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A partial $K_4 \setminus e$-design of order n can be embedded in a $K_4 \setminus e$-design of order at most $8n + 16\sqrt{n} + 82$

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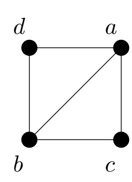
Abstract

We show that a partial $K_4 \setminus e$-design of order n can be embedded in a $K_4 \setminus e$-design of order at most $8n + 26$ if the design has even degree, and of order at most $8n + 16\sqrt{n} + 82$ otherwise.

1 Introduction

In everything that follows we will denote the graph $K_4 \setminus e$ (= the complete undirected graph on 4 vertices with one edge deleted) by ⧄ . A ⧄ -design of order n is a pair (S, K), where K is a collection of edge disjoint copies of the graph ⧄ which partitions the edge set of K_n with vertex set S. It is well-known that the spectrum for ⧄ -designs is precisely the set of all $n \equiv 0$ or $1 \pmod 5 \geq 6$ (see [2] for example). A *partial* ⧄ -design of order n is a pair (X, P), where P is a collection of edge disjoint copies of ⧄ of the edge set of K_n with vertex set X. The difference between a partial ⧄ -design and a ⧄ -design is that the copies of ⧄ in a partial ⧄ -design do *not necessarily* partition the edges of K_n.

In what follows we will denote the graph



by any one of (a, b, c, d), (a, b, d, c), (b, a, c, d), (b, a, d, c) and call these graphs *blocks*.

Given a partial ⧄ -design (X, P) of order n a natural question to ask is whether or not it can be completed; i.e., can $E(K_n) \setminus E(P)$ be partitioned into copies of ⧄ ?

Clearly this cannot be done in general, since no partial $\boxslash$ -design of order $n \not\equiv 0$ or $1 \pmod 5$ can be completed.

Example 1.1 (partial $\boxslash$ -design of order 8) $X = \{1,2,3,4,5,6,7,8\}$ and $P = \{(2,5,3,6),(7,8,1,2),(1,4,2,5)\}$.

Given the fact that a partial $\boxslash$ -design cannot necessarily be completed, the next question to ask is whether or not a partial $\boxslash$ -design can be *embedded.* The partial $\boxslash$ -design (X,P) is said to be *embedded* in the (partial) $\boxslash$ -design (S,K) provided $X \subseteq S$ and $P \subseteq K$.

Example 1.2 ($\boxslash$ -design of order 10) $X = \{1,2,3,4,5,6,7,8,9,10\}$ and

$$\begin{aligned} K = \{&(\mathbf{1},\mathbf{4},\mathbf{2},\mathbf{5}),\ (\mathbf{2},\mathbf{5},\mathbf{3},\mathbf{6}),\ (3,6,4,1),(\mathbf{7},\mathbf{8},\mathbf{1},\mathbf{2}),\ (9,10,1,2),\\ &(7,9,3,4),\ (8,10,3,4),\ (7,10,5,6),\ (8,9,5,6)\}. \end{aligned}$$

The partial $\boxslash$ -design of order 8 in Example 1.1 is embedded in the $\boxslash$ -design of order 10 in Example 2.1.

Naturally, if an embedding is possible, we would like the size of the containing $\boxslash$ -design to be as small as possible.

In [4] it is shown that a partial $\boxslash$ -design of order n can always be embedded in a $\boxslash$ -design of order at most $15n + 46$. The purpose of this paper is to very much improve this bound to $\leq 8n + 26$ if the partial $\boxslash$ -design has even degree and $8n + 16\sqrt{n} + 82$ otherwise.

2 Preliminaries

In 1974 Allan Cruse proved the following important theorem.

Theorem 2.1 (A. B. Cruse [1]) *A partial idempotent commutative quasigroup of order n can be embedded in an idempotent commutative quasigroup of every odd order $\geq 2n+1$.* □

In 1993 this result was generalized as follows: a *partial* groupoid $(P,\circ)$ is said to be idempotent provided $x \circ x = x$ for all $x \in P$. In other words, the quantification "partial" refers to products of the form $x \circ y$ where $x \neq y$. A (partial) groupoid $(P,\circ)$ is called a (partial) *embedding groupoid* provided:

(1) $(P, \circ)$ is idempotent,

(2) if $x \neq y$ either both $x \circ y$ and $y \circ x$ are defined or neither is defined,

(3) $(P, \circ)$ is row latin, and

(4) each $x \in P$ occurs as a product an odd number of times.

We remark that in the case where both $x \circ y$ and $y \circ x$ are defined it is *not necessary* that $x \circ y = y \circ x$.

Example 2.2 (A partial embedding groupoid of order 8)

$\circ$	1	2	3	4	5	6	7	8
1	1	2			5			
2	1	2	3			6		8
3		2	3	4				
4			3	4	5		7	8
5	1			4	5			
6		2				6	7	
7				4		6	7	
8		2		4				8

Theorem 2.3 (C. C. Lindner and C. A. Rodger [3]) *A partial embedding groupoid of order n can be embedded in an idempotent groupoid of every odd order $\geq 2n+1$ which is* (1) *row latin and* (2) *the main diagonal plus all products not defined in the embedding groupoid is a partial $x^2 = x$, $xy = yx$ quasigroup.* □

We will now use Theorem 2.3 to produce an embedding construction for partial ⧄ -designs of even degree.

3 Embedding partial designs of even degree

The technique of embedding requires that the partial ⧄ -design is a graph of even degree; i.e., all vertices have even degree. In the next section we will give an embedding for a partial ⧄ -design with vertices of odd degree into a partial ⧄ -design of even degree.

Let (X, P) be a partial ⧄ -design of *even* degree of order n and define a binary operation "$\circ$" on X by:

(1) $x \circ x = x$, for all $x \in X$, and

(2) if $x \neq y$, $x \circ y$ and $y \circ x$ are defined and $x \circ y = y$ and $y \circ x = y$ if and only if the edge $\{x, y\}$ belongs to a block of P.

It is straightforward to see that $(P, \circ)$ is a partial embedding groupoid. By Theorem 2.3 we can embedded $(P, \circ)$ into an idempotent groupoid $(Q, \circ)$ of any odd order $m \geq 2n+1$ which is (1) row latin and (2) the main diagonal plus all products not defined in the embedding groupoid is a partial $x^2 = x$, $xy = yx$ quasigroup. Let m be the smallest $m \equiv 1$ or $5 \pmod{10} \geq 2n+1$.

Set $S = H \cup (Q \times \{1,2,3,4\})$, where $H = \{1,2,3,4,5,6\}$. Define a collection of blocks K as follows:

(1) Let $\infty \in Q$ and define a copy of the ⊠ -design of order 10 in Example 1.2 on $H \cup (\{\infty\} \times \{1,2,3,4\})$ and put these blocks in K.

(2) For each $a \in Q \setminus \{\infty\}$ define a copy of Example 1.2 on $H \cup (\{a\} \times \{1,2,3,4\})$, delete the blocks $\{(1,4,2,5), (2,5,3,6), (3,6,4,1)\}$, and place the remaining 6 blocks in K.

(3) For each $(a,b,c,d) \in P$ let $(\{1,2,3,4\}, \otimes)$ be an idempotent quasigroup and for each $i, j \in \{1,2,3,4\}$, except for $i = j = 3$, place $((a,i),(b,j),(c,i\otimes j),(d,i\otimes j))$ in K. (So each block $(a,b,c,d) \in P$ contributes 15 blocks to K.)

(4) For each edge $\{a,b\}$ not belonging to a block of P, place the three blocks $((a,1),(b,1),(a\circ b,2),(a\circ b,3))$, $((a,2),(b,2),(a\circ b,3),(a\circ b,4))$, and $((a,4),(b,4),(a\circ b,1),(a\circ b,3))$ in K.

(5) Define a ⊠ -design of order m on $Q \times \{3\}$ and place these blocks in K.

It is straightforward and not difficult to show that the sum of the blocks in (1), (2), (3), (4), and (5) is $\binom{4m+6}{2}/5$ and that each edge of K_{4m+6} with vertex set S belongs to a block of type (1), (2), (3), (4), or (5). This proves that (S, K) is a ⊠ -design of order $4m+6$.

Since the quasigroup $(\{1,2,3,4\}, \otimes)$ in (3) is idempotent and K contains for each $(a,b,c,d) \in P$ the 15 blocks $((a,i),(b,j),(c,i\otimes j),(d,i\otimes j))$, all $i,j \in \{1,2,3,4\}$, except for $i = j = 3$, K contains the three blocks $((a,1),(b,1),(c,1),(d,1))$, $((a,2),(b,3),(c,3),(d,3))$, and $((a,4),(b,4),(c,4),(d,4))$ and hence 3 disjoint copies of the ⊠ -design (X, P).

Lemma 3.1 *A partial ⊠ -design of even degree of order n can be embedded in a ⊠ -design of order $4m+6$ for every $m \equiv 1$ or $5 \pmod{10} \geq 2n+1$.* □

Corollary 3.2 *If m is as small as possible in Lemma 3.1, the containing ⊠ -design has order $\leq 8n+26$.* □

It remains to make the containing $\boxslash$ -design as small as possible when the partial $\boxslash$ -design has vertices of odd degree. This depends on first embedding the partial $\boxslash$ -design into a "small" partial $\boxslash$ -design of even degree. This is the topic of the next section.

4 Embedding partial designs

In this section we give a construction for embedding partial $\boxslash$ -designs containing vertices of odd degree.

To begin with a $K_{1,4}$ design of order n is a pair (Q, F), where F is a collection of copies of the graph $K_{1,4}$ which partitions the edge set of K_n with vertex set Q. The spectrum of such designs is the set of all $n \equiv 0$ or $1 \pmod 8$ [5].

Let (X, P) be a partial $\boxslash$ -design of order n with vertices of odd degree. Then of course there are an even number of such vertices which we pair up (in any manner) $\{a_1, b_1\}$, $\{a_2, b_2\}, \ldots, \{a_t, b_t\}$. Let (Q, F) be a $K_{1,4}$-design of order x, where $\binom{x}{2}/4 \geq n/2 \geq t$. (The number of blocks of (Q, F) is $\binom{x}{2}/4$.) Choose any t blocks in F, say $k_1, k_2, \ldots, k_t$ and pair the block k_i with the pair of odd vertices $\{a_i, b_i\}$.

If k_i consists of the edges $\{e_1, e_2\}$, $\{e_1, e_3\}$, $\{e_1, e_4\}$, $\{e_1, e_5\}$ form the two copies of $\boxslash$: (a_i, e_1, e_2, e_3) and (b_i, e_1, e_4, e_5). Denote the set of $2t$ copies of $\boxslash$ constructed in this manner by E. Then $(X \cup Q, P \cup E)$ is a partial $\boxslash$ -design of even degree of order $n + x$. By Lemma 3.2 $(X \cup Q, P \cup E)$ can be embedded in a $\boxslash$ -design of order $\leq 8(n+x)+26$. It remains to estimate x. If we choose x to be the *smallest* positive integer such that $x \equiv 0$ or $1 \pmod 8$ and $\binom{x}{2}/4 \geq n/2 \geq t$, then $2\sqrt{n}+7 \geq x \geq 2\sqrt{n}$, and so $8(n+x)+26 \leq 8(n+2\sqrt{n}+7)+26 = 8n+16\sqrt{n}+82$.

Lemma 4.1 *A partial $\boxslash$ -design of order n having vertices of odd degree can be embedded in a $\boxslash$ -design of order $\leq 8n + 16\sqrt{n} + 82$.* □

5 Summary of results

We can combine Corollary 3.2 and Lemma 4.1 into the following theorem.

Theorem 5.1 *A partial $\boxslash$ -design can be embedded in a $\boxslash$ -design of order $\leq 8n+26$ if the design has even degree and of order $\leq 8n + 16\sqrt{n} + 82$ if it contains vertices of odd degree.* □

Regardless of whether or not the partial $\boxslash$ -design has even degree or not, the results in Theorem 5.1 are a dramatic improvement over the results in [4]. Although a vast improvement over the previous best known embedding, Theorem 5.1 is still not the best possible embedding. (The authors are not quite sure what the best possible embedding is.) Much work remains on this very interesting problem, made all the more interesting by the fact that $\boxslash$ -designs are as close as one can get to block designs with block size 4, and to date there are no "small" embeddings for such block designs.

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