Halving Block Designs with Block Size Four

K. T. Phelps*
Department of Algebra, Combinatorics and Analysis
Auburn University, Alabama 36849, U.S.A.

Abstract

A block design (S, B) is said to have the halving property if the blocks in B can be partitioned into two isomorphic sets. In this paper we construct block designs with block size four with the halving property, thus settling (with the exception of 6 orders) a problem raised by A. Rosa (cf [4]).

1 Introduction

A Steiner system S(2,4,v) is a pair (V,B) where V is a v-set and B is a collection of 4-subsets of V (usually called blocks) such that every pair is contained in exactly one block of B. In general, halving a design (V,B) is a partition of the blocks B into two isomorphic parts, B' and B'_α ; that is, $B = B' \cup B'_\alpha$ and α is a permutation of V which maps the blocks of B' onto those of B'_α. Designs which admit such a partition are said to have the halving property and it is natural to ask what are the necessary and sufficient conditions for the existence of designs with this property. The halving problem has been investigated for complete designs [3], and Steiner triple systems [4], among others. P.K. Das and A. Rosa [4] raised the question of halving designs S(2,4,v). The obvious necessary condition is that $v \equiv 1$ or 16 mod 24. P.K. Das and A. Rosa established the existence of designs S(2,4,v) which can be halved for v = 16, 25 and 40 (and infinitely many other orders). In this note we establish the existence of S(2,4,v) which can be halved for all $v \equiv 1$ or 16 mod 24 (with the possible exception of six orders).

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2 Basic Construction

We start with a group divisible design (V, G, B) with the halving property. In this case, there is a partition of the groups G and the blocks B and a permutation α of V mapping the groups and blocks in one part to those in the other part. We denote such a partition by $G = G' \cup G'_\alpha$ and $B = B' \cup B'_\alpha$ where $\alpha : B' \to B'_\alpha$ and $\alpha : G' \to G'_\alpha$.

Assume that there exists an S(2,4,u), (U, B_u), which has the halving property with the isomorphism α_u and a transversal design TD(k,4) (on $4k$ points, $Z_4 \times K$, with groups of size k and blocks of size 4) (cf [1]). Given a group divisible design (V, G, B) with the halving property and isomorphism α, we construct a design S(2,4,n), $n = u + vk$ on the set $U \cup V \times K$ which will have the halving property as follows:

(1) for each $b \in B'$ take the blocks of a TD(k,4) constructed on the set $b \times K$ where the groups are the sets $\{a\} \times K$,for each $a \in b$;

(2) for each $g \in G'$ take the blocks of a design S(2,4,m) constructed on the set $U \cup g \times K$, having the subdesign (U, B_u), where $m = u + |g|\, k$.

The above collection of blocks, denoted by D', will form half of the blocks of the S(2,4,n) design. We define the permutation α' by $\alpha'(x, i) = (\alpha(x), i)$ for $x \in V$, $i \in K$ and define $\alpha'(s) = \alpha_u(s)$, $s \in U$, where α and α_u are the halving isomorphisms as noted above. Then the other half of the blocks will be simply the image of D' under α'. Clearly the two halves are isomorphic. It is a tedious but easy matter to check that every pair is covered exactly once.

The key ingredients in the above construction are, first, designs S(2,4,m) which contain subdesigns of order u. Such designs exist for any admissible pair u, m as long as $m \geq 3u + 1$ [6]. The second key ingredients are transversal designs TD(k,4) which are equivalent to pairs of orthogonal latin squares of order k. These exist for all k, $k \neq 2$ or 6. The last ingredient is a group divisible design with the halving property and our next lemma establishes the existence of two such designs.

Lemma 1 *There exists a group divisible design with the halving property on* 16 *points having group size* 4 *and block size* 4, *and one on* 24 *points with group size* 3 *and block size* 4.

Proof: The examples constructed by A. Rosa [4] provide both group divisible designs. The S(2,4,16) example has a parallel class which is halved: this parallel class will be our groups. The S(2,4,25) example has an isomorphism with a fixed point; deleting this fixed point gives a design on 24 points with group size 3 and block size 4 which has the halving property. □

Theorem 1 *There exists a* $S(2,4,n)$ *design with the halving property for all* n, *(A)* $n = u + 24k$, $k \geq (2u+1)/3$ *and* $u + 3k \equiv 1$ *or* 4 *mod* 12, *or (B)* $n = u + 16k$, $k \geq (2u+1)/4$ *and* $u + 4k \equiv 1$ *or* 4 *mod* 12, *where* $u = 0, 1, 16, 25$ *and* 40, *with* $k \neq 2$ *or* 6.

Proof: Apply the above construction using the two group divisible designs from the previous lemma. □

The following table summarizes all the various values of u, k and n.

u	k	n	
0	$3t+1$	$48t+16$	$t \geq 0$
1	$4t$	$96t+1$	$t \geq 0$
1	$4t+1$	$96t+25$	$t \geq 0$
1	$3t$	$48t+1$	$t \neq 2$, $t \geq 0$
16	$4t$	$96t+16$	$t \geq 3$
16	$4t+3$	$96t+88$	$t \geq 2$
40	$4t$	$96t+40$	$t \geq 7$
25	$3t$	$48t+25$	$t \geq 5$

From the above table we see that we are missing only nine admissible orders.

Corollary 1 *There exists a design* $S(2,4,n)$ *with the halving property for all* $n \equiv 1$ *or* 16 *mod* 24 *with the possible exception of* $n = 73$, 169, 184 *and* $n = 96t + 40$ *for* $t = 1, 2, \ldots, 6$.

Lemma 2 *There exists a design* $S(2,4,n)$ *with the halving property for* $n = 73$, 169, 520.

Proof: The Bose finite field construction of $S(2,4,n)$ for $n = 73$, 169 (cf [2]) gives systems in which the halving automorphism is a multiplicative element of the field. A cyclic S(2, 4, 520) was constructed by R. Mathon [5] with $520 = 96.5 + 40$. □

Corollary 2 *There exists a design $S(2,4,n)$ with the halving property for all $n \equiv 1$ or 16 mod 24 with the possible exception of $n = 136$, 184, 232, 328, 424, 616.*

3 Final Comments

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References

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