On the partition function of a finite set

T. C. Brown

Department of Mathematics
Simon Fraser University
Burnaby, BC
Canada V5A 1S6
tbrown@sfu.ca

Wun-Seng Chou

Institute of Mathematics
Academia Sinica
Nankang, Taipei 11529
R.O. CHINA
macws@ccvax.sinica.edu.tw

Peter J.-S. Shiue*

Department of Mathematical Sicences
University of Nevada at Las Vegas
Las Vegas, Nevada 89154-4020
U.S.A.
shiue@unlv.edu

Abstract

Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k relatively prime positive integers. Let $p_A(n)$ denote the partition function of n with parts in A, that is, p_A is the number of partitions of n with parts belonging to A.

We survey some known results on $p_A(n)$ for general k, and then concentrate on the cases $k = 2$ (where the exact value of $p_A(n)$ is known for all n), and the more interesting case $k = 3$. We also describe an approach using the cycle indicator formula.

Let $A = \{a, b, c\}$, where a, b, c are pairwise relatively prime. It has long been known (Ehrhart, J. Reine Angew. Math. **226** (1967), 1–29)

* Most of the work for this paper was done during a stay at the Institute of Mathematics, Academia Sinica, Taipei, Taiwan. The author thanks the Institute of Mathematics, Academia Sinica, Taipei, Taiwan for support and a pleasant stay.

that the problem of finding the value of $p_A(n)$ reduces to the problem of finding the value of $p_A(r)$, where $0 \leq r < abc$. Sertöz and Özlük (Istanbul Tek. Üniv. Bül. **39** (1986), 41–51) have handled the case $abc - a - b - c < r < abc$. Our main contribution is a recursive method for computing the value of $p_A(r)$ in the case $r \leq abc - a - b - c$.

1 Introduction

Let n be a positive integer. A *partition* of n is a representation of n as a sum of positive integers. The order of the terms of this sum does not matter. The *partition function*, denoted by $p(n)$, counts the number of partitions of n. For example, $p(4) = 5$, since 4 has exactly 5 partitions: $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4.

Now, let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k relatively prime positive integers. A *partition of n with parts in A* is a representation of n as a sum of not necessarily distinct elements of A. Again, the order of the terms of this sum does not matter. The *partition function* in this situation, denoted by $p_A(n)$, counts the number of partitions of n with parts in A; see Stanley [38]. Obviously, $p_A(n)$ is the number of non-negative integer solutions $(x_1, x_2, \ldots, x_k)$ of the equation

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = n$$

as mentioned by Comtet [8]. It is well known that for sufficiently large n the equation has a solution. Trivially, if $A = \{1, 2, \ldots, n\}$, then $p_A(n) = p(n)$ (see [25]).

The famous problem of Frobenius is to find the largest natural number g such that $p_A(g) = 0$, that is, the largest natural number g which cannot be expressed in the form $a_1x_1 + a_2x_2 + \cdots + a_kx_k$, where the x_i are non-negative integers.

The Frobenius problem has a long history. See, for example, [16] and [31]. Sylvester [37] completely solved the problem for $k = 2$ in 1882, and Glaisher [15] simplified the proof in 1909. When $A = \{a_1, a_2\}$ and a_1, a_2 are relatively prime, then every $n \geq (a_1 - 1)(a_2 - 1)$ can be expressed in the form $n = a_1x + a_2y$, where x and y are non-negative integers, and $a_1a_2 - a_1 - a_2$ cannot be so expressed. Thus the number g in this case is $g = a_1a_2 - a_1 - a_2$.

When $k = 3$, no closed-form expression for g is known, except in some special cases, although there do exist explicit algorithms for calculating it. See for example [7], [9], [14], [19], [20], [32], and [33].

It seems very difficult to calculate g when $k \geq 4$ (however, see [35]). In the general case, various upper bounds are known (for instance, see [6]), and closed-form expressions are known in a few special cases, for example in the case that $a_1, a_2, \cdots, a_k$ is an arithmetic progression (See [31]). In fact, it was long conjectured that the Frobenius problem is NP-hard, and was finally proved by Ramirez-Alfonsin [29].

This paper is devoted to the study of $p_A(n)$ when $k = 2$ and 3. Our main contribution is a recursive method for computing the value $p_A(n)$ when $n \leq a_1a_2a_3 - a_1 - a_2 - a_3$ where a_1, a_2, a_3 are pairwise relatively prime integers. We also provide a short proof of a known result when $k = 2$ (see Theorem 4.1). Our proof yields a complete explicit formula for $p_A(n)$ in the case $k = 2$ (see Corollary 4.3).

In Sections 2 and 3, we survey some known results on $p_A(n)$ for general k. In Section 4, we focus our attention on the cases $k = 2$ and $k = 3$ (see [10] and [11] for some results concerning the case $k = 4$). Section 5 describes an approach using the cycle indicator formula.

2 Asymptotic formulas for $p_A(n)$ and $p(n)$

If $A = \{a_1, a_2, \ldots, a_k\}$ is a set of k relatively prime positive integers, it is known that

$$p_A(n) \sim \frac{n^{k-1}}{a_1 a_2 \cdots a_k (k-1)!}$$

(see [40]). A proof of this result appears in [26], Problem 27. The proof there is based on the generating function of $p_A(n)$. Elementary proofs are given in [24], [36], and [41]. For the case $A = \{1, 2, \cdots, k\}$, an elementary proof of this formula was given by Erdös [12].

For the unrestricted partition function $p(n)$, Rademacher [28] (see also [2]) gives the asymptotic formula

$$p(n) \sim \frac{\exp(\pi (2/3)^{1/2} n^{1/2})}{4\sqrt{3}n},$$

a result which was proved earlier by Hardy and Ramanujan [17]. Erdös [12] gave an elementary proof of the relation

$$p(n) \sim \frac{a \cdot \exp(\pi (2/3)^{1/2} n^{1/2})}{n},$$

but was unable to show that $a = \frac{1}{4\sqrt{3}}$. Krätzel [21] proved the bound $p(n) \leq 5^{n/4}$, with equality only when $n = 4$.

3 Recurrence relations for $p_A(n)$ and $p(n)$

Apostol [2] (see also [1]) shows by analytical methods that

$$n p_A(n) = \sum_{k=1}^{n} \sigma_A(k) p_A(n-k),$$

where $\sigma_A(n)$ denotes the sum of those divisors of n which belong to A.

This result generalizes a result of Euler, who proves this identity for the case $A = \{1, 2, \ldots, k\}$. This result holds for an arbitrary set A of positive integers, not necessarily finite. Hence when A is the set of all positive integers, this becomes

$$n p(n) = \sum_{k=1}^{n} p(n-k) \sigma(k).$$

Bell [4] shows that if $A = \{a_1, a_2, \ldots, a_k\}$ and a is the least common multiple of $\{a_1, a_2, \ldots, a_k\}$, then

$$p_{_A}(an + b) = c_0 + c_1 n + c_2 n^2 + \cdots + c_{k-1} n^{k-1},$$

where $c_0, c_1, c_2, \ldots, c_k$ are constants dependent of a and b, $0 \le b < a$. (See also [27] and [41].)

The constants are fully determined if $p_{_A}(an + b)$ is known for k different values of n. This can be done using Lagrange's interpolation formula. For example, if $A = \{a_1, a_2, a_3\}$, then

$$\begin{aligned} 2p_{_A}(an + b) = {} & (n-2)(n-3)p_{_A}(a+b) - 2(n-1)(n-3)p_{_A}(2a+b) \\ & + (n-1)(n-2)p_{_A}(3a+b). \end{aligned}$$

This formula does not however provide an effective way to calculate $p_{_A}(n)$. Later, Kuriki [22] proves a somewhat different recursion formula for $p_{_A}(n)$.

Although there are a number of algorithms for finding the largest number not representable in the form $a_1x_1 + a_2x_2 + \cdots + a_kx_k$ (see for example [13], [23], and [35]), it would be of interest to find a fast algorithm for calculating $p_{_A}(n)$.

4 The cases $|A| = 2$ and $|A| = 3$

In the first part of this section, we consider the case $|A| = 2$. It is quite well known that $p_{_A}(n) = \left[\frac{n}{ab}\right]$ or $\left[\frac{n}{ab}\right] + 1$ (see [25]). However, one unified formula has been obtained as stated in the following theorem. This theorem was proved independently by Sertöz in 1998 [34], Tripathi in 2000 [39] and Beck and Robins [3]. Their proofs involve generating functions. There is also a simple direct proof, which we give below. We then give a simple algorithm for calculating $p_{_A}(n)$ based on the proof of this theorem.

Theorem 4.1 *Let $A = \{a, b\}$ with $(a, b) = 1$. Define $a'(n)$ and $b'(n)$ by $a'(n)a \equiv -n \mod b$ with $1 \le a'(n) \le b$ and $b'(n)b \equiv -n \mod a$ with $1 \le b'(n) \le a$, respectively. Then for all $n \ge 1$,*

$$p_{_A}(n) = \frac{n + aa'(n) + bb'(n)}{ab} - 1.$$

Proof. It is well known (see for example Brown and Shiue [5]) that for all $n \ge 0$, if $n = qab+r$ with $0 \le r < ab$ then $p_{_A}(n) = q+p_{_A}(r)$, that for all $0 < n < ab$, $p_{_A}(n) = 0$ or 1, that $p_{_A}(n) = 1$ for $ab - a - b < n < ab$, and that $p_{_A}(n) = 0$ if $n = ab - a - b$. Therefore to prove the theorem we may assume that $0 < n < ab - a - b$.

Note that ab divides $aa'(n) + bb'(n) + n$, since each of a and b divides $aa'(n) + bb'(n) + n$. Also, $0 < aa'(n) + bb'(n) + n < 3ab$, so that either $aa'(n) + bb'(n) + n = ab$ or $aa'(n) + bb'(n) + n = 2ab$. Now we only need to show that

(i) $aa'(n) + bb'(n) + n = ab$ implies $p_{_A}(n) = 0$;

(ii) $aa'(n) + bb'(n) + n = 2ab$ implies $p_{_A}(n) = 1$.

If $aa'(n)+bb'(n)+n = ab$ and $as+bt = n$ for some $s,t \geq 0$, then $aa'(n)+bb'(n)+as+bt = ab$, or $a(a'(n)+s)+b(b'(n)+t) = ab$, so $a|(b'(n)+t)$ and $b|(a'(n)+s)$. Since $0 < b'(n)+t \leq a$ and $0 < a'(n)+s \leq b$, this gives $a = b'(n)+t$ and $b = a'(n)+s$, hence $2ab = ab$, a contradiction. This proves (i). To prove (ii), simply note that if $aa'(n)+bb'(n)+n = 2ab$, then $n = a(b-a'(n))+b(a-b'(n))$. □

This theorem is easy to generalize to the case $(a,b) = d$ in the following corollary. We omit its trivial proof.

Corollary 4.2 *Let $A = \{a,b\}$ with $(a,b) = d$. If d divides n, define $a'(n)$ and $b'(n)$ by $a'(n)\frac{a}{d} \equiv -\frac{n}{d} \mod \frac{b}{d}$ and $b'(n)\frac{b}{d} \equiv -\frac{n}{d} \mod \frac{a}{d}$, respectively, as in Theorem 4.1. Then for all $n \geq 1$,*

$$p_{_A}(n) = \begin{cases} 0 & \text{if } d \text{ does not divide } n \\ \frac{n+aa'(n)+bb'}{lcm\{a,b\}} - 1 & \text{if } d \text{ divides } n. \end{cases}$$

From the statement and the proof of Theorem 4.1, if $(a,b) = 1$, we can compute $p_{_A}(n)$ in the following

Corollary 4.3 *Let $A = \{a,b\}$ with $(a,b) = 1$ and let $n = qab + r$ with $0 \leq r < ab$. Then*

$$p_{_A}(n) = \begin{cases} q+1 & \text{if } ab-a-b < r < ab, \\ q & \text{if } r = ab-a-b, \\ q+1 & \text{if } r < ab-a-b \text{ and } aa'(r)+bb'(r)+r = 2ab, \\ q & \text{if } r < ab-a-b \text{ and } aa'(r)+bb'(r)+r = ab, \end{cases}$$

where $a'(r)$ and $b'(r)$ are defined as in Theorem 4.1.

We now give an example using this corollary. We do not write down all partitions and only compute the number $p_{_A}(n)$ instead.

Example 4.4 [34] *Let $n = 123456789012345$ and $A = \{a,b\}$, where $a = 1234567$, $b = 12345678$. Write $q = 8$ and $r = 1524255800937$. Then we have $n = q \cdot ab + r$. Moreover, $a'(r) = 462963$ and $b'(r) = 1064806$. Hence, $aa'(r)+bb'(r)+r = 15241566651426 = ab$. By Corollary 4.3, we have $p_{_A}(n) = 8$.*

We now consider the case $|A| = 3$ in the remaining part of this section. The case is a little bit more complicated. First of all, we need the following lemma. In this lemma and afterwards, $u'_v(t)$ will denote the number $1 \leq u'_v(t) \leq v$ satisfying $uu'_v(t) \equiv -t \mod v$, whenever $u,v \geq 1$ and t are integers satisfying $(u,v) = 1$.

Lemma 4.5 *Let $A = \{a,b,c\}$, where a, b, and c are relatively prime positive integers. Write $d_3 = (a,b)$, $d_1 = (b,c)$, and $d_2 = (c,a)$. Then for any integer $n > 0$, the number $n' = n-(d_1-a'_{d_1}(n))a-(d_2-b'_{d_2}(n))b-(d_3-c'_{d_3}(n))c$ is divisible by $d_1d_2d_3$. Moreover, $p_{_A}(n) = p_{_{A'}}(\frac{n'}{d_1d_2d_3})$, where $A' = \{\frac{a}{d_2d_3}, \frac{b}{d_3d_1}, \frac{c}{d_1d_2}\}$ and where we use the convention that $p_{_{A'}}(0) = 1$ and $p_{_{A'}}(\frac{n'}{d_1d_2d_3}) = 0$ if $n' < 0$.*

Proof. If $ax+by+cz=n$ with $x,y,z\geq 0$, then d_3 divides $n-cz=ax+by$. Since $d_3-c'_{d_3}(n)$ is the smallest nonnegative integer u such that d_3 divides $n-uc$, $z=d_3z'+(d_3-c'_{d_3}(n))$ for some nonnegative integer z'. Similarly, $x=d_1x'+(d_1-a'_{d_1}(n))$ and $y=d_2y'+(d_2-b'_{d_2}(n))$ for some nonnegative integers x' and y', respectively. So, $ax+by+cz=n$ with $x,y,z\geq 0$ if and only if $a(x-(d_1-a'_{d_1}(n)))+b(y-(d_2-b'_{d_2}(n)))+c(z-(d_3-c'_{d_3}(n)))=n'$ with $x-(d_1-a'_{d_1}(n)), y-(d_2-b'_{d_2}(n)), z-(d_3-c'_{d_3})\geq 0$. This implies that $d_1d_2d_3$ divides n'. Morever,

$$\frac{a(x-(d_1-a'_{d_1}(n)))}{d_1d_2d_3}+\frac{b(y-(d_2-b'_{d_2}(n)))}{d_1d_2d_3}+\frac{c(z-(d_3-c'_{d_3}(n)))}{d_1d_2d_3}=\frac{n'}{d_1d_2d_3}.$$

This implies $p_A(n)=p_{A'}(\frac{n'}{d_1d_2d_3})$. □

From this lemma, it is enough to consider a set $A=\{a,b,c\}$ such that the positive integers a,b, and c are pairwise relatively prime, i.e., $(a,b)=(b,c)=(c,a)=1$. The following two theorems are quite well-known.

Theorem 4.6 (Ehrhart [10]) *Let $A=\{a,b,c\}$, where positive integers a,b, and c are pairwise relatively prime. Let $n=q\cdot abc+r$ with $0\leq r<abc$. Then*

$$p_A(n)=p_A(r)+\frac{q(n+r+a+b+c)}{2}.$$

In particular,

$$p_A(abc)=\frac{abc+a+b+c}{2}+1.$$

Theorem 4.7 (Sertöz and Özlük [36]) *Let $A=\{a,b,c\}$ where positive integers a, b, and c are pairwise relatively prime. Then, for $1\leq x\leq a+b+c-1$,*

$$p_A(abc-x)=\frac{abc+a+b+c}{2}-x.$$

In particular,

$$p_A(abc-a-b-c+1)=\frac{abc-a-b-c}{2}+1.$$

It seems that it is not easy to find a "simple" closed form for computing $p_A(n)$ when $n\leq abc-a-b-c$. Here, we are going to give a method to compute such $p_A(n)$. For this purpose, we need the following

Proposition 4.8 *Let $A=\{a,b,c\}$ where positive integers a, b, c are pairwise relatively prime and let n be a non-negative integer. Then*

$$p_A(n)=\begin{cases} p_A(n-a-b-c)+q_A(n) & \text{if } n\geq a+b+c,\\ q_A(n) & \text{if } 1\leq n<a+b+c\end{cases}$$

where $q_A(n)=p_{A\setminus\{a\}}(n)+p_{A\setminus\{b\}}(n)+p_{A\setminus\{c\}}(n)-\epsilon_a(n)-\epsilon_b(n)-\epsilon_c(n)$ with

$$\epsilon_d(m)=\begin{cases} 1 & \text{if } d|m,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Write $E_{\{a,b,c\}}(n) = \{(x,y,z)|x,y,z \geq 0$ are integers, and $xa+yb+zc=n\}$. Let $(x_1,y_1,z_1) \in E_{\{a,b,c\}}(n)$. If $0 < n < a+b+c$ then $x_1y_1z_1 = 0$. Thus, $p_A(n-a-b-c) = |E_{\{a,b,c\}}(n) \setminus \{E_{\{a,b,0\}}(n) \cup E_{\{a,0,c\}}(n) \cup E_{\{0,b,c\}}(n)\}|$ and the results follows by the inclusion-exclusion formula. □

In the following corollary the values $p_A(abc-a-b-c)$ and $p_A(abc-a-b-c-1)$ are obtained as particular cases of Proposition 4.8.

Corollary 4.9 *Let $A = \{a,b,c\}$ where $a,b,$ and c are positive pairwise relatively prime integers. Then*

$$p_A(abc-a-b-c) = \frac{abc-a-b-c}{2} + 1.$$

and

$$p_A(abc-a-b-c-1) = \frac{abc-a-b-c}{2} - 1.$$

Proof. From Proposition 4.8, we have $p_A(abc-a-b-c) = p_A(abc) - p_{A\setminus\{a\}}(abc) - p_{A\setminus\{b\}}(abc) - p_{A\setminus\{c\}}(abc) + \epsilon_a(abc) + \epsilon_b(abc) + \epsilon_c(abc)$. By Theorem 4.6, we have that $p_A(abc) = \frac{(abc+a+b+c)}{2} + 1$ and, by Corollary 4.3, we obtain that $p_{A\setminus\{a\}}(abc) = a+1$, $p_{A\setminus\{b\}}(abc) = b+1$, and $p_{A\setminus\{c\}}(abc) = c+1$. Since $\epsilon_a(abc) = \epsilon_b(abc) = \epsilon_c(abc) = 1$ then $p_A(abc-a-b-c) = \frac{(abc-a-b-c)}{2} + 1$.

Now again, from Proposition 4.8, we have $p_A(abc-a-b-c-1) = p_A(abc-1) - p_{A\setminus\{a\}}(abc-1) - p_{A\setminus\{b\}}(abc-1) - p_{A\setminus\{c\}}(abc-1) + \epsilon_a(abc-1) + \epsilon_b(abc-1) + \epsilon_c(abc-1)$. By Theorem 4.7, we have that $p_A(abc-1) = \frac{(abc+a+b+c)}{2} - 1$ and, by Corollary 4.3, we obtain that $p_{A\setminus\{a\}}(abc-1) = p_{A\setminus\{a\}}((a-1)bc+(bc-1)) = a$ (similarly, $p_{A\setminus\{b\}}(abc-1) = p_{A\setminus\{b\}}((b-1)ac+(ac-1)) = b$ and $p_{A\setminus\{c\}}(abc-1) = p_{A\setminus\{c\}}((c-1)ab+(ab-1)) = c$). Since $\epsilon_a(abc-1) = \epsilon_b(abc-1) = \epsilon_c(abc-1) = 0$ then $p_A(abc-a-b-c-1) = \frac{(abc-a-b-c)}{2} - 1$. □

Using Proposition 4.8, we will give a method to compute $p_A(n)$ for $n \leq abc-a-b-c$ in the following theorem. For this purpose, we need the notation that for positive integers u and v with $(u,v)=1$, write $v'_u(n)$ instead of $v'(n)$ as in Theorem 4.1.

Theorem 4.10 *Let $A = \{a,b,c\}$ where positive integers a, b, and c are pairwise relatively prime. Let n be a positive integer and let t be the largest integer such that $n - t(a+b+c) \geq 0$. Then,*

$$\begin{aligned} p_A(n) = {} & \frac{2n(t+1)s_3 - t(t+1)s_3^2}{2abc} + \frac{1}{a}\sum_{i=0}^{t}(b'_a(n-is_3) + c'_a(n-is_3)) \\ & + \frac{1}{b}\sum_{i=0}^{t}(c'_b(n-is_3) + a'_b(n-is_3)) + \frac{1}{c}\sum_{i=0}^{t}(a'_c(n-is_3) + b'_c(n-is_3)) \\ & - 3(t+1) - \sum_{i=0}^{t}(\epsilon_a(n-is_3) + \epsilon_b(n-is_3) + \epsilon_c(n-is_3)) \end{aligned}$$

where $s_3 = a+b+c$ with $\epsilon_d(m)$ defined as in Proposition 4.8.

Proof. By applying recursively Proposition 4.8, we have that

$$p_A(n) = \sum_{i=0}^{t-1} q_A(n - is_3) + p_A(n - ts_3) = \sum_{i=0}^{t} q_A(n - is_3)$$

where $q_A(m)$ is defined as in Proposition 4.8. Hence,

$$\begin{aligned}\sum_{i=0}^{t} q_A(n - is_3) = & \sum_{i=0}^{t}(p_{A\setminus\{a\}}(n - is_3) + p_{A\setminus\{b\}}(n - is_3) + p_{A\setminus\{c\}}(n - is_3)) \\ & - \sum_{i=0}^{t}(\epsilon_a(n - is_3) + \epsilon_b(n - is_3) + \epsilon_c(n - is_3)).\end{aligned}$$

The result follows by using Theorem 4.1. □

We give the following example as an illustration of this theorem.

Example 4.11 *Consider* $A = \{5, 7, 11\}$ *and* $n = 41$. *Write* $a = 5$, $b = 7$, *and* $c = 11$ *for convenience. Then,* $s_3 = a + b + c = 23$. *Since* $41 = 1 \times 23 + 18$, $t = 1$. *It is easy to see that the first term in the theorem equals*

$$\frac{2n(t+1)s_3 - t(t+1)s_3^2}{2abc} = \frac{1357}{385}.$$

For positive integers u *and* v *with* $(u, v) = 1$, *let* u_v^{-1} *be the multiplicative inverse of* u *modulo* v. *It easy to see that* $a_b^{-1} = 3$, $a_c^{-1} = 9$, $b_a^{-1} = 3$, $b_c^{-1} = 8$, $c_a^{-1} = 1$, *and* $c_b^{-1} = 2$. *Write* $k = 18$. *Then,* $a'_b(k + is_3) \equiv -a_b^{-1}k - i(1 + a_b^{-1}c) \equiv 2 + i \mod 7$ *for* $i = 0, 1$. *Also,* $a'_c(k + is_3) \equiv 3 + 2i \mod 11$, $b'_a(k + is_3) \equiv 1 + i \mod 5$, $b'_c(k + is_3) \equiv 10 + 3i \mod 11$, $c'_a(k + is_3) \equiv 2 + 2i \mod 5$, *and* $c'_b(k + is_3) \equiv 6 + 3i \mod 7$ *for* $i = 0, 1$. *So,* $\frac{1}{a}\sum_{i=0}^{1}(b'_a(k+is_3)+c'_a(k+is_3) = \frac{9}{5}$, $\frac{1}{b}\sum_{i=0}^{1}(a'_b(k+is_3)+c'_b(k+is_3)) = \frac{13}{7}$, *and* $\frac{1}{c}\sum_{i=0}^{1}(a'_c(k + is_3) + b'_c(k + is_3)) = \frac{20}{11}$. *Moreover, neither* 18 *nor* 41 *is divided by any one of* 5, 7 *and* 11. *Hence,* $\epsilon_a(k + is_3) = \epsilon_b(k + is_3) = \epsilon_c(k + is_3) = 0$ *for* $i = 0, 1$. *Combining all results above together, we have*

$$p_A(41) = \frac{1357}{385} + \frac{9}{5} + \frac{13}{7} + \frac{20}{9} - 3(1 + 1) - 0 = 3.$$

Indeed, there are exactly 3 *partitions of* 41 *with parts in* A, *namely*

$$\begin{aligned}41 &= 5 + 5 + 5 + 5 + 7 + 7 + 7 \\ &= 5 + 5 + 5 + 5 + 5 + 5 + 11 \\ &= 5 + 7 + 7 + 11 + 11.\end{aligned}$$

5 The cycle indicator formula

The cycle indicator C_n of the symmetric permutation group of n letters is an effective tool in enumerative combinatorics, which may be written in the form (cf. [30])

$$C_n(t_1, t_2, \ldots, t_n) = \sum \frac{n!}{k_1! k_2! \cdots k_n!} \left(\frac{t_1}{1}\right)^{k_1} \left(\frac{t_2}{2}\right)^{k_2} \cdots \left(\frac{t_n}{n}\right)^{k_n},$$

where $t_1, t_2, \ldots, t_n$ are real numbers and the summation is over all non-negative integer solutions $k_1, k_2, \ldots, k_n$ of the equation $k_1 + 2k_2 + \cdots + nk_n = n$.

Let $\sigma(n) = \sum_{d|n} d$. Then Hsu and Shiue [18] obtain

$$p(n) = \frac{1}{n!} C_n(\sigma(1), \sigma(2), \ldots, \sigma(n)),$$

where $p(n)$ is the unrestricted partition function from Section 1 above. From this, they obtain by purely combinatorial methods the previously mentioned recurrence relation

$$np(n) = \sum_{k=1}^{n} \sigma(k) p(n-k).$$

The cycle indicator equality above can be generalized in the following way. Let A be any given set of positive integers. (A can be finite or infinite.) Define $p_A(0) = 1$ and $\sigma_A(n) = \sum_{d|n, d\in A} d$. Then Hsu and Shiue [18] obtain

$$p_A(n) = \frac{1}{n!} C_n(\sigma_A(1), \sigma_A(2), \ldots, \sigma_A(n)),$$

and consequently they deduce, again by purely combinatorial methods,

$$np_A(n) = \sum_{k=1}^{n} \sigma_A(k) p_A(n-k).$$

As a particular instance, let us take $H = \{2^0, 2^1, 2^2, \ldots\}$, so that $b(n) = p_H(n)$ is the number of *binary partitions* of n. Let $\beta(n) = \sum_{2^i|n} 2^i$. Then the above equations become $b(n) = \frac{1}{n!} C_n(\beta(1), \beta(2), \ldots, \beta(n))$ and $nb(n) = \sum_{k=1}^{n} \beta(k) b(n-k)$.

Acknowledgment
The authors wish to express their sincere thanks to the referee for very helpful comments and suggestions that led to a considerable improvement of this paper.

References

[1] G. E. Andrews, The Theory of Partition, Encyclopedia of Mathematics and its Applications, Vol. 2. Addison-Wesley, Reading, Mass., 1976.

[2] T. M. Apostol, Introduction to Analytic Number Theory, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.

[3] M. Beck and S. Robins, *A formula related to the Frobenius problem in two dimensions*, Number Theory, Springer-Verlag, New York, to appear.

[4] E. T. Bell, *Interpolated denumerants and Lambert series*, Amer. J. Math. **65** (1943), 382–386.

[5] T. C. Brown and P. J.-S. Shiue, *A remark related to the Frobenius problem*, Fibonacci Quart. **31** (1993), 32–36.

[6] Z.-M. Chen, *A theorem on linear form with integral coefficient* (in Chinese), Sichuan Daxue Xuebao, (1956), No. 1, 1–3.

[7] Z.-M. Chen, *An algorithm to find* M_3 (in Chinese), J. Southwest Teachers College **3** (1984), 2–8.

[8] L. Comtet, Advanced Combinatorics, The Art of Finite and Infinite Expansions, Revised and enlarged edition, D. Reidel, Dordrecht, 1974.

[9] J. L. Davison, *On the linear Diophantine problem of Frobenius*, J. Number Theory **48** (1994), 353–363.

[10] E. Ehrhart, *Sur un problème de géométrie diophantienne linéaire. I. Polyèdres et réseaux* (French), J. Reine Angew. Math. **226** (1967), 1–29.

[11] E. Ehrhart, *Sur un problème de géométrie diophantienne linéaire. II. Systèmes diophantiens linéaires* (French), J. Reine Angew. Math. **227** (1967), 25–49.

[12] P. Erdös, *On an elementary proof of some asymptotic formulas in the theory of partition*, Ann. of Math. (2) **43** (1942), 437–450.

[13] H. Greenberg, *An algorithm for a linear Diophantine equation and a problem of Frobenius*, Numer. Math. **43** (1980), 349–352.

[14] H. Greenberg, *Solution to a linear Diophantine equation for nonnegative integers*, J. Algorithms **9** (1988), 343–353.

[15] J. W. L. Glaisher, *Formulae for partitions into given elements, derived from Sylvester's theorem*, Quart. J. Math. **40** (1909), 275–348.

[16] R. K. Guy, Unsolved Problems in Number Theory, Second edition. Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, I, Springer-Verlag, New York, 1994.

[17] G. H. Hardy and S. Ramanujan, *Asymptotic formulae in combinatory analysis*, Proc. London Math. Soc. **17** (1918), 75–115.

[18] L. C. Hsu and P. J.-S. Shiue, *Cycle indicator and special polynomials*, Annals of Combin. **5** (2001), 179–196.

[19] M. Hujter and B. Vizvári, *The exact solutions to the Frobenius problem with three variables*, J. Ramanujan Math. Soc. **2** (1987), 117–143.

[20] I. D. Kan, B. S. Stechkin and I. V. Sharkov, *On the Frobenius problem for three arguments* (Russian), Mat. Zametki **62** (1997), 626–629; translation in Math. Notes **62** (1997), 521–523.

[21] E. Krätzel, *Die maximale Ordnung der Anzahl der wesentlich verschiedenen Abelschen Gruppen n-ter Ordnung* (German), Quart. J. Math. Oxford Ser. (2) **21** (1970), 273–275.

[22] S. Kuriki, *Sur une méthode de calcul du dénumérant*, TRU Math. **14** (1978), 47–48.

[23] M. Lewin, *An algorithm for a solution of a problem of Frobenius*, J. Reine Angew. Math. **276** (1975), 68–82.

[24] M. B. Nathanson, *Partition with parts in a finite set*, Proc. Amer. Math. Soc. **128** (2000), 1269–1273.

[25] I. Niven, H. S. Zuckerman and H. L. Montgomery, An Introduction to The Theory of Numbers, fifth ed., Wiley, New York, 1991.

[26] G. Pólya and G. Szegö, Problems and Theorems in Analysis. I. Series, Integral Calculus, Theory of Functions. Translated from the German by D. Aeppli. Corrected printing of the revised translation of the fourth German edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 193. Springer-Verlag, Berlin-New York, 1978.

[27] M. Raczunas and P. Chrzpolhkastowski-Wachtel, *A Diophantine problem of Frobenius in terms of the least common multiple*, Selected papers in honor of Paul Erdös on the occasion of his 80th birthday (Keszthely, 1993), Discrete Math. **150** (1996), 347–357.

[28] H. Rademacher, *On the partition function $p(n)$*, Proc. London Math. Soc. **43** (1937), 241–254.

[29] J. L. Ramirez-Alfonsin, *Complexity of the Frobenius problem*, Combinatorica **16** (1996), 143–147.

[30] J. Riordan, An Introduction to Combinatorial Analysis, reprint of the 1958 edition, Princeton University Press, Princeton, N.J., 1980.

[31] J. B. Roberts, *Note on linear forms*, Proc. Amer. Math. Soc. **7** (1956), 465–469.

[32] Ö. J. Rödseth, *On a linear diophantine problem of Frobenius*, J. Reine Angew. Math. **301** (1978), 171–178.

[33] E. S. Selmer, *On the linear Diophantione problem of Frobenius*, J. Reine Angew. Math. **293/294** (1977), 1–17.

[34] S. Sertöz, *On the number of solutions of a Diophantine equation of Frobenius*, Discrete Math. Appl. **8** (1998), 153–162.

[35] S. Sertöz and A. E. Özlük, *On a Diophantine problem of Frobenius*, Istanbul Tek. Üniv. Bül. **39** (1986), 41–51.

[36] S. Sertöz and A. E. Özlük, *On the number of representations of an integer by a linear form*, Istanbul Üniv. Fen Fak. Mat. Derg. **50** (1991), 67–77.

[37] J. J. Sylvester, *Excursus on rational fractions and partitions*, Amer. J. Math. **5** (1882), 119–136.

[38] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Wadsworth & Brooks Co., Monterey, California, 1986.

[39] A. Tripathi, *The number of solutions to* $ax + by = n$, Fibonacci Quart. **38** (2000), 290–293.

[40] J. H. van Lint and R. M. Wilson, A Course in Combinatorics, Cambridge University Press, Cambridge, 1992 (p. 134, problem 15C).

[41] E. M. Wright, *A simple proof of a known result in partitions*, Amer. Math. Monthly **68** (1961), 144–145.

(Received 12/12/2001)