On homomorphic images of edge transitive directed graphs

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ABSTRACT: Let Γ be an infinite, connected, vertex transitive and edge transitive, directed graph with finite but unequal in-valency and out-valency. Then there is an epimorphism ϕ from the vertex set of Γ to the set of integers $\mathbb{Z}$ such that (α, β) is an edge of Γ if and only if $\phi(\beta) = \phi(\alpha) + 1$. Thus the natural directed graph on $\mathbb{Z}$ is a homomorphic image of Γ. Moreover, for each $i \in \mathbb{Z}$ the inverse image $\phi^{-1}(i)$ is infinite.

1. Introduction

Let Γ be an infinite connected, vertex transitive and edge transitive, directed graph with finite but unequal in-valency and out-valency. We shall show that there is a graph epimorphism from Γ onto the *integer directed graph,* that is the directed graph Z with vertex set $\mathbb{Z}$ such that (i, j) is an edge if and only if $j = i + 1$. This result partially explains a phenomenon observed in [1] for highly arc transitive directed graphs.

For $s \geq 0$, an *s-arc* in a directed graph Γ is a sequence $\alpha = (\alpha_0, \ldots, \alpha_s)$ of $s + 1$ vertices of Γ such that $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s$, and (α_{i-1}, α_i) is an edge for $1 \leq i \leq s$; and Γ is said to be *s-arc transitive* if its automorphism group acts transitively on the set of s-arcs of Γ. Thus edge transitive directed graphs, the subject of this note, are 1-arc transitive. A directed graph Γ is called *highly arc transitive* if it is s-arc transitive for all $s \geq 0$. It was observed in [1] that a large class of highly arc transitive directed graphs had the integer directed graph Z as a homomorphic image. The theorem below shows that this is a property of all such directed graphs if the in-valency and out-valency are finite and unequal.

Theorem. Let Γ be an infinite, connected, vertex transitive and edge transitive, directed graph with finite, but unequal, in-valency and out-valency. Then there is a graph epimorphism ϕ from Γ to the integer directed graph Z and, for each $i \in \mathbb{Z}$, the inverse image $\phi^{-1}(i)$ is infinite.

In [1, Remark 3.4 (b)] it was asked whether the inverse images $\phi^{-1}(i)$ were finite for a certain class of highly arc transitive directed graphs with finite in- or out-valency. The theorem shows that the answer to this question is 'no' when the in- and out-valencies are finite but unequal.

2. Transitive permutation groups with finite subdegrees.

Let G be a transitive permutation group on a set Ω. Then G has a natural action on $\Omega \times \Omega$ defined by

$$(\alpha, \beta)^g := (\alpha^g, \beta^g)$$

for $(\alpha, \beta) \in \Omega \times \Omega$ and $g \in G$. The orbits of G in $\Omega \times \Omega$ are called *orbitals* for G, and, for each $\alpha \in \Omega$ and each orbital Δ, the set

$$\Delta(\alpha) := \{\beta \mid (\alpha, \beta) \in \Delta\}$$

is an orbit for the stabilizer G_α of α. Moreover each orbit of G_α in Ω is equal to $\Delta(\alpha)$ for some orbital Δ. The cardinality of $\Delta(\alpha)$ is independent of α and is called a *subdegree* of G. To each orbital Δ corresponds a *paired orbital* Δ^*, namely

$$\Delta^* := \{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$$

which may or may not be equal to Δ. If all subdegrees of G are finite then the following function

$$\psi : G \to \mathbb{Q} \setminus \{0\}$$

is well-defined. Let $\alpha \in \Omega$. For $g \in G$ let $\Delta := (\alpha, \alpha^g)^G$, the orbital of G containing the pair (α, α^g). Then $\psi(g)$ is defined as

$$\psi(g) := \frac{\mid \Delta(\alpha) \mid}{\mid \Delta^*(\alpha) \mid}.$$

This function was first brought to my attention by G. Bergman, and more recently by Peter Neumann. Peter showed that ψ is a homomorphism:

Lemma 1. Let G be a transitive permutation group on Ω such that all subdegrees of G are finite. Then the map ψ defined above is a homomorphism from G into the multiplicative group of rational numbers.

Proof. Let g, $h \in G$ and let $\Delta := (\alpha, \alpha^g)^G$, $\Gamma := (\alpha, \alpha^h)^G = (\alpha^{h^{-1}}, \alpha)^G$, and $\Sigma := (\alpha, \alpha^{gh})^G = (\alpha^{h^{-1}}, \alpha^g)^G$. Set $\beta := \alpha^{h^{-1}}$ and $\gamma := \alpha^g$. Then

$$\begin{aligned}
\psi(gh) &= \frac{\mid \Sigma(\alpha) \mid}{\mid \Sigma^*(\alpha) \mid} = \frac{\mid \Sigma(\beta) \mid}{\mid \Sigma^*(\gamma) \mid} = \frac{\mid G_\beta : G_{\beta\gamma} \mid}{\mid G_\gamma : G_{\beta\gamma} \mid} \\
&= \frac{\mid G_\beta : G_{\alpha\beta\gamma} \mid}{\mid G_\gamma : G_{\alpha\beta\gamma} \mid} \\
&= \frac{\mid \Gamma(\beta) \mid \cdot \mid G_{\alpha\beta} : G_{\alpha\beta\gamma} \mid}{\mid \Delta^*(\gamma) \mid \cdot \mid G_{\alpha\gamma} : G_{\alpha\beta\gamma} \mid} \\
&= \psi(g)\psi(h) \frac{\mid \Gamma^*(\alpha) \mid \cdot \mid G_{\alpha\beta} : G_{\alpha\beta\gamma} \mid}{\mid \Delta(\alpha) \mid \cdot \mid G_{\alpha\gamma} : G_{\alpha\beta\gamma} \mid} \\
&= \psi(g)\psi(h).
\end{aligned}$$

Further, the function ψ is independent of α.

Lemma 2. Let G be as in Lemma 1, let $\beta \in \Omega$, and let ψ_β be the function defined by

$$\psi_\beta(g) := \frac{|\Gamma(\beta)|}{|\Gamma^*(\beta)|}$$

where $\Gamma := (\beta, \beta^g)^G$, for $g \in G$. Then $\psi_\beta = \psi$.

Proof. Since G is transitive on Ω, $\beta = \alpha^x$ for some $x \in G$. Let $g \in G$. Then $\Delta := (\beta, \beta^g)^G = (\alpha^x, \alpha^{xg})^G = (\alpha, \alpha^{xgx^{-1}})^G$. Hence

$$\begin{aligned}\psi_\beta(g) &= \psi(xgx^{-1}) \\ &= \psi(x)\psi(g)\psi(x)^{-1} \\ &= \psi(g)\end{aligned}$$

since ψ is a homomorphism into the abelian group $Q\backslash\{0\}$.

These are the basic tools we shall use to prove our theorem.

3. Proof of the Theorem

Let Γ be a connected, vertex transitive and edge transitive directed graph, and let G be a group of automorphisms acting transitively on the edges of Γ. Then if (α, β) is an edge of Γ, the G-orbital $(\alpha, \beta)^G$, which we shall denote by $\overline{\Gamma}$, is the set of all edges of Γ. Thus the subdegrees $u = |\overline{\Gamma}(\alpha)|$ and $v = |\overline{\Gamma}^*(\alpha)|$ of G are the out-valency and in-valency of Γ respectively. Moreover, since Γ is connected it is not difficult to show that, if u and v are finite, then all subdegrees of G are finite. We shall show, for such a group G, that the image of the function ψ defined in section 2 is cyclic.

Proposition 3 Let G be a group of automorphisms of a connected directed graph Γ which acts transitively on the vertices and edges of Γ. Suppose that Γ has finite out-valency u and finite in-valency v. Then the function ψ defined in section 2 has image

$$\{(\frac{u}{v})^i \mid i \in \mathbb{Z}\},$$

the cyclic subgroup generated by u/v.

Proof. Let α be a vertex of Γ and let $g \in G$, and $\Delta = (\alpha, \alpha^g)^G$. The proof that $\psi(g) = |\Delta(\alpha)| / |\Delta^*(\alpha)|$ is a power of u/v is by induction on the length of the shortest undirected path in Γ from α to α^g. By an undirected path of length n from α to α^g we mean a sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \alpha^g$ of $n+1$ vertices such that for each $1 \leq i \leq n$, either (α_{i-1}, α_i) or (α_i, α_{i-1}) is an edge of Γ. If the shortest such path has length 0 or 1, then by our remarks above $\psi(g)$ is 1, u/v, or $(u/v)^{-1}$. Suppose then that the shortest such path has length $n \geq 2$ and that $\psi(h)$ is a power of u/v whenever there is an undirected path from α to α^h of length less than n. The penultimate vertex, α_{n-1}, in a path $\alpha = \alpha_0, \ldots, \alpha_{n-1}, \alpha_n = \alpha^g$, is of the form

$\alpha_{n-1} = \alpha^h$ for some $h \in G$, and inductively $\psi(h) = (u/v)^j$ for some $j \in \mathbb{Z}$. If (α^h, α^g) is an edge then also $(\alpha, \alpha^{gh^{-1}})$ is an edge and we have $\psi(gh^{-1}) = u/v$, whence $\psi(g) = \psi(gh^{-1})\psi(h) = (u/v)^{j+1}$. Similarly, if (α^g, α^h) is an edge then $(\alpha^{gh^{-1}}, \alpha)$ is an edge and so $\psi(g) = \psi(gh^{-1})\psi(h) = (u/v)^{j-1}$. Thus the result is proved by induction.

Now we are in a position to prove our theorem. Let Γ and G be as in Proposition 3 and suppose that $u \neq v$ so that ψ has an infinite cyclic image (and hence of course Γ is infinite). Let β be a vertex of Γ, say $\beta = \alpha^g$ for some $g \in G$. Then, if $\beta = \alpha^h$ for some other element $h \in G$, the images $\psi(g)$ and $\psi(h)$ are equal (for $\alpha = \alpha^{gh^{-1}}$ and so, by the definition of ψ, $\psi(gh^{-1}) = 1$, whence $\psi(g) = \psi(h)$). Now define a map ϕ from the vertex set of Γ to $\mathbb{Z}$ by

$$\phi(\beta) = i$$

where, if $\beta = \alpha^g$, then $\psi(g) = (u/v)^i$. By the remarks above this map is well defined. Suppose that (β, γ) is an edge of Γ, and that $\beta = \alpha^g$, $\gamma = \alpha^h$, and $\psi(g) = (u/v)^i$. Then $(\alpha, \alpha^{hg^{-1}})$ is also an edge and consequently $\psi(hg^{-1}) = u/v$, whence $\psi(h) = \psi(hg^{-1})\psi(g) = (u/v)^{i+1}$ so that $\phi(\beta) = i$, $\phi(\gamma) = i + 1$. Thus ϕ is a graph epimorphism from Γ onto the integer directed graph Z.

We shall show that the orbits of the kernel K of ψ are the inverse images $\phi^{-1}(i)$ for $i \in \mathbb{Z}$. It follows from the definition of ϕ that $\alpha^K = \phi^{-1}(0)$. Suppose inductively that, for some non-negative integer i, $\phi^{-1}(i)$ and $\phi^{-1}(-i)$ are K-orbits. We shall show that $\phi^{-1}(i+1)$ is a K-orbit. Let $\beta = \alpha^x \in \phi^{-1}(i)$ and let (β, γ) be an edge of Γ, where $\gamma = \alpha^g$. Then, as above, $\phi(\gamma) = i + 1$. If $\gamma' = \gamma^k$ for some $k \in K$ then $\gamma' = \alpha^{gk}$ and $\psi(gk) = \psi(g)\psi(k) = \psi(g) = (u/v)^{i+1}$ whence $\phi(\gamma') = i + 1$. Thus $\gamma^K \subseteq \phi^{-1}(i+1)$. On the other hand if $\gamma' \in \phi^{-1}(i+1)$, say $\gamma' = \alpha^h$ then $\psi(g^{-1}h) = \psi(g)^{-1}\psi(h) = 1$ so $g^{-1}h \in K$ and $\gamma' = \alpha^h = \gamma^{g^{-1}h} \in \gamma^K$. Therefore $\phi^{-1}(i+1) = \gamma^K$. A similar proof shows that $\phi^{-1}(-i-1)$ is also a K-orbit, and hence by induction the K-orbits are the sets $\phi^{-1}(i)$, $i \in \mathbb{Z}$.

Now suppose that $\phi^{-1}(i)$ is finite for some $i \in \mathbb{Z}$. Since the $\phi^{-1}(i)$ are orbits of the normal subgroup K of G they all have the same cardinality, N say. Then, counting the number of edges from $\phi^{-1}(0)$ to $\phi^{-1}(1)$ we have $Nu = Nv$ whence $u = v$ which is a contradiction. Hence the sets $\phi^{-1}(i)$ are infinite. This completes the proof of the theorem.

Reference

1. P.J. Cameron, C.E. Praeger and N.C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs, submitted.