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Families of 2-critical sets for dihedral groups

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Abstract

The dihedral group is defined by $\langle x, y \mid x^n = y^2 = e,\ xy = yx^{-1}\rangle$ for $n \geq 3$ and has order $2n$. In this note four families of 2-critical sets are presented for the Latin square based on this group for even n.

1 Definitions

A *Latin square* of order n is an $n \times n$ array with entries chosen from a set N of size n such that each element of N occurs precisely once in each row and column. Note that the Cayley table of a group satisfies this requirement and hence a Latin square can be formed from any group. A Latin square can be represented by a set of ordered triples $\{(i, j; k) \mid \text{element } k \text{ occurs in cell } (i, j)\}$. Let $L_1 = \{(i_1, j_1; k_1) \mid i_1, j_1, k_1 \in N\}$ and $L_2 = \{(i_2, j_2; k_2) \mid i_2, j_2, k_2 \in N\}$ be two Latin squares of order n . Then L_1 and L_2 are called *isotopic* if there exist permutations α, β, γ, such that $L_2 = \{(\alpha(i_1), \beta(j_1); \gamma(k_1)) \mid (i_1, j_1; k_1) \in L_1\}$. (We write $L_1 \cong L_2$ when they are isotopic.)

A *partial Latin square* of order n is an $n \times n$ array where the entries in non-empty cells are chosen from a set N of size n in such a way that each element of N occurs

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at most once in each row and at most once in each column of the array. A partial latin square I will be represented as a set of ordered triples $\{(i,j;k) \mid \text{element } k \in N \text{ occurs in cell } (i,j) \text{ of the array}\}$. Let I be a partial Latin square of order n. Then $|I|$ is said to be the *size* of I and the set of cells $\{(i,j) \mid (\exists k \in N)((i,j;k) \in I)\}$ is said to determine the *shape* of I. Let I and I' be two partial Latin squares of the same order, with the same size and shape. Then I and I' are said to be *mutually balanced* if the entries in each row (and column) of I are the same as those in the corresponding row (and column) of I'. They are *disjoint* if no non-empty cell in I' contains the same entry as the corresponding cell of I. A *Latin trade* I is a partial Latin square for which there exists another partial Latin square I', of the same order, size and shape with the property that I and I' are disjoint and mutually balanced. An *intercalate* is a Latin trade of size four, and this is the smallest possible size for a Latin trade.

A partial Latin square $C = \{(i,j;k) \mid i,j,k \in N\}$, of order n is called *uniquely completable* (UC) if there is precisely one Latin square L of order n that has element k in position (i,j) for each $(i,j;k) \in C$. If C is a UC set in a Latin square L, of order n, with the property that any proper subset of C is contained in at least two Latin squares of order n, then C is called a *critical set*. Given a uniquely completable set C in a Latin square L and an element $x \in C$, if there exists an intercalate $I \subset L$ such that $I \cap C = \{x\}$, then x is said to be 2-*essential*. A uniquely completable set C is called 2-*critical* if for all $x \in C$, x is 2-essential (see [17]). An example of a 2-critical set is presented in Figure 1.

The following well-known lemma states the relationship between critical sets and Latin trades in Latin squares.

Lemma 1. A partial Latin square C of order n is a critical set if and only if
1) C is UC to a Latin square L of order n, and
2) for each element e of C there exists a Latin trade I_e in L such that $I_e \cap C = \{e\}$.

By Lemma 1 a 2-critical set is also a critical set. The proof of the following lemma is straightforward.

Lemma 2. Let L be a Latin square of order n and let $L' = \{(\alpha(i), \beta(j); \gamma(k)) \mid (i,j;k) \in L\}$, where α, β and γ are permutations on rows, columns and entries of L, respectively. If C is a 2-critical set in L then $C' = \{(\alpha(i), \beta(j); \gamma(k)) \mid (i,j;k) \in C\}$ is a 2-critical set in L'.

Critical sets in Latin squares have been studied extensively (see for example [1, 2, 3, 5, 7, 8, 9, 13]). Critical sets in Latin squares based on groups have also been studied in the past. Khodkar [14] studied smallest critical sets for the elementary abelian 2-groups. Adams and Khodkar [4] found smallest critical sets for the Latin squares based on the groups of order eight. Burgess [6] found a critical set of size 20 for a Latin square based on the quaternion group of order 8. Stinson and van Rees [17] and Donovan, Fu and Khodkar [10] investigated 2-critical sets in Latin squares based on the elementary abelian 2-groups. The order of these 2-critical sets is 2^n for some n. These 2-critical sets have large sizes and use about $4^n - 3^n$ entries. Howse

[12] and Sittampalam and Keedwell [16] investigated critical sets in Latin squares based on dihedral groups. In this note we give some families of 2-critical sets in Latin squares based on dihedral groups of order $2n$ for n even. Note that the critical sets based on dihedral groups given in [16] are not 2-critical sets. Howse [12] introduced two families, $DH1$ and $DH2$, of critical sets in dihedral groups. We observed that DH1 is 2-critical for n even (see [15]). The 2-critical sets E_2, E_3 and E_4 given in this note for $n \geq 10$ have smaller sizes than $DH1$ given in [12].

2 Circulant Latin Squares

The *back circulant Latin square* of order n is defined by the set of ordered triples $BC(n) = \{(i, j; i + j (\text{mod } n)) \mid 0 \leq i, j \leq n - 1\}$. Similarly, the *forward circulant Latin square* of order n is defined by the set of ordered triples $FC(n) = \{(i, j; j - i (\text{mod } n)) \mid 0 \leq i, j \leq n - 1\}$. (See Figure 1.) Define the permutations α, β and γ on $\mathbb{Z}_n$ by $\alpha(i) = i$ and $\beta(i) = \gamma(i) = n - 1 - i$ for $i \in \mathbb{Z}_n$. Then $FC(n) = \{(\alpha(i), \beta(j); \gamma(k)) \mid (i, j; k) \in BC(n)\}$. So $BC(n)$ and $FC(n)$ are isotopic.

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

0	1		
1			
			2

0			3
			0
		1	

Figure 1: $FC(4)$, $BC(4)$ and 2-critical sets in $BC(4)$.

2.1 2-critical sets in $BC(n)$

In this subsection we define some 2-critical sets in $BC(n)$ which are crucial for the rest of this paper. In the following lemmas addition is modulo n.

Lemma 3. The partial Latin square

$$\begin{aligned} S &= \{(i, j; i + j) \mid 0 \leq i \leq a, 0 \leq j \leq a - i\} \cup \\ &\quad \{(i, j; i + j) \mid a + 2 \leq i \leq n - 1, n + 1 + a - i \leq j \leq n - 1\} \end{aligned}$$

for n even and $a \in \{n/2 - 1, n/2 - 2\}$ is a 2-critical set in $BC(n)$.

Proof. It is well-known (see for example Donovan and Cooper [9]) that S is a critical set in $BC(n)$. For each element $(i, j; i + j)$ in S the intercalate

$$\{(i, j; i + j), (i + n/2, j; i + j + n/2), (i, j + n/2; i + j + n/2), (i + n/2, j + n/2; i + j)\}$$

is in $BC(n)$ and intersects S only in $(i, j; i + j)$. So S is a 2-critical set in $BC(n)$. □

An example of S for $n = 4$ and $a = 1$ is displayed in Figure 1. Note that $|S| = n^2/4$.

Lemma 4. Let n be even. Then the partial Latin square

$$\begin{aligned} S' &= \{(i,j;i+j) \mid \tfrac{n}{2}+1 \le i \le n-1, \tfrac{3n-4}{2}+1-i \le j \le n-2\} \cup \\ & \{(i,j,i+j) \mid 0 \le i \le \tfrac{n-6}{2}+1, 0 \le j \le \tfrac{n-6}{2}+1-i\} \cup \\ & \{(i,n-1;i-1) \mid 0 \le i \le \tfrac{n-4}{2}+1\} \end{aligned}$$

is a 2-critical set for $BC(n)$.

Proof. First note that $\{(i,j-1;i+j-1) \mid (i,j;i+j) \in BC(n)\} = BC(n)$. Moreover, if S is as in Lemma 3 with $a = n/2-1$ then $S' = \{(i,j-1;i+j-1) \mid (i,j;i+j) \in S\}$. So S and S' are isotopic. Now by Lemma 3 we see that S is a 2-critical set in $BC(n)$. So S' is a 2-critical set in $BC(n)$ by Lemma 2. □

An example of S' for $n = 4$ is displayed in Figure 1. Note that $|S'| = n^2/4$.

3 Dihedral groups

The *dihedral group* D_n, $n \ge 3$, is defined by

$$D_n = \langle x, y \mid x^n = y^2 = e\ ,\ xy = yx^{n-1} \rangle.$$

The group D_n is of order $2n$. Order the rows and columns of the Cayley table for D_n thus:

$$e = x^0, x^1, \ldots, x^{n-1}, y, yx, \ldots, yx^{n-1}.$$

Then by removing the headline and sideline and relabelling the elements, the Latin square D_n on elements $0, \ldots, 2n-1$ is formed. The new labelling is assigned by the one-to-one mapping σ, where $\sigma(x^i) = i$ and $\sigma(yx^i) = i+n$ for $0 \le i \le n-1$.

The Latin square D_n can be divided into four quadrants where each quadrant is isotopic to either a back or forward circulant square, on either of the element sets $N_1 = \{0, 1, \ldots, n-1\}$ or $N_2 = \{n, \ldots, 2n-1\}$. These quadrants are labelled Q_1, Q_2, Q_3 and Q_4. More precisely,

$$\begin{aligned} Q_1 &= \{(i,j;(i+j)(\text{mod } n)) \mid i,j \in N_1\} = BC(n), \\ Q_2 &= \{(i,j;(j-i)(\text{mod } n)+n) \mid i \in N_1,\ j \in N_2\} \cong FC(n), \\ Q_3 &= \{(i,j;(i+j)(\text{mod } n)+n) \mid i \in\ N_2, j \in N_1\} \cong BC(n), \\ Q_4 &= \{(i,j;(j-i)(\text{mod } n)) \mid i,j \in N_2\} = FC(n). \end{aligned}$$

4 Families of 2-critical sets in D_n for even n

In this section we present four new families of 2-critical sets in D_n of sizes less than $7n^2/4$.

4.1 The 2-critical set R_1

We define the partial Latin square R_1 in D_n by $R_1 = \cup_{i=1}^{4} E_i$, where

$$
\begin{aligned}
E_1 &= Q_1 \setminus \{(\tfrac{n}{2}-1, 0; \tfrac{n}{2}-1)\};\\[1ex]
E_2 &= \{(i,j;k) \in Q_2 \mid 0 \le i \le \tfrac{n}{2}-1,\ j = n\} \cup\\
&\quad\ \{(i,j;k) \in Q_2 \mid 0 \le i \le \tfrac{n}{2}-2,\ \tfrac{3n}{2}+1+i \le j \le 2n-1\} \cup\\
&\quad\ \{(i,j;k) \in Q_2 \mid \tfrac{n}{2}+1 \le i \le n-1,\ n+1 \le j \le i+\tfrac{n}{2}\};\\[1ex]
E_3 &= \{(i,j;k) \in Q_3 \mid n \le i \le \tfrac{3n}{2}-1,\ 0 \le j \le \tfrac{3n}{2}-i-1\} \cup\\
&\quad\ \{(i,j;k) \in Q_3 \mid \tfrac{3n}{2}+1 \le i \le 2n-1,\ \tfrac{5n}{2}-i \le j \le n-1\};\\[1ex]
E_4 &= \{(i,j;k) \in Q_4 \mid \tfrac{3n}{2} \le i \le 2n-1,\ j = 2n-1\} \cup\\
&\quad\ \{(i,j;k) \in Q_4 \mid n \le i \le \tfrac{3n}{2}-2,\ \tfrac{n}{2}+i \le j \le 2n-2\} \cup\\
&\quad\ \{(i,j;k) \in Q_4 \mid \tfrac{3n}{2}+1 \le i \le 2n-1,\ n \le j \le i-\tfrac{n}{2}-1\}.
\end{aligned}
$$

Note that $|R_1| = \frac{7n^2}{4} - 1$. The shape of R_1 is illustrated in Figure 2 for $n = 8$.

0	1	2	3	4	5	6	7	8					13	14	15
1	2	3	4	5	6	7	0	15						13	14
2	3	4	5	6	7	0	1	14							13
	4	5	6	7	0	1	2	13							
4	5	6	7	0	1	2	3								
5	6	7	0	1	2	3	4		12						
6	7	0	1	2	3	4	5		11	12					
7	0	1	2	3	4	5	6		10	11	12				
8	9	10	11									4	5	6	
9	10	11											4	5	
10	11													4	
11															
															3
							12	3							2
						12	13	2	3						1
					12	13	14	1	2	3					0

Figure 2: The 2-critical set R_1 when $n = 8$

Lemma 5. The partial Latin squares E_2, E_3 and E_4 are 2-critical sets in Q_2, Q_3 and Q_4, respectively.

Proof. Let S' be as in Lemma 4. Then $E_2' = \{(\alpha(i), \beta(j); \gamma(k)) \mid (i,j;k) \in S'\}$, where $\alpha(i) = i$ and $\beta(i) = \gamma(i) = n-1-i$ for $i \in \mathbb{Z}_n$, is a 2-critical set by Lemma 2. Since E_2 can be obtained from E_2' by renaming the entries it follows that E_2 is a

2-critical set. It is easy to see that E_3 is isotopic to the 2-critical set S $(a = n/2-1)$ given in Lemma 3. So E_3 is 2-critical. We note that E_4 is S' as in Lemma 4 with the rows permuted. Therefore it is 2-critical by Lemma 2. □

Lemma 6. For $n = 2m,\ m \geq 2$, the partial Latin square R_1 is UC to D_n.

Proof. First note that cell $(\frac{n}{2}-1, 0)$ is forced to contain $\frac{n}{2}-1$ because the entry $\frac{n}{2}-1$ already appears in all rows $i \geq \frac{3n}{2}$. So the entries $0, 1, \ldots, n-1$ cannot occur in the empty cells of E_2 or E_3. We apply Lemma 5 to complete E_2 and E_3 to Q_2 and Q_3, respectively. This forces the empty cells in E_4 to be filled by entries $0, 1, \ldots, n-1$. By applying Lemma 5 we complete E_4 to Q_4. So R_1 is a UC set in D_n. □

Lemma 7. Let $n = 2m,\ m \geq 2$. Then every element in R_1 is 2-essential.

Proof. By Lemma 5 every element of E_2, E_3 and E_4, is 2-essential. Here we prove each element of E_1 is also 2-essential. Let $(i, j; k)$ be an element of E_1 (note that $k \equiv i + j \pmod{n}$).
If $0 \leq i \leq \frac{n}{2} - 2$, $0 \leq j \leq \frac{n}{2} - 1$ and $i + j \leq \frac{n}{2} - 1$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{3n}{2} + i; \tfrac{3n}{2}), (\tfrac{3n}{2} - j, j; \tfrac{3n}{2}), (\tfrac{3n}{2} - j, \tfrac{3n}{2} + i; k)\}.$$

If $i = \frac{n}{2} - 1$ and $j = 1$, we take the intercalate:

$$\{(i, j; k), (i, n + j; \tfrac{3n}{2} + 2), (\tfrac{3n}{2} + 1, j; \tfrac{3n}{2} + 2), (\tfrac{3n}{2} + 1, n + j; k)\}.$$

If $0 \leq i \leq \frac{n}{2} - 1$ and $\frac{n}{2} - i \leq j \leq n - 1$, except $(\frac{n}{2} - 1,\ 1)$, we take the intercalate:

$$\{(i, j; k), (i, n + 1 + i; n + 1), (2n + 1 - j, j; n + 1), (2n + 1 - j, n + 1 + i; k)\}.$$

If $\frac{n}{2} \leq i \leq n - 1$ and $0 \leq j \leq \frac{3n}{2} - i - 2$, we take the intercalate:

$$\{(i, j; k), (i, n - 2 + i; 2n - 2), (2n - 2 - j, j; 2n - 2), (2n - 2 - j, n - 2 + i; k)\}.$$

If $\frac{n}{2} \leq i \leq n - 1$ and $j = \frac{3n}{2} - i - 1$, we take the intercalate:

$$\{(i, j; k), (i, n - 1 + i; 2n - 1), (2n - 1 - j, j; 2n - 1), (2n - 1 - j, n - 1 + i; k)\}.$$

If $\frac{n}{2} + 1 \leq i \leq n - 1$ and $\frac{3n}{2} - i \leq j \leq n - 1$, we take the intercalate:

$$\{(i, j; k), (i, 2n - 1; 2n - 1 - i), (2n - 1 - k, j; 2n - 1 - i), (2n - 1 - k, 2n - 1; k)\}.$$

□

By applying Lemmas 6 and 7 we obtain the following result.

Theorem 8. R_1 is a 2-critical set in D_n for $n = 2m,\ m \geq 2$.

4.2 The 2-critical set R_2

We define the partial Latin square R_2 in D_n by $R_2 = E_5 \cup E_2 \cup E_6 \cup E_7$, where E_2 is as before (see definition R_1) and

$$
\begin{aligned}
E_5 &= Q_1 \setminus \{(i,j;k) \in Q_1 \mid \tfrac{n}{2}+2 \le i \le n-1, \tfrac{3n}{2}-i+1 \le j \le n-1\};\\
E_6 &= \{(i,j;k) \in Q_3 \mid n \le i \le \tfrac{3n}{2}-2,\ 0 \le j \le \tfrac{3n}{2}-i-2\} \cup \\
&\quad\ \{(i,j;k) \in Q_3 \mid \tfrac{3n}{2} \le i \le 2n-1,\ \tfrac{5n}{2}-i-1 \le j \le n-1\};\\
E_7 &= \{(i,j;k) \in Q_4 \mid \tfrac{3n}{2}+1 \le i \le 2n-1,\ n+1 \le j \le i-\tfrac{n}{2}\} \cup \\
&\quad\ \{(i,j;k) \in Q_4 \mid n \le i \le \tfrac{3n}{2}-2,\ \tfrac{n}{2}+1+i \le j \le 2n-1\} \cup \\
&\quad\ \{(i,j;k) \in Q_4 \mid n \le i \le \tfrac{3n}{2}-1,\ j = n\}.
\end{aligned}
$$

Note that

$$|R_2| = \frac{7n^2}{4} - (1+2+\ldots+(\frac{n}{2}-2)) = \frac{13n^2+6n-8}{8}.$$

The shape of R_2 is illustrated in Figure 3 for $n = 8$.

0	1	2	3	4	5	6	7	8					13	14	15
1	2	3	4	5	6	7	0	15						13	14
2	3	4	5	6	7	0	1	14							13
3	4	5	6	7	0	1	2	13							
4	5	6	7	0	1	2	3								
5	6	7	0	1	2	3	4		12						
6	7	0	1	2	3	4			11	12					
7	0	1	2	3	4				10	11	12				
8	9	10						0					5	6	7
9	10							7						5	6
10								6							5
								5							
							11								
						11	12		4						
					11	12	13		3	4					
				11	12	13	14		2	3	4				

Figure 3: The 2-critical set R_2 when $n = 8$

The proof of the following lemma is similar to that of Lemma 5.

Lemma 9. The partial Latin squares E_6 and E_7 are 2-critical sets in Q_3 and Q_4, respectively.

Lemma 10. For $n = 2m,\ m \ge 2$, the partial Latin square R_2 is UC to D_n.

Proof. The empty cells in R_2 can be filled as follows.
The empty cells in columns j for $j = 0, 1, \dots, \frac{n}{2} + 1$, are forced.
The cell $(n - 1 - i, n)$ for $i = 0, 1, \dots, \frac{n}{2} - 1$ is forced .
The cell $(2n - 1 - i, n)$ for $i = 0, 1, \dots, \frac{n}{2} - 1$ is forced.
The empty cells in row i for $i = 0, 1, \dots, \frac{n}{2} + 1$ are forced.
The empty cells in row i for $i = 2n - 1, 2n - 2, 2n - 3$ can be filled uniquely.
The empty cells in column j of Q_2 for $j = 2n - 1, 2n - 2, 2n - 3, 2n - 4, 2n - 5$ are forced.
Now we repeat the following steps for $k = 0, 1, 2, \dots,$ until one of Q_2 or Q_3 is filled. (Note that once any one quadrant is completed, the unique completion sets in the other quadrants are sufficient to force completion of the rest of the square.)

1. The empty cells in row i of Q_3 for $i = 2n-4-6k, 2n-5-6k, 2n-6-6k, 2n-7-6k, 2n-8-6k, 2n-9-6k$ are forced.
2. The empty cells in row i of Q_4 for $i = 2n-4-6k, 2n-5-6k, 2n-6-6k, 2n-7-6k, 2n-8-6k, 2n-9-6k$ can be filled uniquely.
3. The empty cells in column j of Q_2 for $j = 2n-6-6k, 2n-7-6k, 2n-8-6k, 2n-9-6k, 2n-10-6k, 2n-11-6k$ are forced.

So R_2 has a unique completion. □

Lemma 11. Let $n = 2m,\ m \geq 2$. Then every element in R_2 is 2-essential.

Proof. By Lemmas 5 and 9 every element of E_2, E_6 and E_7 is 2-essential. Here we prove each element of E_5 is also 2-essential. Let $(i, j; k)$ be an element of E_5 (note that $k \equiv i + j \pmod{n}$).
If $0 \leq i \leq \frac{n}{2} - 1$ and $0 \leq j \leq \frac{n}{2}$, where $i + j \leq \frac{n}{2}$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{3n}{2} + i; \tfrac{3n}{2}), (\tfrac{3n}{2} - j, j; \tfrac{3n}{2}), (\tfrac{3n}{2} - j, \tfrac{3n}{2} + i; k)\}.$$

If $0 \leq i \leq \frac{n}{2} - 1$ and $\frac{n}{2} + 1 - i \leq j \leq n - 1$, we take the intercalate:

$$\{(i, j; k), (i, n + 1 + i; n + 1), (2n + 1 - j, j; n + 1), (2n + 1 - j, n + 1 + i; k)\}.$$

If $i = \frac{n}{2}$ and $j = 0$, we take the intercalate:

$$\{(\tfrac{n}{2}, 0; \tfrac{n}{2}), (\tfrac{n}{2}, n; \tfrac{3n}{2}), (\tfrac{3n}{2}, 0; \tfrac{3n}{2}), (\tfrac{3n}{2}, n; \tfrac{n}{2})\}.$$

If $\frac{n}{2} \leq i \leq n - 1$ and $0 \leq j \leq \frac{n}{2} + 1$, except cell $(\frac{n}{2}, 0)$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{n}{2} + i + 1; \tfrac{3n}{2} + 1), (\tfrac{3n}{2} + 1 - j, j; \tfrac{3n}{2} + 1), (\tfrac{3n}{2} + 1 - j, \tfrac{n}{2} + 1 + i; k)\}.$$

If $\frac{n}{2} \leq i \leq n - 2$ and $\frac{n}{2} + 2 \leq j \leq n - 1$, and $i + j \leq \frac{3n}{2}$, we take the intercalate:

$$\{(i, j; k), (i, n + i - 1; 2n - 1), (2n - 1 - j, j; 2n - 1), (2n - 1 - j, n - 1 + i; k)\}.$$

□

By applying Lemmas 10 and 11 we obtain the following result.

Theorem 12. R_2 is a 2-critical set in D_n for $n = 2m,\ m \geq 2$.

4.3 The 2-critical set R_3

We define the partial Latin square R_3 in D_n by $R_3 = E_8 \cup E_9 \cup E_{10} \cup E_{11}$, where

$$
\begin{aligned}
E_8 &= Q_1 \setminus \{(i,j;k) \in Q_1 \mid \tfrac{n}{2}+2 \le i \le n-1, \tfrac{3n}{2}-i \le j \le n-2\};\\
E_9 &= \{(i,j;k) \in Q_2 \mid 0 \le i \le \tfrac{n}{2}-1, \tfrac{3n}{2}+i \le j \le 2n-1\} \cup\\
&\quad\ \{(i,j;k) \in Q_2 \mid \tfrac{n}{2}+1 \le i \le n-1,\ n \le j \le \tfrac{n}{2}+i-1\};\\
E_{10} &= \{(i,j;k) \in Q_3 \mid n \le i \le \tfrac{3n}{2}-2,\ 0 \le j \le \tfrac{3n}{2}-2-i\} \cup\\
&\quad\ \{(i,j;k) \in Q_3 \mid \tfrac{3n}{2}+1 \le i \le 2n-1,\ \tfrac{5n}{2}-1-i \le j \le n-2\} \cup\\
&\quad\ \{(i,j;k) \in Q_3 \mid n \le i \le \tfrac{3n}{2}-1,\ j = n-1\};\\
E_{11} &= \{(i,j;k) \in Q_4 \mid n \le i \le \tfrac{3n}{2}-1, \tfrac{n}{2}+i \le j \le 2n-1\} \cup\\
&\quad\ \{(i,j;k) \in Q_4 \mid \tfrac{3n}{2}+1 \le i \le 2n-1,\ n \le j \le i-\tfrac{n}{2}-1\}.
\end{aligned}
$$

Note that

$$
|R_3| = \frac{7n^2}{4} - (1+2+\ldots+(\frac{n}{2}-2)) = \frac{13n^2+6n-8}{8}.
$$

The shape of R_3 is illustrated in Figure 4 for $n = 8$.

0	1	2	3	4	5	6	7					12	13	14	15
1	2	3	4	5	6	7	0						12	13	14
2	3	4	5	6	7	0	1							12	13
3	4	5	6	7	0	1	2								12
4	5	6	7	0	1	2	3								
5	6	7	0	1	2	3	4	11							
6	7	0	1	2	3		5	10	11						
7	0	1	2	3			6	9	10	11					
8	9	10					15					4	5	6	7
9	10						8						4	5	6
10							9							4	5
							10								4
						11		3							
					11	12		2	3						
				11	12	13		1	2	3					

Figure 4: The 2-critical set R_3 when $n = 8$

The proof of the following lemma is similar to that of Lemma 5.

Lemma 13. The partial Latin squares E_9, E_{10} and E_{11} are 2-critical sets in Q_2, Q_3 and Q_4, respectively.

Lemma 14. For $n = 2m,\ m \geq 2$, the partial Latin square R_3 is UC to D_n.

Proof. The empty cells in R_3 can be filled as follows.
The empty cells in row i for $i = 0, 1, \dots, \frac{n}{2} + 1$ are forced.
The cell $(2n - 1 - i, n)$ for $i = 0, 1, \dots, \frac{n}{2} - 1$ is forced.
The empty cells in columns j for $j = 0, 1, \dots, \frac{n}{2}$ are forced.
The empty cells in row i for $i = 2n - 1, 2n - 2$ can be filled uniquely.
The empty cells in column j of Q_2 for $j = 2n - 1, 2n - 2, 2n - 3, 2n - 4, 2n - 5$ are forced.
Now we repeat the following steps for $k = 0, 1, 2, \dots,$ until one of Q_2 or Q_3 is filled.

1. The empty cells in row i of Q_3 for $i = 2n-3-6k, 2n-4-6k, 2n-5-6k, 2n-6-6k, 2n-7-6k, 2n-8-6k$ are forced.

2. The empty cells in row i of Q_4 for $i = 2n-3-6k, 2n-4-6k, 2n-5-6k, 2n-6-6k, 2n-7-6k, 2n-8-6k$ can be filled uniquely.

3. The empty cells in column j of Q_2 for $j = 2n-6-6k, 2n-7-6k, 2n-8-6k, 2n-9-6k, 2n-10-6k, 2n-11-6k$ are forced.

Now it is easy to see that R_3 has a unique completion. □

Lemma 15. Let $n = 2m, m \geq 2$. Then every element in R_3 is 2-essential.

Proof. By Lemma 13 every element of E_9, E_{10} and E_{11} is 2-essential. Here we prove each element of E_8 is also 2-essential. Let $(i, j; k)$ be an element of E_8 (note that $k \equiv i + j \pmod{n}$).
If $0 \leq i \leq \frac{n}{2} - 1$ and $0 \leq j \leq \frac{n}{2} - 1$, where $i + j \leq \frac{n}{2} - 1$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{3n}{2} + i - 1; \tfrac{3n}{2} - 1), (\tfrac{3n}{2} - j - 1, j; \tfrac{3n}{2} - 1), (\tfrac{3n}{2} - j - 1, \tfrac{3n}{2} + i - 1; k)\}.$$

If $0 \leq i \leq \frac{n}{2} - 1$ and $1 \leq j \leq n - 2$, where $i + j \geq \frac{n}{2}$, we take the intercalate:

$$\{(i, j; k), (i, n + i; n), (2n - j, j; n), (2n - j, n + i; k)\}.$$

If $\frac{n}{2} \leq i \leq n - 1$ and $0 \leq j \leq \frac{n}{2}$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{n}{2} + i; \tfrac{3n}{2}), (\tfrac{3n}{2} - j, j; \tfrac{3n}{2}), (\tfrac{3n}{2} - j, \tfrac{n}{2} + i; k)\}.$$

If $\frac{n}{2} \leq i \leq n-2$ and $\frac{n}{2}+1 \leq j \leq n-2$, where $i+j \leq \frac{3n}{2}-1$, we take the intercalate:

$$\{(i, j; k), (i, n - 1 + i; 2n - 1), (2n - 1 - j, j; 2n - 1), (2n - 1 - j, n - 1 + i; k)\}.$$

If $0 \leq i \leq \frac{n}{2}$ and $j = n - 1$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{3n}{2} - 1 + i; \tfrac{3n}{2} - 1), (\tfrac{3n}{2}, j; \tfrac{3n}{2} - 1), (\tfrac{3n}{2}, \tfrac{3n}{2} - 1 + i; k)\}.$$

If $\frac{n}{2} + 1 \leq i \leq n - 1$ and $j = n - 1$, we take the intercalate:

$$\{(i, j; k), (i, \tfrac{n}{2} + i; \tfrac{3n}{2}), (\tfrac{3n}{2} + 1, j; \tfrac{3n}{2}), (\tfrac{3n}{2} + 1, \tfrac{n}{2} + i; k)\}.$$

□

By applying Lemmas 14 and 15 we obtain the following result.

Theorem 16. R_3 is a 2-critical set in D_n for $n = 2m,\ m \geq 2$.

4.4 The 2-critical set R_4

We define the partial Latin square R_4 in D_n by $R_4 = E_{12} \cup E_9 \cup E_6 \cup E_{11}$, where E_6, E_9 and E_{11} are as before (see the definitions of R_2 and R_3) and

$$E_{12} \;=\; Q_1 \setminus \{(i,j;k) \in Q_1 \mid \tfrac{n}{2}+1 \le i \le n-1, \tfrac{3n}{2} - i \le j \le n-1\}.$$

Note that

$$|R_4| = \frac{7n^2}{4} - (1+2+\ldots+(\frac{n}{2}-1)) = \frac{13n^2+2n}{8}.$$

The shape of R_4 is illustrated in Figure 5 for $n = 8$.

0	1	2	3	4	5	6	7					12	13	14	15
1	2	3	4	5	6	7	0						12	13	14
2	3	4	5	6	7	0	1							12	13
3	4	5	6	7	0	1	2								12
4	5	6	7	0	1	2	3								
5	6	7	0	1	2	3		11							
6	7	0	1	2	3			10	11						
7	0	1	2	3				9	10	11					
8	9	10										4	5	6	7
9	10												4	5	6
10														4	5
															4
							11								
						11	12	3							
					11	12	13	2	3						
				11	12	13	14	1	2	3					

Figure 5: The 2-critical set R_4 when $n = 8$

Lemma 17. For $n = 2m,\ m \ge 2$, the partial Latin square R_4 is UC to D_n.

Proof. The empty cells in R_4 can be filled as follows.
The empty cells in columns j for $j = 0, 1, \ldots, \frac{n}{2}$ are forced.
The empty cells in row i for $i = 0, 1, \ldots, \frac{n}{2}$ can be filled uniquely.
The empty cells in row i for $i = 2n-1, 2n-2$ are forced.
The empty cell (i,j) is forced for $i = \frac{n}{2}+1, \ldots, n-1$ and $j = 2n-4, 2n-3, 2n-2, 2n-1$.
Now we repeat the following steps for $k = 0, 1, 2, \ldots,$ until one of Q_2 or Q_3 is filled.

1. The empty cells in row i of Q_2 for $i = 2n-3-4k, 2n-4-4k, 2n-5-4k, 2n-6-4k$ are forced.

2. The empty cells in row i of Q_4 for $i = 2n-3-4k, 2n-4-4k, 2n-5-4k, 2n-6-4k$ can be filled uniquely.

3. The empty cells in column j of Q_3 for $j = 2n-5-4k, 2n-6-4k, 2n-7-4k, 2n-8-4k$ are forced.

Now it is easy to see that R_4 has a unique completion. □

Lemma 18. Let $n = 2m,\ m \geq 2$. Then every element in R_4 is 2-essential.

Proof. By Lemmas 9 and 13 every element of E_6, E_9 and E_{11} is 2-essential. Here we prove each element of E_{12} is also 2-essential. Let $(i, j; k)$ be an element of E_{12} (note that $k \equiv i + j \pmod{n}$).
If $0 \leq i \leq \frac{n}{2} - 1$ and $0 \leq j \leq \frac{n}{2} - 1$, where $i + j \leq \frac{n}{2} - 1$, we take the intercalate:

$$\{(i,j;k),(i,\tfrac{3n}{2}-1+i;\tfrac{3n}{2}-1),(\tfrac{3n}{2}-1-j,j;\tfrac{3n}{2}-1),(\tfrac{3n}{2}-1-j,\tfrac{3n}{2}-1+i;k)\}.$$

If $0 \leq i \leq \frac{n}{2}$ and $\frac{n}{2} - i \leq j \leq n-1$, except the cell $(\frac{n}{2}, 0)$, we take the intercalate:

$$\{(i,j;k),(i,n+i;n),(2n-j,j;n),(2n-j,n+i;k)\}.$$

If $i = \frac{n}{2}$ and $j = 0$, we take the intercalate:

$$\{(\tfrac{n}{2},0;\tfrac{n}{2}),(\tfrac{n}{2},n;\tfrac{3n}{2}),(\tfrac{3n}{2},0;\tfrac{3n}{2}),(\tfrac{3n}{2},n;\tfrac{n}{2})\}.$$

If $\frac{n}{2} + 1 \leq i \leq n-1$ and $0 \leq j \leq \frac{n}{2}$, we take the intercalate:

$$\{(i,j;k),(i,\tfrac{n}{2}+i+1;\tfrac{3n}{2}+1),(\tfrac{3n}{2}+1-j,j;\tfrac{3n}{2}+1),(\tfrac{3n}{2}+1-j,\tfrac{n}{2}+1+i;k)\}.$$

If $\frac{n}{2}+1 \leq i \leq n-2$ and $\frac{n}{2}+1 \leq j \leq n-2$, and $i+j \leq \frac{3n}{2}-1$, we take the intercalate:

$$\{(i,j;k),(i,n-1+i;2n-1),(2n-1-j,j;2n-1),(2n-1-j,n-1+i;k)\}.$$

□

By applying Lemmas 17 and 18 we obtain the following result.

Theorem 19. R_4 is a 2-critical set in D_n for $n = 2m,\ m \geq 2$.

5 Concluding remarks

In this paper we constructed four new families of 2-critical sets in Latin squares based on dihedral groups of order $2n$ for even n. The sizes of these 2-critical sets satisfy $|R_4| < |R_3| = |R_2| < |R_1|$. Moreover, for $n \geq 10$ we observe that $|R_2| < |DH1|$ (see [12] for the critical set $DH1$). The existence of 2-critical sets in dihedral groups of order $2n$ for odd n is still under investigation.

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