated in Etzion [2] and Yin et al. [10], an optimal $(g+1)$-ary (v,k,d) constant weight code (CWC) over Z_{g+1} can be constructed from a given $H(v,g,k,t)$. For convenience, when two codewords obtained from blocks B_1 and B_2 have distance d, we simply say that B_1 and B_2 have distance d. In the code which is related to an $H(v,g,k,t)$, it is not difficult to see that $k-t+1 \leq d \leq 2(k-t)+1$. An $H(v,g,k,t)$ which forms a code with minimum Hamming distance d is denoted by $\mathrm{GS}_d(t,k,v,g)$ and called *a generalized Steiner system.* If $d = 2(k-t)+1$, it is simply denoted by $\mathrm{GS}(t,k,v,g)$.

Much work has been done for the existence of $\mathrm{GS}(t,k,v,g)$ when $t=2$ and $k=3$. However, not much is known for other cases. Especially, for the case of $t=2$ and $k=4$, there are only partial results on $\mathrm{GS}(2,4,v,2)$. In order to save space, we omit these references; the interested reader may see [7] and the references therein.

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The concept of maximum distance holey packing MDHP$(2, k, v, g)$ was first introduced by Yin (see [8]), and was used to construct $(g+1)$-ary $(v, k, 2(k-2)+1)$ CWC. The definition of holey packing was also first introduced by Yin (see [9]). Let k, g and $v \geq k$ be integers. A *holey packing* k-HP of type g^v is an ordered triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$, where $\mathcal{X}$ is a gv-set (of points), $\mathcal{G}$ is a partition of $\mathcal{X}$ into v *hole*s (or *group*s) of g points, and $\mathcal{B}$ is a collection of k-subsets (called *block*s) of $\mathcal{X}$ such that any pair of points from distinct groups occurs in at most one of the blocks and no block contains two distinct points from the same group. A *maximum distance holey packing* MDHP$(2, k, v, g)$, is defined as a k-HP of type g^v with $g > 1$ and BN$(2, k, v, g)$ blocks whose derived code has minimum Hamming distance $d = 2(k-2)+1$, where

$$\mathrm{BN}(2,k,v,g) = \begin{cases} \left\lfloor \frac{vg}{k} \left\lfloor \frac{(v-1)g}{k-1} \right\rfloor \right\rfloor - 1, & \text{if } (v-1)g \equiv 0 \pmod{k-1} \text{ and} \\ & \quad v(v-1)g^2 \not\equiv 0 \pmod{k(k-1)}; \\ \left\lfloor \frac{vg}{k} \left\lfloor \frac{(v-1)g}{k-1} \right\rfloor \right\rfloor, & \text{otherwise.} \end{cases}$$

Let PN$(2, k, v, g)$ denote the packing number, that is, the maximum number of blocks in a k-HP of type g^v. The value of PN$(2, k, v, g)$ is bounded above by BN$(2, k, v, g)$ (see [8]), that is,

$$\mathrm{PN}(2,k,v,g) \leq \mathrm{BN}(2,k,v,g). \tag{1}$$

Similar to the way that a $(g+1)$-ary (v, k, d) CWC can be constructed from a GS$_d(2, k, v, g)$, we can also construct a $(g+1)$-ary CWC from a k-HP of type g^v with some extra properties. An *optimal holey packing* OHP$_d(2, k, v, g)$ is defined as a k-HP of type g^v with $g > 1$ and BN$(2, k, v, g)$ blocks whose derived code has minimum Hamming distance d. In what follows, a $(g+1)$-ary (v, k, d) CWC with $g > 1$ is said to be *optimal* if it has BN$(2, k, v, g)$ codewords. Note that if $d = 2k-3$, then an OHP$_d(2, k, v, g)$ is just the same as an MDHP$(2, k, v, g)$. It is easy to see that a GS$(2, k, v, g)$ is a special MDHP$(2, k, v, g)$. Similarly, a GS$_d(2, k, v, g)$ is a special OHP$_d(2, k, v, g)$. The existence of MDHP$(2, 3, v, g)$ for $g = 2, 3$ has been completely solved (see [10], [5]). The existence of GS$_4(2, 4, v, g)$ for $g = 2, 3, 6$ was also completely solved in [7]. So, it is natural to determine the existence of OHP$_4(2, 4, v, g)$ for $g = 2, 3, 6$. In this paper, it is proved that there exists an OHP$_d(2, k, v, g)$ for any integer $v \equiv 2 \pmod 3$ and $v \geq 5$. We state the main result as follows.

Theorem 1.1 *There exists an OHP*$_4(2, k, v, 2)$ *for any integer* $v \equiv 2 \pmod 3$ *and* $v \geq 5$.

For general background on design theory, see [1].

2 Product Constructions

In this section, we will give some recursive constructions, which will be used to prove Theorem 1.1 in the next section.

In order to establish recursive constructions for MDHP$(2, k, v, g)$, Wang et al. [5] defined the notion of incomplete MDHP. Similarly, we define an incomplete OHP as follows. An *incomplete optimal holey packing*, denoted by IOHP$_4(2, 4, (n+u, u), g)$, is a quadruple $(\mathcal{X}, \mathcal{G}_1,\ \mathcal{G}_2,\ \mathcal{B})$, where $\mathcal{X}$ is a $g(n+u)$-set (of *points*), $\mathcal{G}_1 = \{G_1,\ G_2,\ \cdots,\ G_{n+u}\}$ is a partition of $\mathcal{X}$ into $n+u$ point classes (called *groups*) of size g, $\mathcal{G}_2 = \{H_1,\ H_2,\ \cdots,\ H_u\} \subseteq \mathcal{G}_1$ and $\mathcal{B}$ is a collection of 4-subsets (called *blocks*) of $\mathcal{X}$ which satisfies the following properties :
(1) each block of $\mathcal{B}$ intersects each group of $\mathcal{G}_1$ in at most one point;
(2) no block contains two distinct points of $Y = \bigcup_{i=1}^{u} H_i$;
(3) every pair of points $\{x,\ y\}$ from distinct groups, such that at least one of x, y is in $\mathcal{X} \setminus Y$, occurs in at most one block;
(4) $u \geq 0$ and $g(n+u-1) \equiv g(u-1) \equiv c \pmod{(k-1)}$, where c is a certain integer satisfying $0 \leq c \leq k-1$;
(5) the number of pairs of points (not both in Y) from distinct groups which do not occur in any block of $\mathcal{B}$ is $\frac{cng}{2}$; and
(6) the derived code has minimum Hamming distance 4.

It is clear that if $u = 0$ or 1, then an IOHP$_4(2, 4, (n+u, u), g)$ is just an OHP$_4(2, 4, n+u, g)$. The following result is similar to Theorem 4.1 in [8].

Lemma 2.1 *An IOHP*$_4(2, 4, (n+u, u), g)$ *contains BN*$(2, 4, n+u, g) - BN(2, 4, u, g)$ *blocks.*

Similar to Lemma 6.9 and Lemma 6.7 in [7], we have the following.

Lemma 2.2 *Let m, n, u be integers such that $u = 0$ or 1, $n \notin \{2, 6\}$. Suppose there exist both a GS*$_4(2, 4, m, g)$ *and an OHP*$_4(2, 4, n+u, g)$*. Then there exist both an IOHP*$_4(2, 4, (mn+u, n+u), g)$ *and an OHP*$_4(2, 4, mn+u, g)$.

Lemma 2.3 *Let m, t, u, h, s, w and a be integers such that $h = sg$, $n = sw$, $w \geq 2a$, $0 \leq sa \leq u$, $1 \leq t \leq w$ and $(w, a) \neq (5, 1)$. Suppose the following designs exist:*
(1) A 4-GDD(h^m) with the property that its blocks can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$ and each group can be partitioned into s subgroup of size g such that the minimum distance in S_r, $0 \leq r \leq t-1$, is 4 with respect to the subgroups.
(2) An IOHP$_4(2, 4, (n+u, u), g)$.
Then there exists an IOHP$_4(2, 4, (e, f), g)$*, where $e = mn + (m-1)sa + u$, $f = (m-1)sa + u$ or $(m-1)sa + n + u$. Further, if there exists an OHP*$_4(2, 4, f, g)$*, then there exists an OHP*$_4(2, 4, e, g)$.

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, some direct constructions are needed.

For $v \equiv 2 \pmod 6$, to construct an $\mathrm{OHP}_4(2,4,v,2)$ in Z_{2v}, it suffices to find a set of base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, $s = \frac{v-2}{6}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms an $\mathrm{OHP}_4(2,4,v,2)$, where $\mathcal{V} = Z_{2v}$, $G = \{G_0, G_1, \cdots, G_{v-1}\}$, $G_i = \{i + vj : 0 \le j \le 1\}$, $0 \le i \le v-1$, and $\mathcal{B} = \{B + j : B \in \mathcal{A},\ 0 \le j \le v-1\}$. For convenience, we write $\mathcal{A} = \{\{0, x, y, z\} : \{x, y, z\} \in S\}$. So, for each $\mathcal{A}$ we need only display the corresponding S.

Lemma 3.1 *There exists an* $OHP_4(2,4,v,2)$ *for each* $v \in \{8, 14, 20, 38, 68, 74\}$.

Proof For each v, with the aid of a computer, we have found a set of base blocks. We list the corresponding S below.

$v = 8$
S: $\{1, 3, 7\}$.

$v = 14$
$S : \{\{1, 3, 10\}, \{4, 12, 17\}$.

$v = 20$
$S : \{1, 3, 9\}, \{4, 11, 15\}, \{5, 17, 27\}$.

$v = 38$
$S : \{28, 40, 43\}, \{8, 45, 67\}, \{16, 30, 66\}, \{13, 14, 19\}, \{2, 27, 34\}, \{11, 29, 52\}$.

$v = 68$
$S : \{24, 34, 135\}, \{5, 8, 99\}, \{50, 119, 123\}, \{19, 49, 77\}, \{18, 92, 103\}, \{20, 56, 113\}, \{15, 27, 98\}, \{2, 41, 72\}, \{16, 22, 104\}, \{9, 41, 90\}, \{7, 21, 47\}$.

$v = 74$
$S : \{1, 27, 100\}, \{8, 87, 93\}, \{77, 102, 107\}, \{43, 50, 83\}, \{2, 20, 96\}, \{14, 58, 95\}, \{64, 86, 124\}, \{12, 47, 51\}, \{10, 92, 137\}, \{78, 106, 135\}, \{9, 32, 68\}, \{3, 117, 132\}$.

□

For $v \equiv 5 \pmod 6$, to construct an $\mathrm{OHP}_4(2,4,v,2)$ in Z_{2v}, it suffices to find a set of base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, $s = \frac{v-2}{3}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms an $\mathrm{OHP}_4(2,4,v,2)$, where $\mathcal{V} = Z_{2v}$, $G = \{G_0, G_1, \cdots, G_{v-1}\}$, $G_i = \{i + vj : 0 \le j \le 1\}$, $0 \le i \le v-1$, and $\mathcal{B} = \{B + 2j : B \in \mathcal{A},\ 0 \le j \le v-1\}$. For convenience, we write $\mathcal{A} = \bigcup_{i=0}^{1} \{\{i, x, y, z\} : \{x, y, z\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding S_i, $0 \le i \le 1$.

Lemma 3.2 *There exists an* $OHP_4(2,4,v,2)$ *for each* $v \in \{5, 11, 17, 23, 47, 59, 83\}$.

Proof For each v, with the aid of a computer, we have found a set of base blocks. We list the corresponding S_i, $0 \le i \le 1$, below.

$v = 5$
$S_0 : \{1, 3, 4\}$; $S_1 : \emptyset$.

$v=11$
$S_0:\{1,2,5\},\{4,13,16\};\quad S_1:\{3,8,15\}.$

$v=17$
$S_0:\{1,2,5\},\{4,10,18\},\{7,12,21\};\quad S_1:\{3,12,27\},\{4,17,23\}.$

$v=23$
$S_0:\{13,34,39\},\{6,8,24\},\{29,31,32\},\{9,27,36\};$
$S_1:\{12,32,33\},\{5,39,44\},\{7,23,36\}.$

$v=47$
$S_0:\{51,58,72\},\{6,23,38\},\{39,43,52\},\{3,26,70\},\{10,75,92\},\{5,25,93\},$
$\{1,33,64\},\{7,67,91\};$
$S_1:\{23,60,76\},\{6,46,66\},\{15,45,53\},\{37,49,77\},\{17,42,50\},\{80,84,93\},$
$\{12,40,58\}.$

$v=59$
$S_0:\{48,73,106\},\{8,52,61\},\{1,63,117\},\{89,93,103\},\{4,15,54\},\{34,47,55\},$
$\{7,31,82\},\{83,88,104\},\{3,6,98\},\{22,87,94\};$
$S_1:\{12,51,68\},\{19,41,102\},\{61,81,93\},\{48,67,86\},\{45,82,114\},\{14,24,42\},$
$\{17,44,89\},\{7,43,91\},\{50,90,92\}.$

$v=83$
$S_0:\{71,86,134\},\{8,9,54\},\{51,84,98\},\{73,107,108\},\{3,49,56\},\{20,137,147\},$
$\{26,132,154\},\{27,35,47\},\{18,28,70\},\{16,66,91\},\{24,105,126\},\{65,67,122\},$
$\{43,68,74\},\{7,55,162\};$
$S_1:\{14,104,137\},\{15,18,115\},\{72,87,109\},\{66,138,143\},\{74,127,136\},$
$\{5,148,150\},\{37,55,78\},\{12,73,111\},\{83,110,151\},\{7,103,128\},\{52,82,88\},$
$\{45,107,135\},\{38,51,125\}.$

□

Lemma 3.3 *There exists an* $IOHP_4(2,4,(8,2),2)$.

Proof Let $\mathcal{X}=Z_{12}$, $\mathcal{G}_1=\{\{i,i+6\}:0\le i\le 5\}$. Let

$$\mathcal{B}=\{\{0,9,10\},\{1,2,6\},\{3,5,7\},\{4,8,11\}\}.$$

Developing $\mathcal{B}$ +3 mod 12, we obtain a 3-RGDD(2^6).

Let $\mathcal{A}_i$ be the blocks obtained by adjoining ∞_i to B_i, where $B_i=\{B+3i: B\in\mathcal{B}\}$, $0\le i\le 3$. Let $\mathcal{V}=\mathcal{X}\cup\{\infty_0,\cdots,\infty_3\}$, $\mathcal{G}=\mathcal{G}_1\cup\{\{\infty_0,\infty_1\},\{\infty_2,\infty_3\}\}$, $\mathcal{H}=\{\{\infty_0,\infty_1\},\{\infty_2,\infty_3\}\}$, $\mathcal{A}=\bigcup_{i=0}^{3}\mathcal{A}_i$. Then $(\mathcal{V},\mathcal{G},\mathcal{H},\mathcal{A})$ is the desired packing. This completes the proof. □

Lemma 3.4 *If* $v\equiv 5 \pmod{12}$ *and* $v\ge 29$*, then there exists an* $OHP_4(2,4,v,2)$.

Proof Write $v = 12s + 5$; then $s \geq 2$. Take $m = 3s + 1$, $n = 4$, $u = 1$ in Lemma 2.2; there exists an $\mathrm{OHP}_4(2, 4, v, 2)$. The existence of $\mathrm{GS}_4(2, 4, m, 2)$ comes from Theorem 1.6 in in [7], and the $\mathrm{OHP}_4(2, 4, 5, 2)$ from Lemma 3.2. This completes the proof. □

The following result was stated in [6] (note that there exists a $\mathrm{GS}_4(2, 4, 7, 4)$ from [6]).

Lemma 3.5 *For* $m = 4, 7$, *there exists a* 4-*GDD*(4^m) *whose groups can be partitioned into two subgroups of size* 2 *each and whose blocks can be partitioned into two sets* S_0 *and* S_1 *such that the minimum distance of* S_i, $0 \leq i \leq 1$, *is* 4 *with respect to the subgroups.*

Lemma 3.6 *There exist both an IOHP*$_4(2, 4, (26, 8), 2)$ *and an OHP*$_4(2, 4, 26, 2)$.

Proof With the 4-GDD(4^4) from Lemma 3.5 and the $\mathrm{IOHP}_4(2, 4, (8, 2), 2)$ from Lemma 3.3, take $m = 4$, $h = 4$, $g = 2$, $s = 2$, $w = 3$, $u = 2$, $t = 2$ and $a = 0$ in Lemma 2.3 to obtain an $\mathrm{IOHP}_4(2, 4, (26, 8), 2)$. Since there exists an $\mathrm{OHP}_4(2, 4, 8, 2)$ from Lemma 3.1, then an $\mathrm{OHP}_4(2, 4, 26, 2)$ exists from Lemma 2.3. This completes the proof. □

Lemma 3.7 *Suppose* $N(n) = p \geq 5$, $0 \leq a \leq n - 1$, $0 \leq b \leq p - 5$. *If there exists an OHP*$_4(2, 4, 3(a + b) + 8, 2)$, *then there exists an OHP*$_4(2, 4, 18n + 3(a + b) + 8, 2)$.

Proof Since $N(n) = p \geq 5$, there exists a $\mathrm{TD}(p + 2, n)$. From $b \leq p - 5$, we have $b + 7 \leq p + 2$, and hence there exists a $\mathrm{TD}(b + 7, n)$. Delete point x and another $n - a - 1$ points from the first group of the $\mathrm{TD}(b + 7, n)$, and delete $n - 1$ points from each of the next b groups. Use x to redefine groups. We obtain a $\{6, 7, n\}$-GDD$(6^n (a+b)^1)$. Since $N(n) \geq 5$, we have $n > 6$. So, from Theorem 1.8 in [7], there exists a $\mathrm{GS}_4(2, 4, q, 6)$ for $q = 6$, 7 and n. Give weight 6 to each point of the GDD, partition each group of size 36 into 18 subgroup of size 2. From Lemma 3.6, there exists an $\mathrm{IOHP}_4(2, 4, (26, 8), 2)$. Adjoining another 8 groups of size 2, there exists an $\mathrm{OHP}_4(2, 4, 18n+3(a+b)+8, 2)$; the $\mathrm{OHP}_4(2, 4, 3(a+b)+8, 2)$ comes from assumption and the $\mathrm{IOHP}_4(2, 4, (26, 8), 2)$ from Lemma 3.6. This completes the proof. □

Lemma 3.8 *There exists an OHP*$_4(2, 4, v, 2)$ *for all* $v \equiv 2 \pmod 3$ *and* $5 \leq v \leq 59$.

Proof From Lemmas 3.1– 3.2, Lemma 3.4 and Lemma 3.6, we need only deal with the values v for $v \in \{32, 35, 44, 50\}$. With the 4-GDD(4^4) from Lemma 3.5, take $m = 4$, $h = 4$, $g = s = 2$, $w = 4$, $u = 0$, $t = 2$ and $a = 0$ in Lemma 2.3. We obtain an $\mathrm{OHP}_4(2, 4, 32, 2)$; the input design $\mathrm{OHP}_4(2, 4, 8, 2)$ comes from Lemma 3.1. Similarly, with the 4-GDD(4^7) from Lemma 3.5, take $m = 7$, $h = 4$, $g = 2$, $s = 2$, $w = 3$, $u = 2$, $t = 2$ and $a = 0$ in Lemma 2.3; there exists an $\mathrm{OHP}_4(2, 4, 44, 2)$. Take $m = 7$, $(n, u) = (5, 0)$ or $(7, 1)$ in Lemma 2.2; an $\mathrm{OHP}_4(2, 4, f, 2)$ exists, where $f = 35$ or 50. This completes the proof. □

Lemma 3.9 *Suppose n_0 is the smallest number r satifying the following property: $N(r) \geq 5,\ r \geq 13$; if $r' > r$ and $N(r') < 5$, then $N(r'-1) \geq 5$ and $N(r'+1) \geq 5$. Then there exists an $OHP_4(2,4,v,2)$ for all $v \geq 18n_0 + 8$.*

Proof For each $v \geq 18n_0 + 8$, write $v = 18n + 3a + 8$, where $0 \leq a \leq 11$. If $N(n) \geq 5$, then by taking $b = 0$ in Lemma 3.7, one gets an $\mathrm{OHP}_4(2,4,v,2)$ since there exists an $\mathrm{OHP}_4(2,4,w,2)$ for all $8 \leq w \leq 41$ from Lemma 3.8. If $N(n) < 5$, we distinguish two cases. If $a < 6$, then $v = 18(n-1)+3a+26$. Since $N(n-1) \geq 5$ and $3a+26 \leq 41$, there exists an $\mathrm{OHP}_4(2,4,v,2)$. If $a \geq 6$, then $v = 18(n+1)+3a-10$. Since $N(n+1) \geq 5$ and $8 \leq 3a-10 \leq 23$, there exists an $\mathrm{OHP}_4(2,4,v,2)$. This completes the proof. ∎

It was stated in [1] that $N(n) \geq 5$ for any $n > 5$ and $n \notin F = \{6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60, 62\}$. So, from Lemma 3.9, one can obtain the following result by taking $n_0 = 16$.

Lemma 3.10 *If $v \equiv 2 \pmod 3$ and $v \geq 296$, then there exists an $OHP_4(2,4,v,2)$.*

In the following, we will show that there exists an $\mathrm{OHP}_4(2,4,v,2)$ for all $v \equiv 2 \pmod 3$ and $v < 296$.

For convenience, let $[x,y]_a^b$ denote the set of integers v for $x \leq v \leq y$ and $v \equiv b \pmod a$.

Lemma 3.11 *If $v \equiv 2 \pmod 3$ and $5 \leq v < 296$, then there exists an $OHP_4(2,4,v,2)$.*

Proof For $v \in [98,110]_3^2 \cup [134,194]_3^2 \cup [206,278]_3^2$, take $n \in \{5,7,8,9,11,12,13\}$, $b = 0$ in Lemma 3.7, and we obtain the result.

For $v \in [197,203]_3^2$, take $n = 9$, $b = 3$ and $6 \leq a \leq 8$ in Lemma 3.7 to obtain the result. The case $v \in [278,293]_3^2$ is obtained from the same lemma with $n = 13$, $b = 6$ and $6 \leq a \leq 11$.

For $v \in [80,92]_6^2$, from Lemma 3.6, there exists an $\mathrm{IOHP}_4(2,4,(26,8),2)$. Take $m = 4$, $h = 4$, $g = 2$, $s = 2$, $w = 9$, $u = 8$, $t = 2$ and $0 \leq a \leq 2$ in Lemma 2.3, and the result is obtained. The 4-GDD(4^4) comes from Lemma 3.5, and the input designs are from Lemma 3.8.

For $v \in [116,128]_6^2$, take $m = 4$, $h = 4$, $g = s = 2$, $w = 4$, $u = 0$ and $a = 0$ in Lemma 2.3 to obtain an $\mathrm{IOHP}_4(2,4,(32,8),2)$. Applying Lemma 2.3 with $m = 4$, $h = 4$, $g = 2$, $s = 2$, $w = 12$, $u = 8$, $t = 2$ and $2 \leq a \leq 4$, we obtain the result, and the input designs are from Lemma 3.8.

From Lemmas 3.1–3.2, Lemma 3.4, Lemma 3.6 and the above results, only the values $v \in Q = \{62, 71, 95, 119, 131\}$ remain to be dealt with. For each $v \in Q$, write $v = mn+u$, where $m \in \{7,10,19\}$, $n \in \{5,6,10,13,17\}$ and $n+u \in \{5,8,11,14,17\}$. So, the result is obtained from Lemma 2.2. This completes the proof. ∎

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1 Combine Lemma 3.10 and Lemma 3.11. ∎

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