2-COLOURING $K_4 - e$ DESIGNS

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Abstract

In this paper, necessary and sufficient conditions are found for the existence of a 2-colourable $K_4 - e$ design of λK_n.

1. Introduction.

Let G be a *simple* graph; i.e., a subgraph of K_n (the complete undirected graph on n vertices). A *λ-fold G-design* (of order n) is a pair (P, B), where B is an edge-disjoint decomposition of λK_n (λ copies of K_n) with vertex set P into copies of the graph G. The *number* n is called the *order* of the G-design (P, B) and, of course, $|B| = \lambda\binom{n}{2}/|E(G)|$ where $|E(G)|$ is the number of edges belonging to G. When $\lambda = 1$

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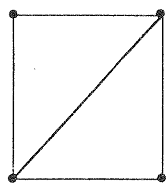
we will abbreviate "1-fold G-design" to simply "G-design". So, for example, a Steiner triple system is a K_3-design and a block design with block size 4 is a K_4-design.

Now let (P, B) be a λ-fold G-design. The subset X of P is called a 2-*colouring* of (P, B) if and only if for each $g \in B, V(g) \cap X \neq \emptyset$ *and* $V(g) \cap (P \backslash X) \neq \emptyset$, where $V(g)$ is the vertex set of the graph g. (The subset X is also called a blocking set. However, in what follows we will stick with calling X a 2-colouring rather than a blocking set.)

It is quite easy to see that the only λ-fold K_3-designs admitting a 2-colouring have orders 3 or 4 (regardless of λ). See [6] for example. Things are considerably different for λ-fold K_4-designs. In a series of two papers [4, 5] D. G. Hoffman, C. C. Lindner, and K. T. Phelps gave a complete solution (modulo a handful of *possible* exceptions) of the problem of constructing λ-fold K_4-designs which can be 2-coloured. In particular, the combined work in [4, 5] guarantees the existence of a λ-fold K_4-design of order n which can be 2-coloured for every admissible (n, λ) except possibly for $(n \in \{37, 40, 73\}, \lambda = 1), (n = 37, \lambda \equiv 1$ or $5 \pmod 6 \geq 5)$, *and* $(n \in \{19, 34, 37, 46, 58\}, \lambda \equiv 2$ or $4 \pmod 6)$. In a forthcoming paper, necessary and sufficient conditions are found for the existence of a 2-colourable G-design of K_n for all connected, simple graphs G with at most 5 edges, $G \neq K_4 - e$ [2].

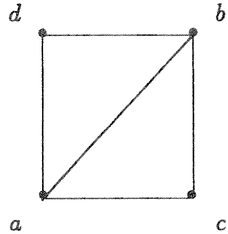
The purpose of this paper is to give a *complete* solution of the existence problem of λ-fold $K_4 - e$ designs which admit a 2-colouring, where

$K_4 - e =$ 

Clearly the spectrum for λ-fold $K_4 - e$ designs is contained in the set of all (i) $n \equiv 0$ or $1 \pmod 5) \geq 6$ for $\lambda = 1$, (ii) $n \equiv 0$ or $1 \pmod 5$ for $\lambda \equiv 1, 2, 3,$ or $4 \pmod 5) \geq 2$, and (iii) $n \geq 4$ for $\lambda \equiv 0 \pmod 5$. We show that these necessary conditions are not only sufficient for the existence of a λ-fold $K_4 - e$ design *but* for the existence of a λ-fold $K_4 - e$ design which can be 2-coloured as well. Here goes!

2. $K_4 - e$ designs.

In what follows we will denote

$K_4 - e \; =$ 

by any one of $(a, b, c, d), (a, b, d, c), (b, a, c, d)$, or (b, a, d, c).

To begin with it is trivial to see that there does not exist a $K_4 - e$ design of order 5. Now for some *necessary* examples.

Example 2.1. The following are examples of $K_4 - e$ designs ($\lambda = 1$), which can be 2-coloured.

$\underline{n = 6}$.

1	4	2	5
2	5	3	6
3	6	1	4

2-colouring {1, 2, 3}

$\underline{n = 10}$.

1	2	3	4	7	9	3	4
3	4	5	6	8	10	3	4
5	6	1	2	7	10	5	6
7	8	1	2	8	9	5	6
9	10	1	2				

2-colouring {1, 2, 3, 5}

$\underline{n = 11}$.

1	10	2	5	5	7	8	11
2	11	3	6	6	8	1	9
1	3	4	7	7	9	2	10
2	4	5	8	8	10	3	11
3	5	6	9	9	11	1	4
6	4	7	10				

2-colouring {1, 2, 3, 4, 5, 6}

$n = 13$.

2	3	1	4	5	8	13	14	2	15	12	13
1	4	5	6	5	9	10	15	3	13	7	11
5	6	2	3	6	11	8	14	3	12	8	9
1	14	13	15	6	7	9	15	3	10	14	15
1	11	10	12	6	12	10	13	4	15	8	11
1	8	7	9	2	10	7	8	4	14	7	12
5	7	11	12	2	9	11	14	4	13	9	10

2-colouring {2, 3, 4, 7, 10, 14}

$n = 15$ (with *hole* = {11, 12, 13, 14, 15} = decomposition of $K_{15} \backslash K_5$ into copies of $K_4 - e$, with K_5 based on {11, 12, 13, 14, 15}).

1	11	2	10	1	4	12	13	1	7	5	8
3	9	1	11	6	9	12	13	2	6	3	10
6	11	5	7	5	10	14	15	3	4	5	10
4	8	6	11	3	7	14	15	8	9	5	10
2	5	12	13	2	8	14	15	2	7	4	9
3	8	12	13	1	6	14	15				
7	10	12	13	4	9	14	15				

2-colouring {2, 3, 4, 5, 6, 7}

$n = 20$.

2	12	1	11	2	13	4	14	15	9	20	10	20	18	7	17
3	13	6	16	2	16	8	18	16	7	14	4	2	3	5	15
4	14	10	20	4	18	9	19	17	9	12	2	2	6	10	20
5	15	4	14	5	19	10	20	18	10	13	3	4	8	1	11
6	16	1	11	6	17	4	14	13	12	5	15	5	9	1	11
7	17	1	11	7	19	2	12	16	12	10	20	6	7	5	15
8	18	5	15	8	20	3	13	18	14	1	11	7	9	3	13
9	19	6	16	12	3	14	4	19	15	1	11	8	10	7	17
10	20	1	11	12	6	18	8	17	16	5	15				
1	11	3	13	14	8	19	9	19	17	3	13				

2-colouring {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

$\underline{n = 21}$.

$i, 8+i, 5+i, 7+i$
$10+i, 12+i, i, 6+i$
$i \in Z_{21}$ (mod 21)

2-colouring {1, 2, 4, 7, 10, 11, 14, 15, 18, 19}.

$\underline{n = 25}$. Let (∞, B) be the $K_4 - e$ design of order 11 (in this example) where

$$\infty = \{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8, \infty_9, \infty_{10}\}$$

with 2-colouring $\{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$.

B	
$(0,i),(0,1+i),(1,4+i),(1,6+i)$	
$(0,i),(1,2+i),(1,1+i),\infty_0$	
$i \in Z_7$ (mod 7)	
$(j,0),(j,2),\infty_1,\infty_6$	$(j,3),(j,6),\infty_3,\infty_8$
$(j,4),(j,6),\infty_1,\infty_6$	$(j,2),(j,5),\infty_3,\infty_8$
$(j,1),(j,3),\infty_1,\infty_6$	$(j,0),(j,3),\infty_4,\infty_9$
$(j,2),(j,4),\infty_2,\infty_7$	$(j,2),(j,6),\infty_4,\infty_9$
$(j,1),(j,6),\infty_2,\infty_7$	$(j,1),(j,5),\infty_4,\infty_9$
$(j,3),(j,5),\infty_2,\infty_7$	$(j,0),(j,5),\infty_5,\infty_{10}$
$(j,0),(j,4),\infty_3,\infty_8$	$(j,1),(j,4),\infty_5,\infty_{10}$
$j \in Z_2$ (mod 2)	
$(0, 5), (1, 5), \infty_1, \infty_6$	$(0, 2), (1, 2), \infty_5, \infty_{10}$
$(0, 0), (1, 0), \infty_2, \infty_7$	$(0, 3), (1, 3), \infty_5, \infty_{10}$
$(0, 1), (1, 1), \infty_3, \infty_8$	$(0, 6), (1, 6), \infty_5, \infty_{10}$
$(0, 4), (1, 4), \infty_4, \infty_9$	

2-colouring $\{\infty_0, \infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup \{(0,i) | i \in Z_7\}$

With the above examples in hand we proceed to the main constructions for $K_4 - e$ designs.

The $10k$ *Construction.* Let $(X, \circ)$ be a quasigroup and $H = \{h_1, h_2, \ldots, h_m\}$ a partition of X. The subsets $h_i \in H$ are called *holes*. If for each hole $h_i \in H, (h_i, \circ)$ is a subquasigroup of $(X, \circ)$, then $(X, \circ)$ is called a *quasigroup with holes* H. Let $(X, \circ)$ be a commutative quasigroup of order $2k$ with holes H all of size 2. Set $P = X \times \{1, 2, 3, 4, 5\}$ and define a collection of graphs B as follows: (1) For each hole $h \in H$, let $(h \times \{1, 2, 3, 4, 5\}, h^*)$ be the $K_4 - e$ design order 10 in Example 2.1 with 2-colouring $h \times \{1, 2\}$ and place the graphs of h^* in B, and

(2) if x and y belong to different holes of H, place the 5 graphs

$$((x,1),(y,1),(x \circ y,2),(x \circ y,4)),$$

$$((x,2),(y,2),(x \circ y,3),(x \circ y,5)),$$

$$((x,3),(y,3),(x \circ y,4),(x \circ y,1)),$$

$$((x,4),(y,4),(x \circ y,5),(x \circ y,2)), \text{ and}$$

$$((x,5),(y,5),(x \circ y,1),(x \circ y,3)) \text{ in } B.$$

Then (P, B) is a $K_4 - e$ design of order $10k$ and $X \times \{1, 2\}$ is a 2-colouring. □

The $10k + 1$ *Construction.* In the $10k$ Construction set $P = \{\infty\} \cup (X \times \{1, 2, 3, 4, 5\})$ and replace (1) by: For each hole $h_i \in H$, let

$$(\{\infty\} \cup (h_i \times \{1, 2, 3, 4, 5\}), h_i^*)$$

be the $K_4 - e$ design of order 11 in Example 2.1 with 2-colouring $h_i \times \{1, 2, 3\}$, and place the graphs of h_i^* in B.

Then (P, B) is a $K_4 - e$ design of order $10k+1$ and $X \times \{1, 2, 3\}$ is a 2-colouring. □

The $10k + 5$ *Construction.* In the $10k$ Construction set

$$P = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (X \times \{1, 2, 3, 4, 5\})$$

and replace (1) by: (i) for the hole h_1, let

$(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h_1 \times \{1,2,3,4,5\}), h_1^*)$

be the $K_4 - e$ design of order 15 in Example 2.1 with the 2-colouring $h_1 \times \{1,2,3\}$ and place the graphs of h_1^* in B, and (ii) for each of the holes $h_2, h_3, \ldots, h_k$, let $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (h_i \times \{1,2,3,4,5\}), h_i^*)$ be the $K_4 - e$ design of order 15 with *hole* $= \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ in Example 2.1 with 2-colouring $h_i \times \{1,2,3\}$ and place the graphs of h_i^* in B.

Then (P, B) is a $K_4 - e$ design of order $10k+5$ and $X \times \{1,2,3\}$ is a 2-colouring. □

The $10k+6$ *Construction.* Let $(X, \circ)$ be an *idempotent* $(x^2 = x)$ and commutative quasigroup of order $2k+1$, set $P = \{\infty\} \cup (X \times \{1,2,3,4,5\})$, and define a collection of graphs B as follows:

(1) For each $a \in X$, let $(\{\infty\} \cup (\{a\} \times \{1,2,3,4,5\}), a^*)$ be the $K_4 - e$ design of order 6 in Example 2.1 with 2-colouring $\{a\} \times \{1,2,3\}$, and place the 6 graphs of a^* in B, and

(2) the same as the $10k$ Construction.

Then (P, B) is a $K_4 - e$ design of order $10k+6$ and $X \times \{1,2,3\}$ is a 2-colouring. □

We can now combine the examples in Example 2.1 and the above four constructions to determine the spectrum of $K_4 - e$ designs which can be 2-coloured.

Theorem 2.2. *The spectrum of* $K_4 - e$ *designs which can be 2-coloured is precisely the set of all* $n \equiv 0$ *or 1 (mod 5)* ≥ 6.

Proof: It is well-known (see [3, 7], for example) that the spectrum for commutative quasigroups of order $2k$ with holes all of size 2 is precisely the set of all $2k \geq 6$. Hence if $n = 10k, 10k+1$, or $10k+5 \geq 30$, the $10k$, $10k+1$, and $10k+5$ Constructions produce a $K_4 - e$ design which can be 2-coloured. If $n = 10k+6 \geq 16$, the $10k+6$ Construction produces a $K_4 - e$ design which can be 2-coloured. The cases $n = 6, 10, 11, 15, 20, 21$ and 25 are taken care of in Example 2.1. □

3. $\lambda \equiv 1, 2, 3$ or 4 (mod 5).

As noted in Section 1, it is obvious that the spectrum for λ-fold $K_4 - e$ designs for $\lambda \equiv 1, 2, 3$ or 4 (mod 5) ≥ 2 is contained in the set of all $n \equiv 0$ or 1 (mod 5). Hence to settle the existence problem for λ-fold $K_4 - e$ designs when $\lambda \equiv 1, 2, 3$, or 4 (mod 5) ≥ 2 we need to take care of the case $n = 5$ *only*, since we can just take λ copies of a $K_4 - e$ design of order $n \geq 6$ (admitting a 2-colouring) in every other case. The following two examples dispose of λ-fold $K_4 - e$ designs of order 5 (which can be 2-coloured) for all $\lambda \equiv 1, 2, 3$, or 4 (mod 5) ≥ 2.

Example 3.1.

$n = 5$ and $\lambda = 2$.

1	2	3	4
3	5	2	4
1	2	4	5
3	5	1	4

2-colouring {1, 2}

$n = 5$ and $\lambda = 3$.

5	1	2	3
5	2	3	4
5	3	4	1
5	4	1	2
1	3	2	4
2	4	1	3

2-colouring {1, 2}

Theorem 3.2. *The spectrum of λ-fold $K_4 - e$ designs with $\lambda \equiv 1, 2, 3$, or 4 (mod 5) ≥ 2 which can be 2-coloured is precisely the set of all $n \equiv 0$ or 1 (mod 5).* □

4. $\lambda \equiv 0$ (mod 5).

The spectrum for λ-fold $K_4 - e$ designs for $\lambda \equiv 0$ (mod 5) is *precisely* the set of *all* $n \geq 4$. The following Folk Construction packs the spectrum.

Folk Construction. Let $(P, \circ)$ be an idempotent *anti-symmetric* $(a \circ b \neq b \circ a, a \neq b \in P)$ quasigroup of order $n \geq 4$. Let $B = \{(a, b, a \circ b, b \circ a) | \text{ all } a \neq b \in P\}$. Then (P, B) is a 5-fold $K_4 - e$ design. Taking k copies of (P, B) produces a $5k$-fold $K_4 - e$ design of order n.

While the above construction produces λ-fold $K_4 - e$ designs with $\lambda \equiv 0 \pmod 5$) galore, it is not apparent (at least not to the authors) how to 2-colour such designs. So, in order to pack the spectrum with λ-fold $K_4 - e$ designs, $\lambda \equiv 0 \pmod 5$, which can be 2-coloured we take the following tack. We 2-colour a handful of idempotent anti-symmetric quasigroups of small orders, and use these 5-fold $K_4 - e$ designs in five different recursive constructions: $n \equiv 0, 1, 2, 3$, and $4 \pmod 5$, with $\lambda = 5$.

The cases $n \equiv 0$ or $1 \pmod 5$ are taken care of by Theorem 2.2 (just take 5 copies of a $K_4 - e$ design), with the exception of $n = 5$. It is less than trivial to 2-colour an idempotent anti-symmetric quasigroup of order 5. So much for $n \equiv 0$ or $1 \pmod 5$. We now move on to the cases $n = 2, 3$, and $4 \pmod 6$, $\lambda = 5$.

Example 4.1. The following four examples are necessary for the $n \equiv 2 \pmod 5$ constructions.

n=7.

$\circ_1$	1	2	3	4	5	6	7
1	1	6	4	2	7	5	3
2	4	2	7	5	3	1	6
3	7	5	3	1	6	4	2
4	3	1	6	4	2	7	5
5	6	4	2	7	5	3	1
6	2	7	5	3	1	6	4
7	5	3	1	6	4	2	7

2-colouring {1, 2, 3, 5}

n=7 (with hole = {1, 2}).

$\circ_2$	1	2	3	4	5	6	7
1	1	2	4	5	6	7	3
2	2	1	5	6	7	3	4
3	5	7	3	1	2	4	6
4	6	3	7	4	1	2	5
5	7	4	6	3	5	1	2
6	3	5	2	7	4	6	1
7	4	6	1	2	3	5	7

2-colouring {1, 2, 3, 5}

n=12.

o_3	1	2	3	4	5	6	7	8	9	10	11	12
1	1	3	2	7	9	8	10	12	11	4	5	5
2	12	2	1	3	8	7	6	11	10	9	5	4
3	11	10	3	2	1	9	5	4	12	8	7	6
4	10	12	11	4	6	5	1	3	2	7	9	8
5	6	11	10	9	5	4	12	2	1	3	8	7
6	5	4	12	8	7	6	11	10	3	2	1	9
7	4	6	5	10	12	11	7	9	8	1	3	2
8	9	5	4	6	11	10	3	8	7	12	2	1
9	8	7	6	5	4	12	2	1	9	11	10	3
10	7	9	8	1	3	2	4	6	5	10	12	11
11	3	8	7	12	2	1	9	5	4	6	11	10
12	2	1	9	11	10	3	8	7	6	5	4	12

2-colouring {1, 2, 3, 4, 5, 6}

n=12 (with hole = {1, 2}).

o_4	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	10	7	4	12	9	6	3	11	8	5
2	2	1	6	10	3	7	11	4	8	12	5	9
3	7	10	3	2	11	4	8	12	5	9	1	6
4	12	7	11	4	8	2	5	9	1	6	10	3
5	6	4	8	12	5	9	1	2	10	3	7	11
6	11	12	5	9	1	6	10	3	7	2	4	8
7	5	9	1	6	10	3	7	11	4	8	12	2
8	10	6	2	3	7	11	4	8	12	5	9	1
9	4	3	7	11	2	8	12	5	9	1	6	10
10	9	11	4	8	12	5	2	1	6	10	3	7
11	3	8	12	5	9	1	6	10	2	7	11	4
12	8	5	9	1	6	10	3	7	11	4	2	12

2-colouring {2, 4, 8, 9, 10, 12}

The $10k+7$ *Construction.* Let $(X,\circ)$ be an idempotent commutative quasigroup of order $2k+1$, set $P = \{\infty_1, \infty_2\} \cup (X \times \{1,2,3,4,5\})$, and define a collection of graphs B as follows :

(1) Let $a \in X$, and let $(\{\infty_1,\infty_2\} \cup (\{a\} \times \{1,2,3,4,5\}), a^*)$ be the 5-fold $K_4 - e$ design of order 7 defined by $\circ_1$ in Example 4.1 with 2-colouring $\{\infty_1,\infty_2\} \cup (\{a\} \times \{1,2\})$, and place the 21 graphs of a^* in B;

(2) for each $b \in X \backslash \{a\}$, let $(\{\infty_1,\infty_2\} \cup (\{b\} \times \{1,2,3,4,5\}, b^*)$ be the 5-fold $K_4 - e$ design of order 7 with *hole* $\{\infty_1,\infty_2\}$ defined by $\circ_2$ in Example 4.1 with 2-colouring $\{\infty_1,\infty_2\} \cup (\{b\} \times \{1,2\})$ and place the 20 graphs in b^* in B; and

(3) if $x \neq y \in X$, place 5 copies of each of the graphs

$$((x,1),(y,1),(x\circ y,2),(x\circ y,4)),$$
$$((x,2),(y,2),(x\circ y,3),(x\circ y,5)),$$
$$((x,3),(y,3),(x\circ y,4),(x\circ y,1)),$$
$$((x,4),(y,4),(x\circ y,5),(x\circ y,2)), \text{ and}$$
$$((x,5),(y,5),(x\circ y,1),(x\circ y,3)) \text{ in } B.$$

Then (P,B) is a 5-fold $K_4 - e$ design of order $10k+7$ and $\{\infty_1,\infty_2\} \cup (X \times \{1,2\})$ is a 2-colouring. □

The $10k+2$ *Construction.* Let $(X,\circ)$ be a commutative quasigroup of order $2k$ with holes $H = \{h_1, h_2, \ldots, h_k\}$ all of size 2, set $P = \{\infty_1,\infty_2\} \cup (X \times \{1,2,3,4,5\})$, and define a collection of graphs B as follows:

(1) For the hole h_1, let $(\{\infty_1,\infty_2\} \cup (h_1 \times \{1,2,3,4,5\}), h_1^*)$ be the 5-fold $K_4 - e$ design of order 12 defined by $\circ_3$ in Example 4.1 with 2-colouring $h_1 \times \{1,2,3\}$, an d place the 66 graphs of h_1^* in B;

(2) for each of the remaining holes $h_2, h_3, h_4, \ldots, h_k$, let

$$(\{\infty_1,\infty_2\} \cup (h_i \times \{1,2,3,4,5\}), h_i^*)$$

be the 5-fold $K_4 - e$ design of order 12 with *hole* $\{\infty_1,\infty_2\}$ defined by $\circ_4$ in Example 4.1 with 2-colouring $h_i \times \{1,2,3\}$ and place the 65 graphs in h_i^* in B; and

(3) the same as (3) in the $10k+7$ Construction.

Then (P, B) is a 5-fold $K_4 - e$ design of order $10k + 2$ and $X \times \{1, 2, 3\}$ is a 2-colouring. □

Example 4.2. The following examples are necessary for the $n \equiv 3 \pmod 5$ constructions.

$n = 8$.

$\circ_1$	1	2	3	4	5	6	7	8
1	1	6	7	8	2	3	4	5
2	7	2	1	5	3	8	6	4
3	8	5	3	1	6	4	2	7
4	2	8	6	4	1	7	5	3
5	3	4	2	7	5	1	8	6
6	4	7	5	3	8	6	1	2
7	5	3	8	6	4	2	7	1
8	6	1	4	2	7	5	3	8

2-colouring {1, 2, 4, 5}

n=8 (with hole = {1, 2, 3}).

$\circ_2$	1	2	3	4	5	6	7	8
1	1	3	2	5	6	7	8	4
2	3	2	1	6	7	8	4	5
3	2	1	3	7	8	4	5	6
4	6	8	5	4	1	2	3	7
5	7	4	6	8	5	1	2	3
6	8	5	7	3	4	6	1	2
7	4	6	8	2	3	5	7	1
8	5	7	4	1	2	3	6	8

2-colouring {1, 2, 4, 5}

n = 13.

$\circ_3$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	5	6	7	11	12	13	2	3	4	8	9	10
2	11	2	4	3	8	10	9	1	13	12	5	7	6
3	12	13	3	2	4	9	8	7	1	11	10	6	5
4	13	12	11	4	3	2	10	6	5	1	9	8	7
5	8	1	13	12	5	7	6	11	4	3	2	10	9
6	9	7	1	11	10	6	5	13	12	2	4	3	8
7	10	6	5	1	9	8	7	12	11	13	3	2	4
8	5	11	7	6	2	13	12	8	10	9	1	4	3
9	6	10	12	5	7	3	11	4	9	8	13	1	2
10	7	9	8	13	6	5	4	3	2	10	12	11	1
11	2	8	10	9	1	4	3	5	7	6	11	13	12
12	3	4	9	8	13	1	2	10	6	5	7	12	11
13	4	3	2	10	12	11	1	9	8	7	6	5	13

2-colouring {2, 3, 4, 5, 6, 7}

n=13 (with hole = {1, 2, 3}).

$\circ_4$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	3	2	5	6	4	10	12	13	9	7	11	8
2	3	2	1	13	7	5	9	6	12	4	10	8	11
3	2	1	3	8	11	10	12	13	6	5	9	4	7
4	6	12	7	4	10	9	11	5	1	13	8	2	3
5	11	8	4	3	5	2	13	9	10	7	1	6	12
6	9	10	12	7	3	6	2	4	8	11	5	13	1
7	4	13	5	1	8	12	7	11	2	6	3	9	10
8	7	5	9	11	13	1	3	8	4	2	12	10	6
9	10	7	8	6	2	11	4	1	9	12	13	3	5
10	13	6	11	12	9	7	8	3	5	10	2	1	4
11	8	4	6	9	12	13	5	10	3	1	11	7	2
12	5	11	13	10	1	8	6	2	7	3	4	12	9
13	12	9	10	2	4	3	1	7	11	8	6	5	13

2-colouring {4, 5, 6, 8, 9, 10}

The $10k+8$ *Construction.* In the $10k+7$ Construction, set

$$P = \{\infty_1, \infty_2, \infty_3\} \cup (X \times \{1,2,3,4,5\})$$

and use the quasigroups of order 8 defined by $\circ_1$ and $\circ_2$ in Example 4.2 with 2-colourings $\{\infty_1, \infty_2\} \cup (\{a\} \times \{1,2\})$ and $\{\infty_1, \infty_2\} \cup (\{b\} \times \{1,2\})$.

Then (P, B) is a 5-fold $K_4 - e$ design of order $10k+8$ and $\{\infty_1, \infty_2\} \cup (X \times \{1,2\})$ is a 2-colouring. □

The $10k+3$ *Construction.* In the $10k+2$ Construction, set

$$P = \{\infty_1, \infty_2, \infty_3\} \cup (X \times \{1,2,3,4,5\})$$

and use the quasigroups of order 13 defined by $\circ_3$ and $\circ_4$ in Example 4.2 with 2-colourings $h_1 \times \{1,2,3\}$ and $h_i \times \{1,2,3\} (i \geq 2)$.

Then (P, B) is a 5-fold $K_4 - e$ design of order $10k+3$ and $X \times \{1,2,3\}$ is a 2-colouring. □

Example 4.3. The following examples are necessary for the following $n \equiv 4 \pmod 5$ constructions.

n=9 (and n=9 with hole = {1, 2, 3, 4}).

o_1	1	2	3	4	5	6	7	8	9
1	1	3	4	2	6	7	8	9	5
2	4	2	1	3	7	8	9	5	6
3	2	4	3	1	8	9	5	6	7
4	3	1	2	4	9	5	6	7	8
5	7	9	6	8	5	1	2	4	3
6	8	5	7	9	3	6	1	2	4
7	9	6	8	5	4	3	7	1	2
8	5	7	9	6	2	4	3	8	1
9	6	8	5	7	1	2	4	3	9

2-colouring {3, 5, 6, 9}

n=14 (and n=14 with hole = {1, 2, 3, 4}).

o_2	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	4	2	3	9	10	12	7	11	13	5	8	14	6
2	3	2	4	1	7	5	14	13	6	12	10	11	8	9
3	4	1	3	2	14	13	6	10	12	11	8	9	7	5
4	2	3	1	4	6	9	10	11	13	5	7	14	12	8
5	10	12	9	11	5	2	8	1	3	14	13	6	4	7
6	8	7	11	5	12	6	13	14	1	4	9	2	3	10
7	14	13	10	12	3	4	7	9	8	1	2	5	6	11
8	13	5	7	9	2	11	3	8	4	6	14	10	1	12
9	6	10	5	8	13	14	1	12	9	7	3	4	11	2
10	9	6	14	7	8	1	11	3	2	10	12	13	5	4
11	12	8	13	14	10	3	4	6	5	2	11	7	9	1
12	11	14	8	6	4	7	9	5	10	3	1	12	2	13
13	7	9	6	10	11	12	5	2	14	8	4	1	13	3
14	5	11	12	13	1	8	2	4	7	9	6	3	10	14

2-colouring {2, 5, 9, 10, 12, 13, 14}

The $10k+9$ *Construction.* In the $10k+7$ Construction set $P = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (X \times \{1,2,3,4,5\})$ and use the quasigroup of order 9 defined by $\circ_1$ in Example 4.3 with 2-colourings $\{\infty_1\} \cup (\{a\} \times \{1,2,3\})$ and $\{\infty_1\} \cup (\{b\} \times \{1,2,3\})$.

Then (P, B) is a 5-fold $K_4 - e$ design of order $10k+9$ and $\{\infty_1\} \cup (X \times \{1,2,3\})$ is a 2-colouring. □

The $10k+4$ *Construction.* In the $10k+2$ Construction set $P = \{\infty_1, \infty_2, \infty_3, \infty_4\} \cup (X \times \{1,2,3,4,5\})$ and use the quasigroup of order 14 defined by $\circ_2$ in Example 4.3 with 2-colouring $\{\infty_1\} \cup (h_1 \times \{1,2,3\})$ and $\{\infty_1\} \cup (h_i \times \{1,2,3\}) (i \geq 2)$.

Then (P, B) is a 5-fold $K_4 - e$ design of order $10k+4$ and $\{\infty_1\} \cup (X \times \{1,2,3\})$ is a 2-colouring . □

Lemma 4.4. *There exists a 5-fold* $K_4 - e$ *design which can be 2-coloured of every order* $n \geq 4$, *except possibly* $n = 22, 23,$ *and* 24.

Proof: The cases $n \equiv 0$ or $1 \pmod 5$ are taken care of at the beginning of this section. Since there exists an idempotent commutative quasigroup of every odd order and a commutative quasigroup with holes of size 2 of every even order ≥ 6, the above six constructions produce a 5-fold $K_4 - e$ design which can be 2-coloured of every order $n \equiv 2, 3,$ or $4 \pmod 5$, except $4, 22, 23,$ and 24. The case $n = 4$ is trivial, leaving only $22, 23,$ and 24. □

5. The Cases $n = 22, 23,$ and 24.

In this section we eliminate the three possible exceptions in the statement of Lemma 4.4.

n=24. Let $T = \{(1,1,1,4), (1,2,3,1), (1,3,4,2), (1,4,2,3), (2,1,4,2), (2,2,2,3), (2,3,1,4), (2,4,3,1), (3,1,2,3), (3,2,4,2), (3,3,3,1), (3,4,1,4), (4,1,3,1), (4,2,1,4), (4,3,2,3), (4,4,4,2)\}$. Let $(X, \circ)$ be an *idempotent anti-symmetric* quasigroup of order 6, set $P = X \times \{1,2,3,4\}$, and define a collection of graphs B, as follows:

(1) For each $a \in X$, let $(\{a\} \times \{1,2,3,4\}, a^*)$ be a 5-fold $K_4 - e$ design of order 4 and place the 6 graphs belonging to a^* in B, and

(2) for all $x \neq y \in X$ and $(i,j,s,t) \in T$ place the graph $((x,i),(y,j),(x \circ y, s),(y \circ x, t))$ in B. Then (P, B) is a 5-fold $K_4 - e$ design and $X \times \{1,2\}$ is a 2-colouring.

$n = 22$. Let $(Q, \circ_1)$ and $(Q, \circ_2)$ be the following two quasigroups.

$\circ_1$	1	2	3	4	5	6
1	1	3	4	5	6	2
2	4	2	1	6	3	5
3	5	6	3	1	2	4
4	6	5	2	4	1	3
5	2	4	6	3	5	1
6	3	1	5	2	4	6

2-colouring {1, 3, 4}

$\circ_2$	1	2	3	4	5	6
1	1	2	4	3	6	5
2	2	1	5	6	3	4
3	6	4	3	5	1	2
4	5	3	6	4	2	1
5	4	6	2	1	5	3
6	3	5	1	2	4	6

2-colouring {1, 3, 4} (hole = {1, 2})

Let $(X, \circ)$ be an idempotent anti-symmetric quasigroup of order 6, set $P = X \times \{1,2,3,4\}$ and define a collection of graphs B as follows:

(1) Let $a \in X$ and let $(\infty_1, \infty_2\} \cup (\{a\} \times \{1,2,3,4\}, a^*)$ be the 5-fold $K_4 - e$ design of order 6 defined by $(Q, \circ)$ with 2-colouring $\{\infty_1\} \cup (\{a\} \times \{1,2\})$ and place these graphs in B,

(2) for each $b \in X \backslash \{a\}$, let $(\{\infty_1, \infty_2\} \cup (\{b\} \times \{1,2,3,4\}, b^*)$ be the 5-fold $K_4 - e$ design of order 6 with *hole* $\{\infty_1, \infty_2\}$ defined by $(Q, \circ_2)$ with 2-colouring $\{\infty_1\} \cup (\{b\} \times \{1,2\})$ and place the graphs of b^* in B, and

(3) the same as (2) in the construction for $n = 24$.

Then (P, B) is a 5-fold $K_4 - e$ design of order 22 and $\{\infty_1\} \cup (X \times \{1,2\})$ is a 2-colouring.

$n = 23$. Unfortunately (for technical reasons) the above two constructions cannot be used to construct a 5-fold $K_4 - e$ design of order 23. We content ourselves with an ad hoc example.

o	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	1	21	22	3	2	11	8	10	12	14	16	13	15	17	19	6	18	20	7	9	23	4	5
2	4	2	23	21	3	15	12	9	6	13	20	17	14	11	18	10	7	19	16	8	22	5	1
3	5	4	3	22	23	14	11	13	10	7	19	16	18	15	12	9	6	8	20	17	1	2	21
4	22	1	5	4	21	8	15	12	14	6	13	20	17	19	11	18	10	7	9	16	2	23	3
5	23	22	1	2	5	7	9	11	13	15	12	14	16	18	20	17	19	6	8	10	3	21	4
6	16	7	9	17	19	6	21	22	3	5	1	18	20	2	4	23	8	10	12	14	11	13	15
7	20	17	8	10	18	1	7	21	22	4	5	2	19	16	3	15	23	9	6	13	12	14	11
8	19	16	18	9	6	5	2	8	21	22	4	1	3	20	17	14	11	23	10	7	13	15	12
9	7	20	17	19	10	22	1	3	9	21	18	5	2	4	16	8	15	12	23	6	14	11	13
10	6	8	16	18	20	21	22	2	4	10	17	19	1	3	5	7	9	11	13	23	15	12	14
11	10	18	20	6	8	23	13	15	7	9	11	3	5	12	14	1	21	22	2	4	16	17	19
12	9	6	19	16	7	10	23	14	11	8	15	12	4	1	13	5	2	21	22	3	17	18	20
13	8	10	7	20	17	9	6	23	15	12	14	11	13	5	2	4	1	3	21	22	18	19	16
14	18	9	6	8	16	13	10	7	23	11	3	15	12	14	1	22	5	2	4	21	19	20	17
15	17	19	10	7	9	12	14	6	8	23	2	4	11	13	15	21	22	1	3	5	20	16	18
16	11	13	15	12	14	2	4	1	18	20	23	21	22	7	9	16	3	5	17	19	6	8	10
17	15	12	14	11	13	16	3	5	2	19	10	23	21	22	8	20	17	4	1	18	7	9	6
18	14	11	13	15	12	20	17	4	1	3	9	6	23	21	22	19	16	18	5	2	8	10	7
19	13	15	12	14	11	4	16	18	5	2	22	10	7	23	21	3	20	17	19	1	9	6	8
20	12	14	11	13	15	3	5	17	19	1	21	22	6	8	23	2	4	16	18	20	10	7	9
21	2	5	4	23	1	18	19	20	16	17	6	7	8	9	10	11	12	13	14	15	21	3	22
22	3	23	21	1	4	19	20	16	17	18	7	8	9	10	6	12	13	14	15	11	5	22	2
23	21	3	2	5	22	17	18	19	20	16	8	9	10	6	7	13	14	15	11	12	4	1	23

2-colouring {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

With the above three examples in hand we can now plug the holes in Lemma 4.4.

Lemma 5.1. *There exists a 5-fold $K_4 - e$ design which can be 2-coloured of every order $n \geq 4$.* □

Theorem 5.2. *The spectrum for λ-fold $K_4 - e$ designs with $\lambda \equiv 0 \pmod 5$ which can be 2-coloured is precisely the set of all $n \geq 4$.*

Proof: Write $\lambda = 5k$ and take k copies of Lemma 5.1. □

6. The main result.

As mentioned in the introduction, the spectrum for λ-fold $K_4 - e$ designs is *precisely*: (i) all $n \equiv 0$ or $1 \pmod 5 \geq 6$ for $\lambda = 1$, (ii) all $n \equiv 0$ or $1 \pmod 5$ for $\lambda \equiv 1, 2, 3,$ or $4 \pmod 5 \geq 2$, and (iii) all $n \geq 4$ for $\lambda \equiv 0 \pmod 5$. Theorems 2.2, 3.2, and 5.2 combine to show that these necessary conditions for the existence of a λ-fold $K_4 - e$ design are, in fact, sufficient for the existence of a λ-fold $K_4 - e$ design which can be 2-coloured.

Theorem 6.1. *The spectrum for λ-fold $K_4 - e$ designs which can be 2-coloured is* precisely*: (i) all $n \equiv 0$ or 1 (mod 5) ≥ 6 for $\lambda = 1$, (ii) all $n \equiv 0$ or 1 (mod 5) for $\lambda \equiv 1, 2, 3,$ or 4 (mod 5) ≥ 2, and (iii) all $n \geq 4$ for $\lambda \equiv 0$ (mod 5).* □

References

[1] J. C. Bermond and J. Schönheim, *G-decomposition of K_n where G has four vertices or less*, Discrete Math., 19 (1977), 113-120.

[2] S. I. El-Zanati and C. A. Rodger, Blocking sets in G-designs, submitted.

[3] C. Fu, The intersection problem for pentagon systems, Ph.D. thesis, Auburn University, 1987.

[4] D. G. Hoffman, C. C. Lindner, and K. T. Phelps, *Blocking sets in designs with block size four*, European Journal of Combinatorics, (to appear).

[5] D. G. Hoffman, C. C. Lindner, and K. T. Phelps, *Blocking sets in designs with block size four II*, Discrete Math., (to appear).

[6] C. C. Lindner, *How to construct a block design with block size four admitting a blocking set*, Invited lecture, Fifteenth Australasian Conference on Combinatorial Mathematics and Combinatorial Computing, University of Queensland, Brisbane, Australia, July 10-14, 1989, Australasian Journal of Combinatorics, 1 (1990), 101-125.

[7] Luc Teirlinck, Generalized idempotent orthogonal arrays, I.M.A. proceedings, to appear.