On the Ramsey number of the quadrilateral versus the book and the wheel

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Abstract

Let G and H be graphs. The Ramsey number $R(G,H)$ is the least integer such that for every graph F of order $R(G,H)$, either F contains G or $\overline{F}$ contains H. Let B_n and W_n denote the book graph $K_2+\overline{K_n}$ and the wheel graph K_1+C_{n-1}, respectively. In 1978, Faudree, Rousseau and Sheehan computed $R(C_4,B_n)$ for $n \leq 8$. In this paper, we compute $R(C_4,B_n)$ for $8 \leq n \leq 12$ and $R(C_4,W_n)$ for $4 \leq n \leq 13$. In particular, we find that $R(C_4,B_8)=17$, not 16 as claimed in 1978 by Faudree, Rousseau and Sheehan. Most of the results are based on computer algorithms.

1. Introduction

For graphs G and H, a (G,H)-graph is a graph F that does not contain G, and such that the complement $\overline{F}$ does not contain H. A $(G,H;n)$-graph is a (G,H)-graph of order n. Let $\mathcal{R}(G,H)$ and $\mathcal{R}(G,H;n)$ denote the set of all (G,H)-graphs and $(G,H;n)$-graphs, respectively. The Ramsey number $R(G,H)$ is defined to be the least integer $n>0$ such that there is no $(G,H;n)$-graph.

In this paper we consider the case where G is a quadrilateral C_4 (cycle of order 4) and H is a book graph B_n or a wheel graph W_n.

In Section 2 we describe the algorithms and computations performed. Section 3 presents all (C_4,B_n)-graphs for $n \leq 8$, and all $(C_4,B_n;R(C_4,B_n)$-1)-graphs for $9 \leq n \leq 12$. Section 4 presents all (C_4,W_n)-graphs for $n \leq 10$, and of all $(C_4,W_n;R(C_4,W_n)-1)$-graphs for $11 \leq n \leq 13$.

A general utility program for graph isomorph rejection, *nauty* [2], written by Brendan McKay, was used extensively. The graphs themselves are available from the author.

The author would like to thank Brendan McKay for verifying the results on $R(C_4, B_n)$.

2. Algorithms and Computations

The algorithm we use is based on an observation made by McKay and Radziszowski in [3] and [4]. A similar approach was used to compute $R(C_4, K_7)$ and $R(C_4, K_8)$ [5]. We first give some notations.

If G is a graph, then VG and EG are its vertex set and edge set, respectively. For $v \in VG$, let $N_G(v) = \{w \in VG \,|\, vw \in EG\}$, and let $deg_G(v) = |N_G(v)|$. The subgraph of G induced by W will be denoted by $G[W]$. Also, for $v \in VG$, define the induced subgraphs $G_v^+ = G[N_G(v)]$ and $G_v^- = G[VG - N_G(v) - \{v\}]$.

We now describe how to compute $R(C_4, B_m)$ (A similar discussion holds for computing $R(C_4, W_m)$.)

If $G \in \mathcal{R}(C_4, B_m; n)$ and $v \in VG$, then $G_v^+ \in \mathcal{R}(P_3, \overline{B_m}; d)$, where $d = deg_G(v)$, and $G_v^- \in \mathcal{R}(C_4, K_{1,m}; n - d - 1)$, where $K_{1,m} = K_1 + \overline{K}_m$. Hence, G_v^+ must be simply a disjoint union of isolated edges and vertices, and G_v^- is of the same type as G, but for $m - 1$. These properties form the basis of one of our algorithms to enumerate graphs in $\mathcal{R}(C_4, B_m; n)$.

Suppose we have a particular $X \in \mathcal{R}(P_3, B_m; s)$ and $Y \in \mathcal{R}(C_4, K_{1,m}; t)$, and we wish to use them to build a graph $G \in \mathcal{R}(C_4, B_m; s + t + 1)$, by adding a new vertex v of degree s, so that $X = G_v^+$ and $Y = G_v^-$. We need to choose the edges between X and Y. A *feasible cone* is a nonempty subset of VY that does not cover both endpoints of any P_3 in Y. To avoid C_4, the neighborhood in Y of each vertex in X must be a feasible cone.

The algorithm assigns in all possible ways feasible cones to vertices in G_v^+, so that C_4 and $\overline{B}_m$ are avoided in G. In particular, no two cones assigned to distinct vertices in G_v^+ may have a nonempty intersection.

We next give two lemmas that speed up our computations.

Lemma 1. *If G is a C_4-free graph with n vertices and has minimum degree d, then*

$$d^2 - d + 1 \leq n.$$

Proof. This lemma is well known (see [5]), and its proof is omitted. ∎

Lemma 2. *Let G be a C_4-free graph with minimum degree $d > 2$, and let v be a vertex of degree d. Then each vertex of G_v^+ can be assigned a feasible cone. Moreover, since G_v^+ is P_3-free, G_v^+ consists of copies of P_2 and isolated points. The feasible cone assigned to the vertex of P_2 has size at least $d-2$ and the feasible cone assigned to the isolated point has size at least $d - 1$.*

Proof. G_v^+ consists of P_2 and isolated points. Thus, if a vertex of G_v^+ were not assigned a feasible cone, then that vertex would have degree (in G) 2 or 1 (depending on whether it is a vertex of P_2 or an isolated point). This contradicts the fact that G has minimal degree $d > 2$. If the feasible cone assigned to the vertex of P_2 has size less than d-2, then that vertex would have degree in G less than d, again a contradiction. Similary, the feasible cone assigned to the isolated point has size at least d-1. ∎

3. Enumerations and Results of $R(C_4, B_n)$

We present here statistics from the enumeration of various families $\mathcal{R}(C_4, B_n)$ obtained by the algorithms and computations outlined in Section 2. Table I gives the number of nonisomorphic (C_4, B_n)-graphs, $2 \le n \le 8$. These detailed data may be useful in future work towards deriving bounds for general Ramsey numbers of the form $R(C_4, B_n)$.

It is computationally infeasible to generate all of $\mathcal{R}(C_4, B_n)$, $9 \le n \le 12$; we only enumerate (C_4, B_n)-graphs on $R(C_4, B_n) - 1$ vertices, and their statistics are presented in Table II.

Theorem 1.

(i) $R(C_4, B_8) = 17$.

(ii) $R(C_4, B_9) = 18$.

(iii) $R(C_4, B_{10}) = 19$.

(iv) $R(C_4, B_{11}) = 20$.

(v) $R(C_4, B_{12}) = 21$.

Proof of (i). Figure 1 presents the adjacency matrix of the $(C_4, B_8; 16)$-graph establishing the lower bound. The nonexistence of $(C_4, B_8; 17)$-graphs, implying the upper bound, follows from the computations described in Section 2.

The proofs of (ii)–(v) use a similar argument. ∎

n	$\lvert\mathcal{R}(C_4, B_n)\rvert$
2	23
3	64
4	191
5	586
6	2402
7	13345
8	95614

Table I. Number of (C_4, B_n)-graphs, $2 \le n \le 8$.

n	m	$\|\mathcal{R}(C_4, B_n; m)\|$
9	17	8
10	18	132
11	19	4185
12	20	195579

Table II. Number of $(C_4, B_n; R(C_4, B_n) - 1)$-graphs, $9 \le n \le 12$.

1	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
2	1	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0
3	1	1	0	0	0	0	1	0	0	0	1	0	0	0	0	0
4	1	0	0	0	0	0	0	1	0	0	0	0	1	0	1	0
5	1	0	0	0	0	0	0	0	1	0	0	1	0	1	0	0
6	0	1	0	0	0	0	0	0	0	1	0	0	1	0	0	1
7	0	0	1	0	0	0	0	0	0	0	1	1	0	0	0	1
8	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1	1
9	0	0	0	0	1	0	0	1	0	0	0	0	0	1	0	1
10	0	1	0	0	0	1	0	0	0	0	0	1	0	0	1	0
11	0	0	1	0	0	0	1	0	0	0	0	0	1	1	0	0
12	0	0	0	0	1	0	1	0	0	1	0	0	0	0	1	0
13	0	0	0	1	0	1	0	0	0	0	1	0	0	1	0	0
14	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0
15	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0	0
16	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0

Figure 1. Adjacency matrix of the only $(C_4, B_8; 16)$-graph.

4. Enumerations and Results of $R(C_4, W_n)$

Table III gives the number of nonisomorphic (C_4, W_n)-graphs, $4 \le n \le 10$. Table IV presents the (C_4, W_n)-graphs on $R(C_4, W_n) - 1$ vertices, $11 \le n \le 13$.

Theorem 2.

(i) $R(C_4, W_4) = 10.$

(ii) $R(C_4, W_5) = 9.$

(iii) $R(C_4, W_6) = 10.$

(iv) $R(C_4, W_7) = 9.$

(v) $R(C_4, W_8) = 11.$

(vi) $R(C_4, W_9) = 12.$

(vii) $R(C_4, W_{10}) = 13.$

(viii) $R(C_4, W_{11}) = 14.$

(ix) $R(C_4, W_{12}) = 16.$

(x) $R(C_4, W_{13}) = 17.$

Proof. The proofs use the same argument as in Theorem 1. ∎

n	$\lvert\mathcal{R}(C_4, W_n)\rvert$
4	109
5	57
6	128
7	200
8	573
9	2003
10	8861

Table III. Number of (C_4, W_n)-graphs, $4 \leq n \leq 10$.

n	m	$\lvert\mathcal{R}(C_4, W_n; m)\rvert$
11	13	503
12	15	2
13	16	1

Table IV. Number of $(C_4, W_n; R(C_4, W_n) - 1)$-graphs, $11 \leq n \leq 13$.

References

[1] R.J. Faudree, C.C. Rousseau and J. Sheehan, More from the Good Book, *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing*, Utilitas Mathematica Publ. (1978), 289–299.

[2] B.D. McKay, nauty users' guide (version 1.5), Technical Report TR-CS-90-02, Computer Science Department, Australian National University, 1990, source code at `http://cs.anu.edu.au/people/bdm/nauty`.

[3] B.D. McKay and S.P. Radziszowski, $R(4,5) = 25$, *Journal of Graph Theory* **19** (1995), 309–322.

[4] B.D. McKay and S.P. Radziszowski, Subgraph counting identities and Ramsey numbers, *Journal of Combinatorial Theory Ser. B* **69** (1997), 193–209.

[5] S.P. Radziszowski and K.K. Tse, A computational approach for the Ramsey numbers $R(C_4, K_n)$, *Journal of Combinatorial Mathematics and Combinatorial Computing* **42** (2002), 195–207.

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