AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **29** (2004), Pages 249–257

Four-regular graphs that quadrangulate both the torus and the Klein bottle

Atsuhiro Nakamoto

Department of Mathematics
Faculty of Education and Human Sciences
Yokohama National University
79-2 Tokiwadai, Hodogaya-ku
Yokohama 240-8501, Japan
nakamoto@edhs.ynu.ac.jp

Abstract

In this paper, we characterize 4-regular graphs which can quadrangulate both the torus and the Klein bottle.

1 Introduction

In this paper, we deal with only connected, undirected, *simple* graphs, that is, without loops and multiple edges. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Let F^2 denote a *surface*, i.e., a connected, compact 2-dimensional manifold without boundaries. When we regard a graph G as a topological space, an injective map $f : G \to F^2$ is called an *embedding* of G into F^2. If each component of $F^2 - f(G)$, called a *face* of $f(G)$, is homeomorphic to a 2-cell, then f is said to be a 2-*cell embedding.* A k-*cycle* means a cycle of length k.

A *quadrangulation* of a closed surface F^2 is a fixed embedding f of a simple graph G on F^2 such that each face of $f(G)$ is bounded by a 4-cycle. We say that a graph G can *quadrangulate* F^2 if there exists a 2-cell embedding f of G into F^2 such that $f(G)$ is a quadrangulation of F^2. In this case, we call f a *quadrangular embedding.*

By Euler's formula, if $f(G)$ is a quadrangulation of F^2, then the equation $2|V(G)|-|E(G)| = 2\chi(F^2)$ holds, where $\chi(F^2)$ denotes the Euler characteristic of F^2. Thus, a necessary condition for a graph G to have a quadrangular embedding into F^2 is that G satisfies $2|V(G)| - |E(G)| = 2\chi(F^2)$. This means that if G can quadrangulate F^2, then G might quadrangulate another closed surface $\tilde{F}^2 \neq F^2$ with $\chi(F^2) = \chi(\tilde{F}^2)$. By the classification of closed surfaces, if $F^2 \neq \tilde{F}^2$ and $\chi(F^2) = \chi(\tilde{F}^2)$, then F^2 and $\tilde{F}^2$ have different orientabilities. In this paper, we focus on the torus, denoted by T^2, and the Klein bottle, denoted by K^2, as F^2 and $\tilde{F}^2$ with the same Euler characteristic.

Let G be a graph. Let $t(G)$ be a quadrangulation of T^2 with a quadrangular embedding t. A simple closed curve l on T^2 is said to be *trivial* if l bounds a 2-cell

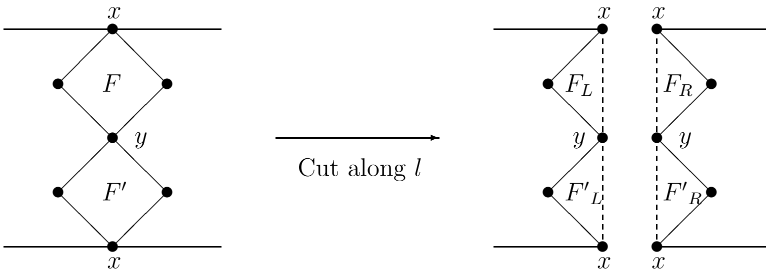

Figure 1: A diagonal 2-curve l

on T^2 and it is *essential* otherwise. A simple closed curve l on $T^2 \supset t(G)$ is said to be a *diagonal* 2*-curve* if l passes through only x, F, y and F' in this order, where F and F' are faces of G, and x, y are two vertices of G which are not adjacent in the boundary 4-cycles of F and F'. Suppose that $t(G)$ has such an essential diagonal 2-curve l. Cutting T^2 along l, we obtain an annulus A with two holes H and H', where we suppose that the right-hand side F_R of F and the right-hand side F'_R of F' are incident to H, and the left-hand side F_L of F and the left-hand side F'_L of F' are incident to H', respectively. See Figure 1.

Now, identify H and H' of A to make the Klein bottle so that $F_R \cup F'_L$ and $F_L \cup F'_R$ are new faces. Then, we obtain an quadrangular embedding of G into the Klein bottle. Thus, a sufficient condition for a quadrangulation $t(G)$ of the torus to have a quadrangular embedding into the Klein bottle is that $t(G)$ has an essential diagonal 2-curve.

We conjecture that this is also necessary, as in the following.

Conjecture 1 *Let G be a graph which has a quadrangular embedding t into the torus. Then, G has a quadrangular embedding of the Klein bottle if and only if $t(G)$ has an essential diagonal* 2*-curve.*

The author has already determined the complete lists of *irreducible* quadrangulations of the torus [5] and those of the Klein bottle [6], where "irreducible" means minimal with respect to a *face contraction* (i.e, identifying a pair of non-adjacent vertices on the boundary 4-cycle of a face and replace two pairs of multiple edges arisen with two single edges respectively). The author has also checked that if $t(G)$ is an irreducible quadrangulation of the torus, then Conjecture 1 is true.

In this paper, we show the following.

Theorem 2 *Conjecture* 1 *is true if G is* 4*-regular.*

A graph which can triangulate the torus and the Klein bottle has been already characterized by Lawrencenko and Negami [4]. This result has been obtained by the

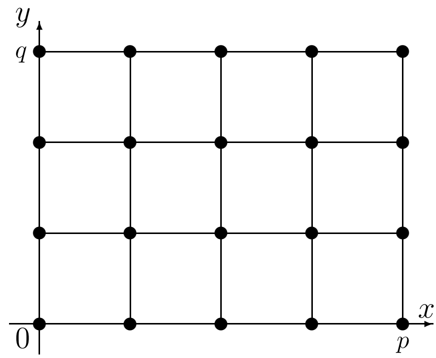

Figure 2: $R_{p,q}$

complete lists of *irreducible* triangulations of torus [2] and the Klein bottle [3], where "irreducible" means minimal with respect to edge contractions. However, it seems to be difficult to use the same method in the current case.

2 Classification of 4-regular quadrangulations of the torus and the Klein bottle

In this section we give preliminaries for proving Theorem 2. The 4-regular quadrangulations of the torus and those of the Klein bottle have been classified in [1] and [7], respectively, and the standard forms of them have been given.

Consider the region $\{(x, y) \in \boldsymbol{R}^2 : 0 \le x \le p, 0 \le y \le q\}$ with a vertex put in each integer point, and arrange $p+1$ vertical segments and $q+1$ horizontal segments passing through them. Then we obtain a configuration $R_{p,q}$ shown in Figure 2. Identifying the top and bottom of $R_{p,q}$, we obtain the annulus denoted by $\Omega_{p,q}$. Moreover, identifying $(0, y)$ and $(p, y + r)$ in $\Omega_{p,q}$ for a fixed r and each y (where the y-coordinates are taken modulo q), we obtain a 4-regular quadrangulation $T(p, r, q)$ of the torus. To keep $T(p, r, q)$ simple, we need $q \ge 3$ and some restriction for p and r.

Theorem 3 (Altshuler [1]) *Every 4-regular quadrangulation of the torus is isomorphic to $T(p, r, q)$ for some integers $p \ge 1, q \ge 3$ and $r \ge 0$.*

By identifying $(0, y)$ and $(p, q - y)$ in $\Omega_{p,q}$ for each y, we obtain a 4-regular quadrangulation $K_h(p, q)$ of the Klein bottle, where this type is called the *handle type.*

In the proof of our main theorem described in the next section, we focus on the cycle of $K_h(p, q)$ passing through $(0, q), (1, q), \dots, (p, q)$ in the form $R_{p,q}$, which is orientation-reversing and straight at each vertex with respect to its rotation. If q is even, then the cycle passing through $(0, q/2), (1, q/2), \dots, (p, q/2)$ has the same property.

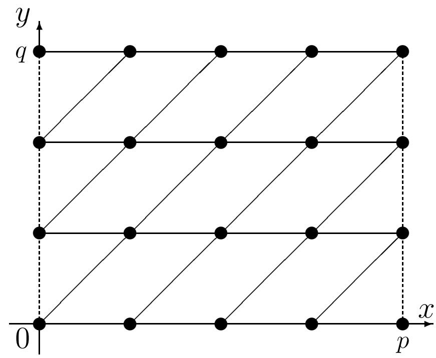

Figure 3: $\tilde{R}_{p,q}$

Consider the annulus $\Omega_{p,2q}$ with $p \geq 1$ and $q \geq 2$, and add a crosscap to each of the boundary components of $\Omega_{p,2q}$. Joining $(0, y)$ and $(0, y + q)$, and (p, y) and $(p, y+q)$, respectively, for each y, through the crosscaps added, we obtain a 4-regular quadrangulation $K_l(p, q)$ of the Klein bottle, called the *ladder type*. We can find in $K_l(p, q)$ two disjoint *Möbius ladders* (i.e., a $2n$-cycle $C = u_1v_1u_2v_2 \cdots u_nv_n$ $(n \geq 2)$ with an edge u_iv_i added for $i = 1, \ldots, n$) induced by the vertices on the boundary components of $\Omega_{p,2q}$.

Consider again the region $\{(x, y) \in \boldsymbol{R}^2 : 0 \leq x \leq p, 0 \leq y \leq q\}$ with a vertex placed on each integer point. Joining (x, y) and $(x+1, y)$, and (x, y) and $(x+1, y+1)$, we obtain $\tilde{R}_{p,q}$ shown in Figure 3. Identifying the top and the bottom of $\tilde{R}_{p,q}$ for each x, we obtain the annulus $\tilde{\Omega}_{p,q}$. Identifying $(0, y)$ and $(p, q-y)$ in $\tilde{\Omega}_{p,q}$ for each y, we obtain a 4-regular quadrangulation $K_m(p, q)$ of the Klein bottle, called the *mesh type*.

In the proof of our main theorem, we focus on the cycle of $K_m(p, q)$ passing through $(0, q), (1, q), \ldots, (p, q)$ in the form $\tilde{R}_{p,q}$, which is orientation-reversing and straight at all vertices, except the vertex on $(0, q) = (p, q)$.

Theorem 4 (Nakamoto and Negami [7]) *Every 4-regular quadrangulation of the Klein bottle is isomorphic to either of $K_h(p, q), K_l(2p, q)$ or $K_m(p, q)$ for some integers p, q. In particular, this expression is unique.*

Note that $K_h(p, q)$ is simple if and only if $p, q \geq 3$, $K_l(p, q)$ is simple if and only if $p \geq 2$ and $q \geq 1$, and $K_m(p, q)$ is simple if and only if $p \geq 3$ and $q \geq 2$.

3 Proof of the Theorem

Throughout this section, let G be a 4-regular graph, and let $k(G)$ be a quadrangulation of the Klein bottle with a quadrangular embedding k.

A cycle C in a graph G is said to be *chordless* if the subgraph of G induced by $V(C)$ is also C. A cycle C in G is said to be *separating* if the graph obtained from G by removing $V(C)$ is not connected.

Lemma 5 *Let C be a chordless non-separating cycle of G with $|C| \neq 4$. For any quadrangular embedding f of G, $f(C)$ is essential in $f(G)$.*

Proof. Suppose that $f(C)$ is not essential, that is, bounding a 2-cell. Since $|C| \neq 4$ and C is chordless, some vertices lie in the interior and the exterior of $f(C)$. This contradicts that C is separating. ■

Let $C = v_1v_2 \cdots v_nu_1u_2 \cdots u_n$ be a cycle of even length $2n \geq 4$. Let M_n denote a Möbius ladder obtained from C by joining v_i and u_i for $i = 1, \ldots, n$. Each edge u_iv_i of M_n is called a *rung* of M_n.

Lemma 6 *Let L be a Möbius ladder M_n with $n \geq 2$ rungs, or a graph obtained from M_n ($n \geq 2$) with only one edge subdivided by exactly two vertices of degree 2. Suppose that a graph G has a quadrangular embedding t into the torus and G includes L as a subgraph. Then the embedding $t|_L : L \to T^2$ is a 2-cell embedding.*

Proof. Suppose that $t|_L : L \to T^2$ is not a 2-cell embedding. Then we can take an essential simple closed curve l on $T^2 \supset G$ which does not intersect L. Cutting open T^2 along l, we obtain an annulus, in which L is embedded. Pasting two disks to the boundary components of the annulus, we have the spherical embedding of L, in which at most two faces (capped off by disks) might have boundary walks of odd length, since other faces of the spherical embedding of L correspond to 2-cell regions of the quadrangulation $t(L)$. If $n \geq 3$, then L has a subdivision of $K_{3,3}$, which is non-planar, and hence this case is impossible. Therefore we have $n = 2$, and then L is isomorphic to a K_4 or a K_4 with only one edge subdivided by exactly two vertices. However, for any spherical embedding of L, each of its facial boundary walks has an odd length, since a unique spherical embedding of K_4 is a triangulation, and adding two vertices to one edge does not change the parity of the length of boundary walks. Therefore, we get a contradiction, and hence the lemma follows. ■

Theorem 7 *Let $k(G) = K_h(p, q)$ with $p \geq 3$ and $q \geq 3$. Then G cannot quadrangulate T^2.*

Proof. Suppose that G has a quadrangular embedding t into T^2. Let C be the cycle of $k(G) = K_h(p, q)$ passing through $(0, q), (1, q), \ldots, (p, q)$ in the form $R_{p,q}$. We first consider the case when $p \neq 4$. Since C is chordless, non-separating and $|C| \neq 4$, $t(C)$ is essential in $t(G)$, by Lemma 5. Cutting open T^2 along $t(C)$, we obtain the annulus A in which $G - C$ is embedded. However, it is impossible since $G - C$ includes a subdivision of a Möbius ladder with at least 3 rungs, which includes a subdivision of $K_{3,3}$. A contradiction.

Now consider the case when $p = 4$. We label the vertices of $K_h(4, q)$ as in the left-hand of Figure 4. If $t(C)$ is essential (where $C = abcd$), then the same argument in the case when $p \neq 4$ follows. Thus, we may assume that $t(C)$ is trivial. Since C is non-separating, $t(C)$ bounds a face. We consider a local structure around the face $abcd$ in $t(G)$. Since $t(G)$ is a 4-regular quadrangulation, the structure will be as in the right-hand of Figure 4. Observe that G has only two 4-cycles $abfe$ and

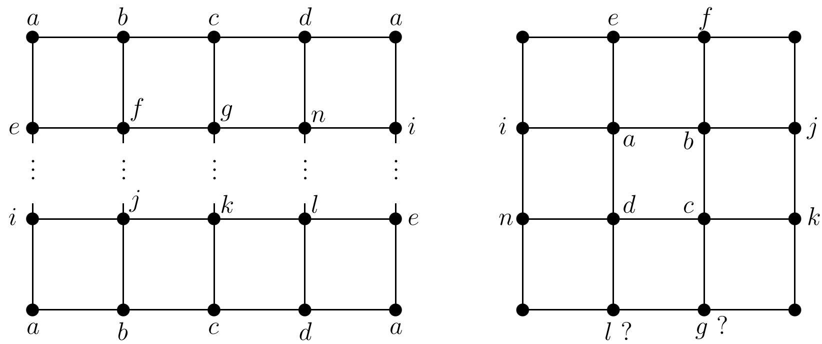

Figure 4: labeling of $K_h(p, 4)$ and a face $abcd$ in $t(G)$

$abji$ containing the edge ab. By symmetry, we may suppose that $abfe$ bounds a face in $t(G)$. Then $adni$ and $bckj$ bound faces adjacent to $abcd$ in $t(G)$. Therefore, the vertex $\neq a, c, n$ neighboring d is l, and the vertex $\neq b, d, k$ neighboring c is g. However, g and l are not adjacent in G, a contradiction. ■

Theorem 8 *Let $k(G) = K_l(p, 2q)$ with $p \geq 1$ and $q \geq 2$. Then G cannot quadrangulate T^2.*

Proof. Suppose that G has a quadrangular embedding $t(G)$ in T^2. Let L and L' be the two disjoint Möbius ladders with q rungs in $K_l(p, 2q)$ (attached to the boundary of $\Omega_{p,2q}$). By Lemma 6, the embeddings $t|_L : L \to T^2$ and $t|_{L'} : L' \to T^2$ are 2-cell embeddings on the torus, but any two graphs 2-cell embedded simultaneously in the same surface must have intersections. This contradicts that L and L' are disjoint. ■

Clearly, $K_m(p, 2)$ with any $p \geq 3$ has an essential diagonal 2-curve, and hence this can quadrangulate T^2. However, the following theorem holds for $K_m(p, q)$ with $q \geq 3$.

Theorem 9 *Let $k(G) = K_m(p, q)$ with $p \geq 3$ and $q \geq 3$. Then G cannot quadrangulate T^2.*

Proof. Suppose that G has a quadrangular embedding t on T^2. Let C be the cycle of $k(G) = K_m(p, q)$ passing through $(0, q), (1, q), \ldots, (p, q)$ in the form $\tilde{R}_{p,q}$. We consider the two cases, according to whether $p = 4$ or not.

CASE 1. $p \neq 4$.

We first suppose that q is even but $q \neq 2$. Then we focus on the cycle C' passing through $(0, q/2), (1, q/2), \ldots, (p, q/2)$ in the form $\tilde{R}_{p,q}$. Since both of C and C' are chordless and $|C| = |C'| \neq 4$, $t(C)$ and $t(C')$ are essential, disjoint and homotopic on T^2. Thus, $t(C) \cup t(C')$ separates T^2 into two annuli A_1 and A_2. Since $G - C \cup C'$ is connected, it must be embedded in one of A_1 and A_2, say A_1. Moreover, by the

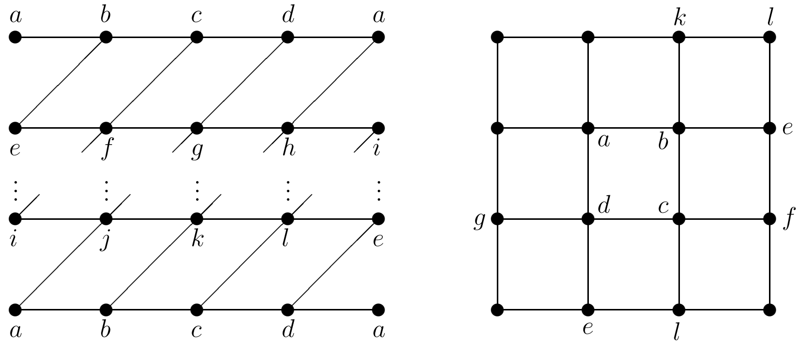

Figure 5: labeling of $K_m(p,4)$ and a face $abcd$ in $t(G)$

assumption $q \geq 4$ here, G has no edge connecting C and C'. Thus, there are no edges in A_2, a contradiction.

Now suppose that q is odd. Focus on the Möbius band surrounded by the cycle $(0,\lfloor q/2\rfloor),(1,\lfloor q/2\rfloor),\ldots,(p,\lfloor q/2\rfloor),\ (0,\lceil q/2\rceil),(1,\lceil q/2\rceil),\ldots,(p,\lceil q/2\rceil)$ in the form $\tilde{R}_{p,q}$. In this Möbius band, we can find a Möbius ladder M_{p-1} with only one edge subdivided by exactly two vertices. Since this graph, denoted by L, is 2-cell embedded in the torus by Lemma 6, L does not embed in the annulus $T^2 - C$, a contradiction.

Case 2. $p = 4$.

We consider $K_m(4,q)$ with $q \geq 3$. We label the vertices of $K_m(4,q)$ as in the left-hand of Figure 5. We may suppose that $C = abcd$ bounds a face in $t(G)$. We consider a local structure around the face $abcd$ in $t(G)$. (See the right-hand of Figure 5.) We first determine the positions of the neighbors e,k of b. Observe that e is not adjacent to a neighbor $\neq b,d$ of a, and hence e,k will be as in the right-hand figure. Moreover, since a common neighbor $\neq b$ of e and k is l, the faces $belk$ and $befc$ are determined. Then the vertex below c in the right-hand figure is determined to be l, and we finally conclude that the face sharing an edge dc with the face $abcd$ must be $dcle$, since the unique neighbor ($\neq a,c$) of d adjacent to l must be e. However, the union of the four faces $cdel$, $abcd$, $bcfe$ and $belk$ is homeomorphic to a Möbius band. This contradicts that $t(G)$ is the torus. Therefore, the theorem follows. ■

By Theorems 7, 8 and 9, we have the following theorem:

Theorem 10 *Let G be a 4-regular graph with a quadrangular embedding k. Then G can quadrangulate T^2 if and only if $k(G) = K_m(p,2)$ for $p \geq 3$.* ■

Lemma 11 *Let G be a 4-regular graph with a quadrangular embedding k such that $k(G) = K_m(p,2)$ for $p \geq 3$. Then, any quadrangular embedding of G into T^2 admits an essential diagonal 2-curve.*

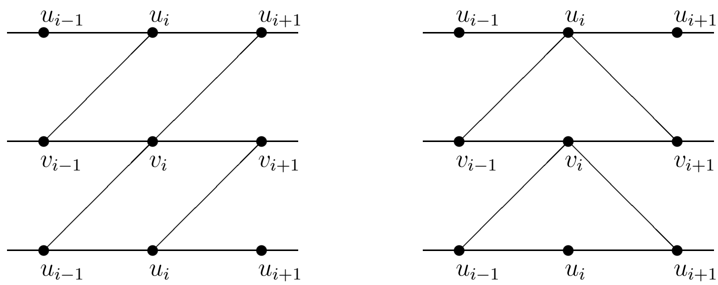

Figure 6: Two ways to arrange edges with respect to u_i and v_i

Proof. If $p = 4$, then $G = K_{4,4}$. It is known in [5] that $K_{4,4}$ has two different embeddings on the torus, each of which admits an essential diagonal 2-curve.

Suppose that $p \neq 4$. Let C and C' be the cycles of $k(G) = K_m(p, 2)$ passing through $(0, 2), (1, 2), \ldots, (p, 2)$, and $(0, 1), (1, 1), \ldots, (p, 1)$, respectively. Then the two cycles C and C' of $K_m(p, 2)$ with $|C| = |C'| = p$ must be mapped into T^2 as two disjoint, essential, homotopic cycles on T^2, by Lemma 5. Since $V(G) = V(C) \cup V(C')$, we have only to arrange $2p$ edges between C and C'. Let $C = v_0 v_1 \cdots v_{p-1}$ and $C = u_0 u_1 \cdots u_{p-1}$. For each i, v_i is adjacent to u_{i-1} and u_{i+1}, and u_i is adjacent to v_{i-1} and v_{i+1}, where the subscripts are taken modulo k. There are essentially two ways to arrange edges with respect to v_i and u_i, as shown in Figure 6. In each case, we can take a diagonal 2-curve through v_i and u_i. ■

Now we show our main theorem.

Proof of Theorem 2. The sufficiency is obvious. Thus, we show the necessity. Suppose that a 4-regular graph G which has quadrangular embedding t into T^2 has also a quadrangular embedding k into K^2. By Theorem 10, we have $k(G) = K_m(p, 2)$ for some $p \geq 3$. Here, by Lemma 11, a quadrangulation $t(G)$ has an essential diagonal 2-curve. ■

References

[1] A. Altshular, Construction and enumeration of regular maps on the torus, *Discrete Math.* **4** (1973), 201–217.

[2] S. Lawrencenko, Irreducible triangulations of a torus, *J. Soviet Math.* **51** (1990), 2537–2543.

[3] S. Lawrencenko and S. Negami, The irreducible triangulations of the Klein bottle, *J. Combin. Theory, Ser. B*, **70** (1997), 265–291.

[4] S. Lawrencenko and S. Negami, Constructing the graphs that triangulate both the torus and the Klein bottle, *J. Combin. Theory, Ser. B* **77** (1999), 211–218.

[5] A. Nakamoto, Irreducible quadrangulations of the torus, *J. Combin. Theory, Ser. B* **67** (1996), 183–201.

[6] A. Nakamoto, Irreducible quadrangulations of the Klein bottle, *Yokohama Math. J.* **43** (1995), 136–149.

[7] A. Nakamoto and S. Negami, Full-symmetric embeddings of graphs on closed surfaces, *Mem. Osaka Kyoiku Univ.* **49** (2000), 1–15.

(Received 13 Dec 2002)