EMBEDDINGS OF SYMMETRIC GRAPHS IN SURFACES

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Abstract. A symmetric graph Σ has a group G of automorphisms which acts transitively on incident vertex-edge pairs. The question of whether such a graph can be embedded in a 2-dimensional surface with the action of G extending to isometries or homeomorphisms of the whole surface is investigated here. It is answered in terms of presentations of G and six different types of embeddings are found.

The context of this paper is the study of symmetry. Every object, concrete or abstract, real or imaginary, has a certain amount of symmetry which is measured by a group. Since the days of Cayley [1], a group has been a mathematical object obeying certain axioms and the question arises: given an abstract group, of what is it a group of symmetries?

A very general answer, described in more detail in Section 1, is: groups tend to act as groups of automorphisms of symmetric graphs. Such a graph is formed, for example, by the vertices and edges of a cube (or any of the regular solids), and the Euclidean group of motions of the cube acts on it as a group of graph automorphisms.

Alternatively, this graph can be formed from the surface of a cube by removing the interiors of its 6 faces. The opposite procedure will be investigated here: given a graph on which a group acts symmetrically, how can some of its cycles be filled in to form a surface on which the action of the members of the group extend to isometries or homeomorphisms? Even for the graph of a cube the answer is not obvious: certainly, six of its 4-cycles can be filled in to give the ordinary cube, but it is also possible to fill in the four 6-cycles obtained by leaving out pairs of opposite vertices and their incident edges, to get a torus. In other words, the graph

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of vertices and edges of a cube can be embedded in both a sphere and a torus in such a way that its group of automorphisms (of order 48), extends to a group of isometries of the surface. The metric is the usual one on the sphere and is Euclidean on the torus.

Consider a 2-dimensional surface S, regarded as a topological manifold, a graph Σ embedded in S and a group, G, of homeomorphisms of S which acts on Σ as a group of automorphisms. Suppose that G acts symmetrically on Σ, i.e. any incident vertex-edge pair of Σ can be mapped on to any other by some element of G. Then all vertices of Σ have the same degree, say p, and the stabilizer in G of any vertex acts transitively on the p edges incident with it. As they are embedded in the surface the action of this stabilizer on the edges is as a dihedral group of order $2p$, a cyclic group of order p or, when p is even, as a dihedral group of order p. As the graph is embedded in the surface, its complement in S is a union of open discs, called faces, which are moved among themselves by the members of G. It is easy to see that G has just one or two orbits on these faces.

The considerations in the last paragraph lead to six types of embeddings. In Section 2, these are analysed in terms of a presentation for the group G. For example, if G acts as a dihedral group of order $2p$ then it has just one orbit on the faces and it is generated by a, b, c which satisfy

$$a^2 = b^2 = c^2 = (ab)^p = (ac)^2 = (bc)^q = 1,$$

where q is the number of edges around each face. The subgroup generated by a and b is the stabilizer of a vertex, that generated by a and c is the stabilizer of an edge and that generated by b and c is the stabilizer of a face.

Section 3 contains the converse of this result: given one of the presentations referred to, it is shown how to construct an appropriate graph and surface. In all cases but one, the construction provides the surface with a metric, relative to which the edges of the graph become geodesics and the members of G become isometries.

Part of the motivation for this paper comes from the use of computers in studying groups and graphs. In particular, the low index subgroups algorithm of a group theory package like John Cannon's CAYLEY accepts presentations like the ones described here and lists homomorphic images of them. The conditions listed in the theorems describe when the corresponding graph gives rise to a tessellation of some surface.

Some examples of the embeddings described in this paper are given in Section 4.

1. Definitions and notations.

All graphs will be without loops or multiple edges. An edge is said to be **incident** with each of its end points, which are said to be **adjacent**. An automorphism of a graph is a permutation of its vertices which maps adjacent vertices onto adjacent vertices. Each automorphism induces a permutation of the edges of the graph on which it acts and a permutation of its incident vertex-edge pairs. A group of automorphisms which acts transitively on the incident vertex-edge pairs of a graph is said to **act symmetrically** on it and the graph is called a **symmetric graph.**

Let A be a group of automorphisms acting symmetrically on a graph Σ. A homomorphism ϕ from a group G **onto** A is called a **symmetric action** of G and G is said to **act symmetrically** on Σ. Usually the homomorphism ϕ will be suppressed and the members of G will be regarded as automorphisms of Σ. At first reading it does no harm to think of the group G as actually being a group of automorphisms of the graph.

A **surface** will always be a 2-dimensional surface without boundary. When a graph is **embedded** in a surface each of its vertices becomes a different point, each of its edges becomes an open arc joining its two endpoints and the complement of the graph in the surface will be a disjoint union of open discs called **faces**, each of which has a finite cycle of the graph as its boundary. Thus no vertex can lie on an edge, pairs of edges are disjoint and the graph has no vertices of valency less than two. A **flag** of the embedding of a graph in a surface is a triple α, β, γ consisting of a vertex α, an edge β and a face γ with α being a vertex of β and both α and β being in the boundary of γ.

All notations for groups are standard. The brackets $\langle \quad \rangle$ will indicate the group generated by whatever is between them; x^y stands for $y^{-1}xy$ and H^y stands for

$y^{-1}Hy$.

2. Surfaces with graphs embedded in them.

Consider a cube embedded in the surface of a sphere and let α, β, γ be one of its flags. Then there are reflections a, b, c of the sphere which leave the cube invariant and are determined uniquely by the following requirements: a fixes α and β but not γ; b fixes α and γ but not β; c fixes β and γ but not α. The following turn out to be true:

$$a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ac)^2 = 1.$$

In fact, these equations are a presentation of the group G of symmetries of the cube.

This group of order 48 has 2 subgroups of index 2 which also act symmetrically on the cube, namely $\langle ab, c\rangle$ and $\langle ab, ac\rangle$ and these can be described, in terms of presentations, as the group generated by x and c with the relations

$$x^3 = c^2 = (x^{-1}cxc)^2 = 1$$

and the group generated by x and y with the relations

$$x^3 = y^4 = (xy)^2 = 1,$$

where $x = ab$, $y = ac$.

Let S be a 2-dimensional surface without boundary in which a connected graph Σ is embedded. Let G be a group of homeomorphisms of S which maps Σ onto itself and acts symmetrically on Σ. The action of G on Σ will be described in terms of presentations.

PROPOSITION 2.1.

G has at most two orbits among the flags, α, β, γ of Σ and any member of G which fixes one such flag fixes them all.

PROOF:

As G acts symmetrically on Σ it has just one orbit on incident vertex-edge pairs of Σ. As each edge of Σ lies in at most two faces, G has at most two orbits among the flags α, β, γ of Σ in S.

Suppose that g is a member of G which fixes a flag of Σ in S. Let α, β, γ be a flag fixed by g. Then g fixes the vertex α, the edge β at α and the face γ having α and β in its boundary. Now, the edges and faces at α are arranged around it in cyclic order: consequently, g fixes every flag having α as its vertex.

Let α_1 be a vertex adjacent to α, suppose that it is joined to α by the edge β_1 and that β_1 is in the boundary of the face γ_1. Then $\alpha, \beta_1, \gamma_1$ is a flag which must be fixed by g. Hence the flag $\alpha_1, \beta_1, \gamma_1$ is also fixed by g.

Thus, by assumption g fixes some flag; if g fixes one flag with a given vertex it fixes all flags with that vertex and it fixes at least one flag at each adjacent vertex. As Σ is connected, it follows that g fixes every flag of Σ.

That proves Proposition 2.1.

In that G acts on Σ, there is a homomorphism ϕ from G to the group of graph automorphisms of Σ; the image $\phi(G)$ acts symmetrically on Σ and it will be the task of the rest of this section to describe the possibilities for $\phi(G)$. The case that $\phi(G)$ has just one orbit on flags is covered in Theorem 2.3. Otherwise $\phi(G)$ has two orbits on flags. The case that $\phi(G)$ has two orbits on faces is covered in Theorem 2.4. Otherwise the stabilizer of a vertex has one or two orbits on the faces incident with it and these cases are covered in Theorems 2.5 and 2.6, respectively.

Let $\alpha_0, \beta_0, \gamma_0$ be a fixed flag of Σ in S and let $V, E, F,$ be the stabilizers of $\alpha_0, \beta_0, \gamma_0$, respectively, in $\phi(G)$. Because Σ is connected it is easy to show that $G = \langle V, E \rangle$ (see [6]). The analysis of $\phi(G)$ is given in terms of these subgroups.

Although, strictly speaking, $\phi(G)$ is not a group of homeomorphisms of S, it still makes sense to think of the action of $\phi(G)$ on faces: in this context a face can be considered as the cycle in Σ which forms its boundary.

The interpretation of Proposition 2.1 for the automorphism group $\phi(G)$ is:

PROPOSITION 2.2.

(a) The only member of $\phi(G)$ which fixes a flag α, β, γ of Σ is the identity.
(b) $V \cap E \cap F = 1$.
(c) V acts faithfully on the edges incident with α_0 and on the faces having α_0 in their boundary.

(d) F acts faithfully on the vertices and on the edges in the boundary of γ_0 .

PROOF:

(a) follows directly from Proposition 2.1.

(b) is true because $V \cap E \cap F$ contains all members of $\phi(G)$ fixing the flag $\alpha_0, \beta_0, \gamma_0$.

(c) If a member v of V fixes all the edges incident with α_0, it also fixes all the faces having α_0 in their boundary. Thus v fixes a flag and must be the identity. Similarly, V acts faithfully on the faces having α_0 in their boundary.

(d) is proved in a similar manner to (c).

That completes the proof of Proposition 2.2.

Here is the first characterization theorem.

THEOREM 2.3.

Suppose that G has just one orbit on the flags of Σ in S. Then $\phi(G) = \langle a, b, c\rangle$ where

$$a^2 = b^2 = c^2 = (ac)^2 = 1$$

and $V = \langle a, b\rangle, E = \langle a, c\rangle, F = \langle b, c\rangle, V \cap V^c = V \cap E = \langle a\rangle, V \cap F = \langle b\rangle, E \cap F = \langle c\rangle$.

PROOF:

The flag $\alpha_0, \beta_0, \gamma_0$ is given. Examination of the embedding of Σ in S shows that there are a uniquely defined vertex α_1, edge β_1 and face γ_1, different from α_0, β_0, γ_0, respectively, such that $\alpha_0, \beta_0, \gamma_1$ and $\alpha_0, \beta_1, \gamma_0$ and $\alpha_1, \beta_0, \gamma_0$ are flags. As G acts transitively on flags, $\phi(G)$ contains members a, b, c which map $\alpha_0, \beta_0, \gamma_0$ onto these flags in the order given. As the only member of $\phi(G)$ fixing a flag is 1, these elements, a, b and c are uniquely determined.

As $a(\alpha_0) = \alpha_0, a(\beta_0) = \beta_0$ and $a(\gamma_0) = \gamma_1$, it follows that a must interchange the two faces γ_0 and γ_1 having β_0 in their boundary. Hence a^2 fixes the flag $\alpha_0, \beta_0, \gamma_0$ so that $a^2 = 1$. Similarly $b^2 = c^2 = (ac)^2 = 1$.

The group V contains both a and b. The element a fixes the edge β_0 and interchanges the two faces on either side of β_0. As a acts on flags, its action on the cycle of edges and faces around the vertex α_0 must be as a reflection fixing the edge β_0. Similarly b must be a reflection fixing the face γ_0. Thus, a and b generate the full group of automorphisms of this cycle, which is a dihedral group of order twice the valency of α_0. As V acts faithfully on the edges and faces around $\alpha_0, V = \langle a, b\rangle, V \cap E = \langle a\rangle$ and $V \cap F = \langle b\rangle$.

As E is the stabilizer of the edge β_0 at α_0 and $\phi(G)$ acts symmetrically on the graph $\Sigma, V \cap E$ has index 2 in E. Hence E has order 4. As E contains a and c and $a \neq c, E = \langle a, c\rangle$. Clearly $E \cap F = \langle c\rangle$.

As the graph Σ is connected, $\phi(G)$ is generated by V and E. Hence $\phi(G) = \langle a, b, c\rangle$. As the subgroup $V \cap V^c$ contains all members of G fixing α_0 and $c(\alpha_0)$, and $V \cap E$ contains all members of G fixing α_0 and the edge β_0 joining α_0 and $c(\alpha_0)$, it follows that $V \cap V^c = V \cap E$.

That completes the proof of Theorem 2.3 and completes the description of the group G when it has just one orbit on the flags of Σ in S. Next, the possibility that G has two orbits on the faces of Σ is considered.

THEOREM 2.4.

Suppose that G has two orbits on the faces of S determined by Σ. Then $\phi(G) = \langle b, c, d\rangle$ where

$$b^2 = c^2 = d^2 = 1$$

and $V = \langle b, d\rangle, E = \langle c\rangle, F = \langle b, c\rangle, V \cap V^c = V \cap E = 1, V \cap F = \langle b\rangle, E \cap F = \langle c\rangle$.

The graph Σ has even valency, the stabilizer in $\phi(G)$ of the other face having β_0 in its boundary is $F' = \langle d, c\rangle$ which is not conjugate to F in $\phi(G)$ and $V \cap F' = \langle d\rangle$, $E \cap F' = \langle c\rangle$.

PROOF: The flag $\alpha_0, \beta_0, \gamma_0$ is given. As in the proof of Theorem 2.3, let $\alpha_0, \beta_0, \gamma_1$, and $\alpha_0, \beta_1, \gamma_0$ and $\alpha_1, \beta_0, \gamma_0$ be flags different from α_0, β_0, γ_0.

Consider the action of G on the faces having α_0 in their boundary. As G acts transitively on the incident vertex-edge pairs of Σ but has two orbits on its faces, G has two orbits on the faces around α_0. This implies that Σ has even valency and that consecutive faces around α_0 are in different orbits.

As $\phi(G)$ acts symmetrically on Σ, $\phi(G)$ contains an element b with $b(\alpha_0) = \alpha_0, b(\beta_0) = \beta_1$. As β_0 and β_1 are consecutive edges around $\alpha_0, b(\gamma_0) = \gamma_0$. Thus $b \in F$. Then b fixes the boundary of γ_0 and, as $b(\alpha_0) = \alpha_0$, b acts as a reflection on this cycle. Thus b^2 fixes the flag $\alpha_0, \beta_0, \gamma_0$ and $b^2 = 1$.

As γ_0, γ_1 are consecutive faces at α_0, they are in different orbits of G. Consequently, any member of $\phi(G)$ which fixes both α_0 and β_0 also fixes γ_0 and γ_1 and, hence, is the identity. Thus $V \cap E = 1$. As G acts symmetrically on $\Sigma, V \cap E$ has index 2 in E so that $E = \langle c \rangle$ where $c^2 = 1$.

As both b and c fix $\gamma_0, \langle b, c \rangle \subseteq F$. As F acts faithfully on the cycle in its boundary and b fixes α_0, c fixes β_0, both b and c must act as reflections on the cycle and $F = \langle b, c \rangle$. Also, $V \cap F = \langle b \rangle$ and $E \cap F = \langle c \rangle$.

As $\alpha_0, \beta_0, \gamma_1$ is a flag there is an edge β_2 at α_0 such that $\alpha_0, \beta_2, \gamma_1$ is another flag. Hence, $\phi(G)$ contains an element d such that $d(\alpha_0) = \alpha_0$ and $d(\beta_0) = \beta_2$. In analogy with b, it follows that $d^2 = 1$ and $d(\gamma_1) = \gamma_1$. As for γ_0 it follows that $\langle c, d \rangle$ is the stabilizer of γ_1.

The involutions b and d fix consecutive faces at α_0 and, as $V \cap E = 1, V = \langle b, d \rangle$.

As F and $\langle d, c \rangle$ are the stabilizers in $\phi(G)$ of γ_0 and γ_1, which lie in different orbits, they are not conjugate. As Σ is connected, $\phi(G) = \langle V, E \rangle = \langle b, c, d \rangle$.

As in Theorem 2.3, $V \cap V^c = V \cap E$. That $V \cap F' = \langle d \rangle$ and $E \cap F' = \langle c \rangle$ follows because of the symmetry between F and F'.

That proves Theorem 2.4.

In all the remaining cases, G has two orbits on flags but only one orbit on faces.

THEOREM 2.5.

Suppose that G has two orbits on the flags, α, β, γ of Σ and one orbit on the faces and that V has one orbit on the faces in which α_0 lies. Then $\phi(G) = \langle v, c \rangle$ where

$$c^2 = 1$$

and either

(1) $V = \langle v \rangle, E = \langle c \rangle, F = \langle v^{-1}c \rangle, V \cap V^c = V \cap E = V \cap F = E \cap F = 1$, or

(2) $V = \langle v \rangle, E = \langle c \rangle, F = \langle c, c^v \rangle, V \cap V^c = V \cap E = V \cap F = 1, E \cap F = \langle c \rangle$.

PROOF:

As G has two orbits on the flags, α, β, γ of Σ and G acts symmetrically on Σ, V cannot act transitively on the flags of Σ in S which contain α_0. However, V does act transitively on the edges and on the faces at α_0. The only possibility is that $V = \langle v \rangle$ is a cyclic group acting transitively on these edges and vertices. Then $V \cap E = V \cap F = 1$. Without loss of generality, suppose that $v(\gamma_0) = \gamma_1$ where γ_1 is the other face having β_0 as an edge.

As $V \cap E = 1$ and $V \cap E$ has index 2 in $E, E = \langle c \rangle$ where $c^2 = 1$.

There are now two cases, depending on whether c fixes γ_0.

1. Suppose $c(\gamma_0) \neq \gamma_0$. Then $c(\gamma_0) = \gamma_1$ and $v^{-1}c(\gamma_0) = \gamma_0$. Now, if α_1 is the other vertex of the edge β_1, then $v^{-1}c(\alpha_1) = \alpha_0$. Thus $v^{-1}c$ fixes the face γ_0 and maps one of its vertices α_1 onto an adjacent one. As $v^{-1}c$ does not fix the edge β_1 joining these two vertices, $\langle v^{-1}c \rangle$ acts cyclically and transitively on the vertices and edges on the boundary of γ_0. As $V \cap F = 1$ this implies that $F = \langle v^{-1}c \rangle$. Clearly $E \cap F = 1$.
2. Suppose $c(\gamma_0) = \gamma_0$ so that $c \in F$. Then, also $c(\gamma_1) = \gamma_1$ and, as $v(\gamma_0) = \gamma_1, c^v(\gamma_0) = \gamma_0$ and $c^v \in F$. Now c^v fixes the edge $v^{-1}(\beta_0)$ which is adjacent to the edge β_0 fixed by c in the boundary of γ_0. Hence $\langle c, c^v \rangle$ is a dihedral group acting transitively on the vertices in the boundary of γ_0. As $V \cap F = 1, \langle c, c^v \rangle = F$. Clearly $E \cap F = \langle c \rangle$.

As the graph Σ is connected, $\phi(G)$ is generated by V and E. Hence $\phi(G) = \langle v, c \rangle$.

As in Theorem 2.3, $V \cap V^c = V \cap E$.

Finally, there remains just one case.

THEOREM 2.6.

Suppose that G has two orbits on the flags of Σ in S and one orbit on the faces and that V has two orbits on the faces which have α_0 in their boundary. Then $\phi(G) = \langle b, c, d \rangle$ where

$$b^2 = c^2 = d^2 = 1$$

and either

(1) $V = \langle b, d\rangle, E = \langle c\rangle, F = \langle b, d^c\rangle, V \cap V^c = V \cap E = E \cap F = 1, V \cap F = \langle b\rangle$,

or

(2) $V = \langle b, d\rangle, E = \langle c\rangle, F = \langle b, c\rangle, V \cap V^c = V \cap E = 1,\ V \cap F = \langle b\rangle, E \cap F = \langle c\rangle$, each face has an even number of vertices in its boundary and the subgroups F and $\langle d, c\rangle$ are conjugate in $\phi(G)$ by an element k with $ck = kc$.

PROOF:

Suppose that γ_1 is the other face having β_0 in its boundary.

As V has one orbit on the edges at α_0 and two on the faces that have α_0 in their boundary, V is generated by two involutions, b and d which act as reflections in γ_0 and γ_1 respectively. Thus $V = \langle b, d\rangle$ where $V \cap F = \langle b\rangle$ and $V \cap E = 1$.

As $V \cap E = 1$ and $V \cap E$ has index 2 in $E, E = \langle c\rangle$ where $c^2 = 1$.

There are now two cases depending on whether c fixes γ_0.

(1) Suppose that $c(\gamma_0) \neq \gamma_0$ i.e. $c \notin F$ and $E \cap F = 1$. As $c(\beta_0) = \beta_0$ and β_0 is the edge between γ_0 and $\gamma_1, c(\gamma_0) = \gamma_1$. Hence $d^c(\gamma_0) = \gamma_0$ and $d^c \in F$. Thus F contains b and d^c acting on the boundary of γ_0 as reflections in adjacent vertices. As $E \cap F = 1$ it follows that $F = \langle b, d^c\rangle$. This is case (1) of the theorem.

(2) Suppose that $c(\gamma_0) = \gamma_0$. Then, also, $c(\gamma_1) = \gamma_1$. As F contains b and c acting on the boundary of γ_0 as reflections in an incident vertex and edge respectively, it follows that $F = \langle b, c\rangle$. Similarly the stabilizer of γ_1 is $F^* = \langle d, c\rangle$. By assumption, $\phi(G)$ contains a member g with $g(\gamma_0) = \gamma_1$. Then c^g fixes γ_0 so that $c^g \in F$. But c^g fixes the edge $g^{-1}(\beta_0)$ of γ_0 so that c and c^g are conjugate in F. Thus $c = c^k$ where $k = gf, f \in F$. Then $k(\gamma_0) = g(\gamma_0) = \gamma_1$ and, as F, F^* are the stabilizers in $\phi(G)$ of γ_0, γ_1, respectively, they are conjugate under k. This is case (2) of the Theorem.

In either case, because γ is a connected graph, $\phi(G) = \langle V, E\rangle = \langle b, c, d\rangle$.

As in Theorem 2.3, $V \cap V^c = V \cap E$ in both cases.

3. Orbit spaces of 2-dimensional tessellations.

In Section 2, four types of embeddings of symmetric graphs in surfaces were identified and described in terms of six types of groups and their subgroups. The presentations given for these groups will now be investigated and surfaces in which symmetric graphs are embedded will be constructed from each of them.

This geometrical study begins with the three simply connected 2-dimensional geometries, the sphere, the Euclidean plane and the hyperbolic plane. The existence of the regular solids show that the surface of the sphere can be tessellated by equilateral triangles with 3, 4 or 5 at a vertex, regular quadrilaterals (i.e squares) with 3 at a vertex and regular pentagons with 3 at a vertex. The Euclidean plane can be tessellated by equilateral triangles with six at a vertex, squares with four at a vertex and hexagons with three at a vertex. In other words, these two geometries can be tessellated by regular p-gons with q at a vertex for some restricted values of p and q. It turns out that, for every other value of p and q with $p > 3, q > 3$, such a tessellation can be achieved in the hyperbolic plane.

Suppose that $p > 3$, $q > 3$ and that T is a tessellation like this in the appropriate geometry. Let α, β, γ be a flag of the tessellation. Then there are uniquely defined symmetries $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of T which are reflections in lines of the geometry, and have the properties:

$\mathbf{a}$ fixes α and β but not γ,

$\mathbf{b}$ fixes α and γ but not β,

$\mathbf{c}$ fixes β and γ but not α.

These three elements satisfy the relations

$$\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = (\mathbf{ac})^2 = (\mathbf{ab})^p = (\mathbf{bc})^q = 1.$$

In fact, $\mathbf{ab}$ is a rotation around α, $\mathbf{ac}$ is a rotation around the midpoint of β and $\mathbf{bc}$ is a rotation around the centre of γ. It turns out that these relations, which define the Coxeter group $[p, q]$, describe the full group of symmetries of this tessellation exactly: they are a presentation of it. For further information on these things see [3, Chapter 5].

In the group $[p, q]$, with this interpretation, the stabilizers of α, β, γ are $\langle \mathbf{a}, \mathbf{b} \rangle$, $\langle \mathbf{a}, \mathbf{c} \rangle$, $\langle \mathbf{b}, \mathbf{c} \rangle$, respectively.

Independent of the value of p and q, the relations that all these groups have in common are

$$\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = (\mathbf{ac})^2 = 1.$$

As the group generated by **a**, **b**, **c** subject to these defining relations is a sort of universal group for them all, denote it by U.

The derived group, U', of U has index 8 and is generated by $(\mathbf{ab})^2$, $(\mathbf{bc})^2$, **abcbca**, all of which are commutators (because **a**, **b**, **c** are all involutions). The quotient group U/U' is elementary abelian, which implies that U has seven subgroups of index 2, namely:

$$U_1 = \langle \mathbf{ab}, \mathbf{c} \rangle,$$
$$U_2 = \langle \mathbf{ab}, \mathbf{ac} \rangle,$$
$$U_3 = \langle \mathbf{b}, \mathbf{b}^a, \mathbf{c} \rangle,$$
$$U_4 = \langle \mathbf{b}, \mathbf{b}^a, \mathbf{ac} \rangle,$$
$$U_5 = \langle \mathbf{a}, \mathbf{b}, \mathbf{b}^c \rangle,$$
$$U_6 = \langle \mathbf{a}, \mathbf{c}, \mathbf{a}^b, \mathbf{c}^b \rangle,$$
$$U_7 = \langle \mathbf{a}, \mathbf{bc} \rangle.$$

Notice that each of these, interpreted as a subgroup of $[p, q]$, has index 2 in $[p, q]$, though some pairs of them may be identical in some cases. (The derived subgroup of $[p, q]$ has index 2 when p and q are both odd, index 4 when one of p and q is odd and index 8 if both are even). Of these subgroups, U_5 contains $\langle \mathbf{a}, \mathbf{b} \rangle$, U_6 contains $\langle \mathbf{a}, \mathbf{c} \rangle$ and U_7 does not contain **c** or **ac**. Hence, the subgroup $\langle \mathbf{a}, \mathbf{b}, \mathbf{b}^c \rangle$ does not act transitively on the vertices of the corresponding tessellation; $\langle \mathbf{a}, \mathbf{c}, \mathbf{a}^b, \mathbf{c}^b \rangle$ does not act transitively on its edges; $\langle \mathbf{a}, \mathbf{bc} \rangle$ does not contain an element which interchanges the vertices of the edge β. Consequently, in these circumstances, none of these subgroups can act symmetrically on the graph of vertices and edges of the tessellation. On the other hand, $U_i\langle \mathbf{a} \rangle = G$ for each $i = 1, \ldots 4$ and U_i acts symmetrically on the graph of the tessellation.

Next, some presentations for the groups U_1 to U_4 will be given.

$$U_1 = \langle x, y \ ; \ y^2 = 1 \rangle, \text{where } x = \mathbf{ab}, \ y = \mathbf{c};$$
$$U_2 = \langle x, y \ ; \ y^2 = 1 \rangle, \text{where } x = \mathbf{ab}, \ y = \mathbf{ac};$$
$$U_3 = \langle x, y, z \ ; \ x^2 = y^2 = z^2 = 1 \rangle, \ \text{where } x = \mathbf{b}, \ y = \mathbf{b^a}, \ z = \mathbf{c};$$
$$U_4 = \langle x, y, z \ ; \ x^2 = y^2 = z^2 = 1 \rangle, \ \text{where } x = \mathbf{b}, \ y = \mathbf{b^a}, \ z = \mathbf{ac}.$$

Groups with the presentations of U_1 to U_4 and U will now be investigated using the interpretation suggested by the above. They will be treated separately, in subsections.

3.1 The group U.

In this subsection, G is a group generated by a, b, c satisfying, among other relations,

(I) $a^2 = b^2 = c^2 = (ac)^2 = 1$.

Let $V = \langle a, b \rangle, E = \langle a, c \rangle, F = \langle b, c \rangle$ and suppose that

(II) ab has order $p \geq 3$ and bc has order $q \geq 3$,

(III) $V \cap V^c = V \cap E = \langle a \rangle, V \cap F = \langle b \rangle, E \cap F = \langle c \rangle$.

Because of these relations, there is a homomorphism $\phi : [p, q] \to G$ which satisfies $\phi(\mathbf{a}) = a, \ \phi(\mathbf{b}) = b, \ \phi(\mathbf{c}) = c$.

The group $[p, q]$ is the full group of symmetries of a tessellation T of one of the three simply connected 2-dimensional geometries into regular p-gons with q faces around each vertex. It is clear that $[p, q]$ acts transitively on the vertices, edges and faces of this tessellation. Hence, those can be identified with the left cosets in $[p, q]$ of the subgroups $\mathbf{V} = \langle \mathbf{a}, \mathbf{b} \rangle, \ \mathbf{E} = \langle \mathbf{a}, \mathbf{c} \rangle, \ \mathbf{F} = \langle \mathbf{b}, \mathbf{c} \rangle$, respectively. The following is easily verified.

PROPOSITION 3.1.

(a) The vertex $\mathbf{xV}$ and the edge $\mathbf{yE}$ are incident if and only if $\mathbf{xV} \subseteq \mathbf{yEV}$ or, equivalently, $\mathbf{yE} \subseteq \mathbf{xVE}$.

(b) The vertex $\mathbf{xV}$ lies on the boundary of the face $\mathbf{zF}$ if and only if $\mathbf{xV} \subseteq \mathbf{zFV}$ or, equivalently, $\mathbf{zF} \subseteq \mathbf{xVF}$.

(c) The edge $\mathbf{yE}$ lies in the boundary of the face $\mathbf{zF}$ if and only if $\mathbf{yE} \subseteq \mathbf{zFE}$ or, equivalently, $\mathbf{zF} \subseteq \mathbf{yEF}$.

(d) The action of $[p, q]$ as a group of symmetries of the tessellation T is given by

$$\mathrm{g} : \mathbf{xV} \to \mathbf{gxV}, \quad \mathbf{yE} \to \mathbf{gyE}, \quad \mathbf{zF} \to \mathbf{gzF},$$

for each $\mathrm{g} \subseteq [p, q]$.

The proof is omitted.

Denote this geometry by X and the tessellation by T. Let $\mathbf{N}$ be the kernel of ϕ and let X^* be the orbit space of $\mathbf{N}$, i.e. the points of X^* are the orbits of points of X under the action of $\mathbf{N}$. As $[p, q]$ is a group of symmetrics of X it moves the vertices, edges and faces of T around among themselves and, hence, $\mathbf{N}$ induces a set of orbits on the vertices, on the edges and on the faces of T. These orbits will be called the vertices, edges and faces of X^* and it will be shown that they form a tessellation T^* of X^* of the same type as T.

This is best done in terms of the characterization of T given in Proposition 3.1. In terms of it, the vertices of T^* can be identified with the left cosets of $\mathbf{VN}$ in $[p, q]/\mathbf{N}$, the edges can be identified with the left cosets of $\mathbf{EN}$ and the faces with the left cosets of $\mathbf{FN}$. The incidences become: $\mathbf{xVN}$ is incident with $\mathbf{yEN}$ if and only if $\mathbf{xVN} \subseteq \mathbf{yEVN}$ or $\mathbf{yEN} \subseteq \mathbf{xVEN}$, with similar expressions for the other incidences. Using the homomorphism theorems, this transfers to an identification of the vertices of T^* with the left cosets of V in G, the edges of T^* with the left cosets of E in G.and the faces of T^* with the left cosets of F in G. Corresponding to Proposition 3.1 is:

PROPOSITION 3.2.

(a) The vertex xV and the edge yE are incident if and only if $xV \subseteq yEV$ or, equivalently, $yE \subseteq xVE$.

(b) The vertex xV lies on the boundary of the face zF if and only if $xV \subseteq zFV$ or, equivalently, $zF \subseteq xVF$.

(c) The edge yE lies in the boundary of the face zF if and only if $yE \subseteq zFE$ or, equivalently, $zF \subseteq yEF$.

(d) X^* is a manifold having the same geometric structure, spherical, Euclidean or hyperbolic, as X and G acts on it as a group of isometries which fixes the tessellation T^* under the action

$$g : xV \to gxV, \ \ yE \to gyE, \ \ zF \to gzF.$$

As such G acts symmetrically on the graph formed by the vertices and edges of T^*.

PROOF:

(a), (b) and (c) follow directly from the same property in $[p, q]/\mathrm{N}$.

(d) It will first be shown that the vertices and edges of of T^* form a graph without loops or multiple edges.

A loop corresponds to a coset yE having identical endpoints yV and ycV, which cannot happen as $c \notin V$. A multiple edge corresponds to two edges y_1E and y_2E having the same endpoints. The conditions for this are

$$y_1V = y_2V \ \text{ and } \ y_1cV = y_2cV,$$

$$\text{or} \quad y_1V = y_2cV \ \text{ and } \ y_1cV = y_2V.$$

In the first case, $y_2^{-1}y_1 \in V \cap V^c \subseteq E$, by hypothesis, and $y_1E = y_2E$. In the second case, $y_2^{-1}y_1 \in cV \cap Vc = c(V \cap V^c) \subseteq E$ and, again, $y_1E = y_2E$.

The assumptions made about the group G imply that the group N acts freely on the tessellation T, i.e. no member of T, except the identity, fixes a point of the manifold X. Thus, following [8, Introduction], the manifold X^* inherits the geometric structure of X.

The rest of (a) is clear.

Finally, to relate this back to Section 2, particularly Proposition 2.2, the following is proved.

PROPOSITION 3.3.

(a) In the tessellation T^* the vertex, edge and face corresponding to the subgroup V, E, F form a flag and their stabilizers in G are V, E, F respectively.

(b) G acts sharply transitively on the flags of T^*.

PROOF.

(a) is clear.

(b) As the vertices of T^* correspond to the left cosets of V in G, G acts transitively on the vertices of T^*. The edges incident with the vertex corresponding to V correspond to the left cosets of E in VE; thus V acts transitively on these vertices. The stabilizer of the incident vertex-edge pair V, E is $V \cap E = \langle a \rangle$. As $a^2 = 1$ and the two faces incident with E are F and aF, it follows that G acts sharply transitively on the flags of X^*.

3.2 The group U_1.

Section 3.1 dealt with the surfaces associated with the group U. As the same methods apply to the group U_1, their consequences will be described here without justification.

In this subsection, G is a group generated by x and y satisfying, among other relations,

(I) $y^2 = 1$.

Rewrite x as ab and y as c. Let $V = \langle ab \rangle, E = \langle c \rangle$, $F = \langle c, c^{ab} \rangle$ and suppose that

(II) ab has finite order $p \geq 3$ and cc^{ab} has finite order $s \geq 2$,

(III) $V \cap V^c = V \cap E = V \cap F = 1, E \cap F = \langle c \rangle$.

Put $\qquad q = 2s.$

Let $\mathbf{K}$ be the subgroup of $[p, q]$ generated by $\mathbf{ab}$ and $\mathbf{c}$ and let $\phi : \mathbf{K} \to G$ be the homomorphism determined by the equations,

$$\phi(\mathbf{ab}) = ab, \quad \phi(\mathbf{c}) = c.$$

Let $\mathbf{N}$ be the kernel of ϕ.

In the tessellation T of the appropriate manifold M, let M^* be the orbit space of $\mathbf{N}$. Then M^* has the same geometric structure as M and G acts on it as a group of symmetries acting symmetrically on the tessellation induced by T. The vertices, edges and faces of the tessellation can be identified with the left cosets of V, E and F, respectively and the group G has two orbits on the flags of T^* and one orbit on its faces. The stabilizer of a vertex has two orbits on the faces incident with that vertex.

This corresponds to the second case of Theorem 2.5.

3.3 The group U_2.

In this subsection, G is a group generated by x and y satisfying, among other relations, (I) $y^2 = 1$.

Rewrite x as ab and y as ac. Let $V = \langle ab \rangle, E = \langle ac \rangle, F = \langle bc \rangle$, where $bc = (ab)^{-1}ac$, and suppose that

(II) V has finite order $p \geq 3$ and F has finite order $q \geq 3$.

(III) $V \cap V^c = V \cap E = V \cap F = E \cap F = 1$.

Let $\mathbf{K}$ be the subgroup of $[p, q]$ generated by $\mathbf{ab}$ and $\mathbf{ac}$ and let $\phi : \mathbf{K} \to G$ be the homomorphism determined by the equations

$$\phi(\mathbf{ab}) = ab, \quad \phi(\mathbf{ac}) = ac.$$

Let $\mathbf{N}$ be the kernel of ϕ.

The same general conclusions hold in this case as in the previous subsection. G has two orbits on the flags of the manifold M^* and one on the faces. The stabilizer of a vertex has one orbit on the faces incident with that vertex.

This corresponds to the first case of Theorem 2.5.

3.3 The group U_3.

In this subsection, G is a group generated by x, y, z satisfying, among other relations,

(I) $x^2 = y^2 = z^2 = 1$.

Rewrite x as b, y as b^a and z as c. Let $V = \langle b, b^a \rangle, E = \langle c \rangle, F = \langle b, c \rangle$, and suppose that

(II) bb^a has finite order $r \geq 2$ and bc has finite order $q \geq 3$,

(III) $V \cap V^c = V \cap E = 1, V \cap F = \langle b \rangle, E \cap F = \langle c \rangle$.

Put $p = 2r$.

Let $\mathbf{K}$ be the subgroup of $[p, q]$ generated by $\mathbf{b}, \mathbf{b}^a, \mathbf{c}$ and let $\phi : \mathbf{K} \to G$ be the homomorphism determined by the equations

$$\phi(\mathbf{b}) = b, \quad \phi(\mathbf{b}^a) = b^a, \quad \phi(\mathbf{c}) = c.$$

The same general conclusions hold as in the previous subsections. G has two orbits on the flags of the tessellation and one orbit on the faces. The stabilizer of each vertex has two orbits on the faces incident with it.

This corresponds to the first case of Theorem 2.6.

3.4 The group U_4.

In this subsection, G is a group generated by x, y, z satisfying, among other relations, (I) $x^2 = y^2 = z^2 = 1$.

Rewrite x as b, y as b^a and z as ac. Let $V = \langle b, b^a \rangle, E = \langle ac \rangle, F = \langle b, b^c \rangle$, where $b^c = (b^a)^{ac}$, and suppose that

(II) bb^a has order $r \geq 2$ and bb^c has order $s \geq 2$,

(III) $V \cap V^c = V \cap E = E \cap F = 1, V \cap F = \langle b \rangle$.

Put $p = 2r$, $q = 2s$.

Let $\mathbf{K}$ be the subgroup of $[p, q]$ generated by $\mathbf{b}, \mathbf{b}^a, \mathbf{ac}$ and let $\phi : \mathbf{K} \to G$ be the homomorphism determined by the equations

$$\phi(\mathbf{b}) = b, \quad \phi(\mathbf{b}^a) = b^a, \quad \phi(\mathbf{ac}) = ac.$$

The same general conclusions hold as in the previous subsections. G has two orbits on the flags of the tessellation, one orbit on the faces and the stabilizer of each vertex has two orbits on the faces incident with it.

This corresponds to the second case of Theorem 2.6.

3.5 The remaining case.

Among the possibilities in Section 2, just one remains, that of Theorem 2.4 where the group had two orbits on the faces. While the other results were proved in terms of geometric manifolds and their isometries, the converse result can only be proved in terms of topological manifolds and homeomorphisms.

In this subsection, G is a group generated by b, c, d satisfying, among other relations,

(I) $b^2 = c^2 = d^2 = 1$.

Let $V = \langle b, d\rangle, E = \langle c\rangle, F = \langle b, c\rangle, F' = \langle d, c\rangle$ and suppose that

(II) bd has finite order $p \geq 3$, bc has finite order $q \geq 3$ and cd has finite order $q' \geq 3$,

(III) $V \cap V^c = V \cap E = 1, V \cap F = \langle b\rangle, V \cap F' = \langle d\rangle, E \cap F = E \cap F' = \langle c\rangle$.

A graph Σ is defined as follows: its vertices are the left cosets of V in G, its edges are the left cosets of E in G and the vertices of the edge yE are the two left cosets, yV and ycV, of V in yEV. Thus, the vertex xV and the edge yE are incident if and only if $xV \subseteq yEV$ or, equivalently, $yE \subseteq xVE$. The group G acts faithfully as a group of automorphisms of Σ by the action

$$g : xV \to gxV, \;\; yE \to gyE$$

for each $g \in G$.

The aim now is to embed Σ in a surface S. This is done by identifying the cycles of Σ which will be the boundaries of the faces of S.

LEMMA 3.4.

For each coset zF, the edges in zF and the vertices in zFV form a cycle. Each edge of the graph lies in exactly one of these cycles.

PROOF:

Notice, first, that $E \subseteq F$ so that $zF = zFE$ and zF is a union of edges of Σ.

As $F = \langle b, c\rangle$ and $E = \langle c\rangle$, the cosets of E in F can be written as $(bc)^i E, i = 1, \ldots, q$. As $V \cap F = \langle b\rangle$, the cosets of V in FV can be written as $(bc)^i V, i = 1, \ldots, q$. The endpoints of the edge $(bc)^i E$ are $(bc)^i V$ and $(bc)^i cV = (bc)^{i-1} bccV = (bc)^{i-1} V$ as $c^2 = 1$ and $b \in V$. Hence, the edges in F and the vertices in FV form a cycle, a typical section of which is

$$\ldots (bc)^{i-1}E, (bc)^{i-1}V, (bc)^i E, (bc)^i V, \ldots$$

If $z \in G$, the automorphism of Σ induced by z shows that the same is true of the edges in zF and the vertices in zFV. Also, each edge yE lies in just one of these cycles, the one defined by the coset yF.

Now, consider each vertex xV as a point and each edge as a Euclidean line segment of length 1 joining its endpoints. A cycle formed from a coset is then a cycle of Euclidean line segmenets of length 1 and can be regarded as being the boundary of a regular Euclidean q-gon. Consider a member g of G. Its action on the graph can be regarded as a mapping of the vertices among themselves which extends to a Euclidean isometry of the edges and to a Euclidean isometry of all the faces.

This extends Σ to a union of regular Euclidean q-gons which meet only at their vertices. The q-gon corresponding to the coset zF will be called the face zF.

Now, as the properties of G are symmetric with respect to F and F', it is also possible to extend Σ to a union of regular Euclidean q'-gons which meet only at their vertices.

Taking both these extensions together gives a covering of Σ by regular q-gons and q'-gons on which G acts as a group of isometrics and each edge lies in the boundary of one polygon from each family. Thus, each point in the interior of an edge, has a neighbourhood which is isometric to the interior of a Euclidean circle. As the same is obviously true for the points in the interiors of the faces, it remains to investigate the vertices.

Consider the vertex V. As $V = \langle b, d\rangle$ and $V \cap E = 1$ the edges incident with V can be written as $xE, x \in V$ and there are $2p$ of them. Consider an edge xE with

$x \in V$. The edge xbE is also incident with the vertex V and xE, xbE are thus consecutive edges of the face xF. Similarly, the edges xE and xdE are consecutive edges of the face xF' The members of the dihedral group V can be written, in order, as

$$1, b, bd, bdb, bdbd, \dots$$

and, consequently, the faces

$$F, bF', bdF, bdbF', \dots$$

form a cycle of faces around the vertex V. Consequently the vertex V, and every other vertex, has a neighbourhood homeomorphic to the interior of a Euclidean circle. As the sum of the angles around this vertex may very well not be 360°, it can only be claimed now that the construction is a 2-dimensional topological manifold.

The action of G on this manifold is described by

$$g : xV \to gxV,\ \ yE \to gyE,\ \ zF \to gzF,\ \ zF' \to gzF'.$$

4. Some examples.

EXAMPLE 1:

The Coxeter groups $[p,q]$, with presentation

$$a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ac)^2 = 1$$

are finite when $[p,q] = [3,3],[3,4],[4,3],[3,5]$ and $[5,3]$ coresponding to the tessellation of the sphere by a tetrahedron, cube, octahedron, dodecahedron and icosahedron. The classical dualities among these objects come by interchanging p and q. Also, the symmetry between c and ac in the equations

$$a^2 = b^2 = c^2 = (ac)^2 = 1$$

leads to another duality. Writing e for ac gives the equations,

$$a^2 = b^2 = e^2 = (ae)^2 = 1$$

In the five groups $[p,q]$ mentioned above, the orders of be are $4,4,4,10,10$ respectively. Thus, these groups are homomorphic images of the Coxeter groups $[3,4]$, $[3,6]$, $[4,6]$, $[3,10]$, $[5,10]$ respectively. In this context, Coxeter and Moser, [3], call the corresponding 4-cycle, 6-cycle or 10-cycle a Petrie polygon: it has the property that all consecutive pairs of edges but no consecutive triples lie together in the boundary of one of the original faces. The corresponding embeddings are of the graph of a tetrahedron into a projective plane, of the graph of a cube into a torus, of the graph of an octahedron into a non-orientable surface of Euler characteristic -2, of the graph of a dodecahedron into an orientable surface of Euler characteristic -4 and of the graph of an icosahedron into a non-orientable surface of Euler characteristic -12.

EXAMPLE 2:

The permutations $a = (12)(45)$, $b = (13)(45)$, $c = (14)(25)$ generate the group A_5 and they satisfy the relations for the Coxeter group $[3,5]$. The resulting graph is the Petersen graph and it is embedded in the projective plane. This shows A_5 to be a homomorphic image of [3, 5] (which is, in fact, isomorphic to $A_5 \times C_2$), the Petersen graph to be an orbit space or quotient graph of the graph of the dodecahedron and the sphere to be a 2-fold covering of the projective plane.

If $e = ac$ then $be = (14253)$ and the construction above leads to another projective plane. The boundaries of the faces of the resulting tessellations in each of these projective planes are six 5-cycles and the two are interchanged by the automorphisms of the Petersen graph which lie in $S_5 - A_5$. The 5-cycles from one plane are homotopic to a projective line in the other.

EXAMPLE 3:

The permutations $a = (35)(46), b = (23)(57), c = (12)(34)(56)(78)$ generate a Sylow subgroup G of A_8 having order 64. As ab and bc both have order 4, the resulting graph has order 8 and valency 4 and the faces are quadrilaterals: it is a complete bipartite graph. The surface arising by interpreting G as satisfying the relations $a^2 = b^2 = c^2 = (ac)^2 = 1$ is a torus.

If $e = ac$ then $be = (1362)(4785)$ which is not a conjugate of $bc = (1342)(5678)$ in G. The result is two embeddings of the complete bipartite graph with 8 vertices into the torus. In each embedding the faces are squares and the boundaries of the faces of the other one form loops not homotopic to the identity. However, be and bc are conjugate under $(23)(46)(57)$ which lies in an extension of G to a Sylow 2-subgroup of G in S_8.

This group G is notable because its generators satisfy $a^2 = b^2 = c^2 = (ac)^2 = 1$ and each of the subgroup U_1 to U_7, defined as in Section 3, has index 2 in G. This need not happen, as is shown in the case of the group [3, 5] of the dodecahedron, which has just one subgroup of index 2.

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