MAXIMAL PLANAR GRAPHS AND DIAGONAL OPERATIONS

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For Professor Ralph G. Stanton, on his 65th birthday

ABSTRACT

Maximal planar graphs embed in the euclidean plane as plane triangulations. Two such plane triangulations are *equivalent* if there is a homeomorphism of the plane which maps the vertices, edges and faces of one onto the corresponding elements of the other. A *diagonal operation* on a plane triangulation deletes an appropriate edge and inserts a related edge. We prove constructively that, given any two plane embeddings of maximal planar graphs of order n, which need not be isomorphic, there is a sequence of diagonal operations which transforms the first into one equivalent to the second. This result was first conjectured by Foulds and Robinson (1979), who made substantial progress towards its demonstration. Lehel (1980) claimed a proof, but unfortunately his argument is erroneous. Ning (1987) first completed the proof with a result which fills the gap in the original work of Foulds and Robinson. The present paper gives a new self-contained proof, from a more geometrical viewpoint.

1. Introduction

Informally, a *planar* graph is an abstract graph which can be realized (drawn) in the euclidean plane without any pair of edges crossing. Any such drawing is a *plane embedding* of the graph. (A more formal treatment of the terminology is given in [1].) Plane embeddings of planar graphs are of considerable importance for applications, such as in printed circuits and in plans for the layout of facilities. (See [4] and [6], for example.)

A planar graph is *maximal* if it has a plane embedding in which every face is triangular, that is, bounded by a simple closed curve comprising three vertices and three edges. By inserting an appropriate number of additional edges in any plane embedding of a planar graph we can "refine" the embedding to a *plane triangulation*, that is, an embedding of a maximal planar graph. (We use these two terms synonymously in the present paper.) Two plane triangulations are *equivalent* if there is a homeomorphism of the plane onto itself which maps the first triangulation onto the second. In this paper we describe a simple operation, called a *diagonal operation*, which can be performed on any plane triangulation, and transforms it into another plane triangulation. We shall show that, up to equivalence, all plane triangulations of *order n* (that is, with n vertices) can be derived from any one of them by an appropriate sequence of diagonalizations.

In work which arose in connection with the facility layout problem, Foulds and Robinson [3] considered an abstract structure called a *deltahedron* which essentially corresponds to an equivalence class of spherical embeddings of a maximal planar graph. They introduced two types of operation, called *α–operations* and *β–operations*, and proved that together these operations suffice to transform any deltahedron of order n into any other. They conjectured that α–operations alone suffice to achieve such transformations: this was proved by Ning [8], who showed that any β–operation can be replaced by a suitable sequence of α–operations. The main result of the present paper essentially corresponds to this conjecture of Foulds and Robinson. (At appropriate points in the paper we shall further discuss details of the work of Foulds and Robinson, of Ning, and of Lehel [7], whose claimed proof of the conjecture of Foulds and Robinson contains a serious error which, to our knowledge, has not previously been reported.)

2. Braced and Unbraced Edges

Let T be any plane triangulation of order n; let V be its set of *vertices*, E its set of *edges*, and F its set of *faces*. It follows from Euler's Polyhedral Formula that $|E| = 3n$-6 and $|F| = 2n$-4. If T has an edge which has the vertices a and b as its endpoints, we shall denote it by ab, and write $ab \in E$. If no edge of T is incident with these two vertices we shall write $ab \notin E$; note that in this case ab does not represent an arc in the plane. We specify paths and cycles in T by listing the sequence of vertices encountered. In particular, a *cycle of order k* in T is a simple closed curve $a_1a_2...a_ka_1$ comprising k vertices and k edges of T.

Any edge $ab \in E$ is incident with precisely two faces, bounded by the 3–cycles $abca$ and $abda$, say. (We may suppose $n \geq 4$, so these two 3–cycles are distinct.) This distinguishes the vertices c and d among all those which are adjacent to both a and b. Let $V^{(2)}$ denote the set of all 2–sets of vertices; we define a map $\varepsilon: E \rightarrow V^{(2)}$ by $\varepsilon(ab) := \{c,d\}$.

The edge $ab \in E$ is *braced* if $cd \in E$, and cd is the *edge which braces* ab; if $cd \notin E$ the edge $ab \in E$ is *unbraced*. Up to equivalence, there is only one plane triangulation of order 4, and all of its edges are braced (Figure 1). However, this is an exceptional case, as shown by the following theorem (corresponding to the first part of Theorem 8 of [3]).

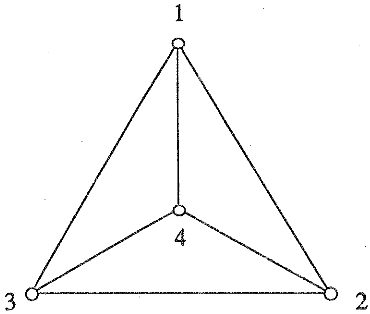

FIGURE 1. *A plane embedding of K_4, in which every edge is braced.*

THEOREM 1. *If T is a plane triangulation of order $n \geq 5$, each edge is either unbraced or the edge which braces it is unbraced.*

Proof. Let $ab \in E$ and $\{c,d\} := \varepsilon(ab)$. If ab is unbraced, it satisfies the theorem. Now suppose ab is braced. Then the 4–cycle $C := acbda$ separates the edges ab and cd, one in its interior and the other in its exterior, by the Jordan Curve Theorem, since no edges cross in T and the sets of endpoints $\{a,b\}$ and $\{c,d\}$ separate each other on C (Figure 2).

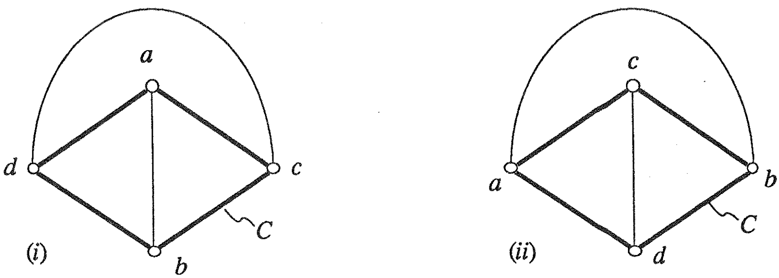

FIGURE 2. *Two possible subdrawings in which ab is braced by cd.*

Let $\{x,y\}:=\varepsilon(cd)$, so the two faces of T incident with cd have boundaries $cdxc$ and $cdyc$. Now $\{x,y\}$ and $\{a,b\}$ cannot coincide, for that would mean we have located all faces of T, contradicting the hypothesis that $n \geq 5$. Consequently $\{x,y\} \neq \{a,b\}$ so we may suppose without loss of generality that $a \notin \{x,y\}$. The 3–cycle $C':=acda$ separates the two faces incident with cd, so x and y lie on opposite sides of C'. By the Jordan Curve Theorem, any arc in the plane with x and y as endpoints would cross C', so $xy \notin E$. Hence cd is unbraced. ¤

COROLLARY 1. *Every plane triangulation of order $n \geq 5$ has at least one unbraced edge.*

COROLLARY 2. *Any maximal planar graph of order $n \geq 5$ has an induced subgraph $K_{1,1,2}$.*

Proof. Let G be a maximal planar graph of order $n \geq 5$, and let T be a planar embedding of G. If ab is an unbraced edge of T, and $\{c,d\}:=\varepsilon(ab)$, the subdrawing induced by the vertex set $\{a,b,c,d\}$ is an embedding of $K_{1,1,2}$. Hence the corresponding vertices of G span $K_{1,1,2}$ as an induced subgraph of G. ¤

The graph $K_{1,1,2}$ results from K_4 by deleting one edge. Because of its shape when embedded in the plane, we call it the *elementary theta graph.* (Subdivisions of it have been called general *theta graphs*; for example, see [4, p.66].) Thus, Corollary 2 may be paraphrased: *Every maximal planar graph of order $n \geq 5$ contains an induced elementary theta graph.*

A single edge can brace more than one edge of a plane triangulation T. For example, Figure 3(*i*) shows an order 6 triangulation in which 7 edges are braced, three of them braced by the single edge ab. It is also possible for a triangulation to have no braced edges, as shown by Figure 3(*ii*).

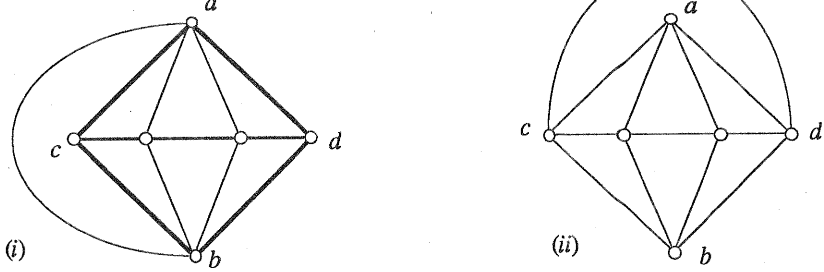

FIGURE 3. *Two plane triangulations of order 6, one with 7 braced edges and one with no braced edges.*

3. Diagonal Operations

Suppose the edge $ab \in E$ is unbraced in the plane triangulation T'. Let $\{c,d\} :=$ $\varepsilon(ab)$. The 4–cycle $C := acbda$ is the boundary of a quadrilateral region R which contains the edge ab. In general R may be the interior or the exterior of C: it depends on T which of these two possible regions actually contains ab.

The *diagonal operation* $\delta(ab)$ applied to T deletes the edge ab and inserts an arc with endpoints c and d in the region R (Figure 4), producing a plane triangulation T'. (Note that it is necessary for ab to be unbraced, otherwise $\delta(ab)$ would produce an embedding of a multigraph having two edges incident with c and d.) We regard the arcs ab and cd as the two possible *diagonals* of R. Any diagonal operation replaces one induced elementary theta graph by another.

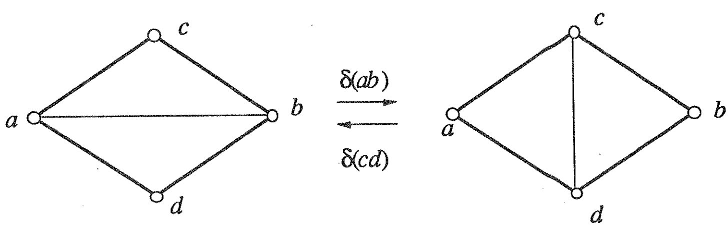

FIGURE 4. *A diagonal operation and its inverse.*

In T' the edge cd is unbraced because a and b are not adjacent in T', the edge ab having been deleted in the passage from T to T'. Thus the diagonal operation $\delta(cd)$ is possible on T', and transforms it back to T (Figure 4). In other words, the inverse of a diagonal operation is another diagonal operation.

A single diagonal operation can alter the status (braced or unbraced) of many edges in a triangulation. For example, the two triangulations in Figure 3 can be transformed into each other by a single diagonal operation.

Our diagonal operation, which replaces an unbraced edge, corresponds to the first case of the α–operation (α_1) of Foulds and Robinson [3]. The second case of their α–operation (α_2) corresponds to replacing a braced edge: if $ab \in E$, $cd \in E$ and $\varepsilon(ab) = \{c,d\}$, the sequence $\delta(cd)$, $\delta(ab)$ of two diagonal operations corresponds to the operation α_2 on the edge ab. It seems to us that the geometrical viewpoint, based on identifying braced edges and the edges which brace them, makes these operations more transparent.

4. Main Result

Because any plane triangulation of order $n \geq 5$ has at least one unbraced edge, it is natural to consider just which triangulations can be derived from it by iterating diagonal operations. We shall prove the following (corresponding to Conjecture 2 of [3]).

THEOREM 2. *Let A and B be any two plane triangulations of order $n \geq 5$. There exists a sequence of diagonal operations which transforms A into a triangulation equivalent to B.*

In other words, if we take any two maximal planar graphs of order n, whether or not they are isomorphic, and take any two plane embeddings of them, a suitable sequence of diagonal operations will transform the first embedding into an embedding equivalent to the second.

For example, consider the plane triangulations A and B of order 6 shown in Figure 5. If the sequence of four diagonal operations

$$\Delta := \delta(25), \delta(35), \delta(34), \delta(16)$$

equivalent to B', also shown in Figure 6. It can readily be seen that B' is equivalent to B, so the sequence Δ achieves the desired result.

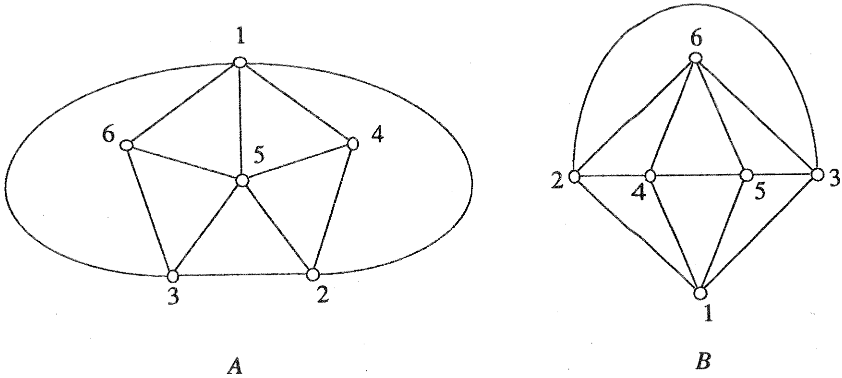

FIGURE 5. *Two plane triangulations of order* 6.

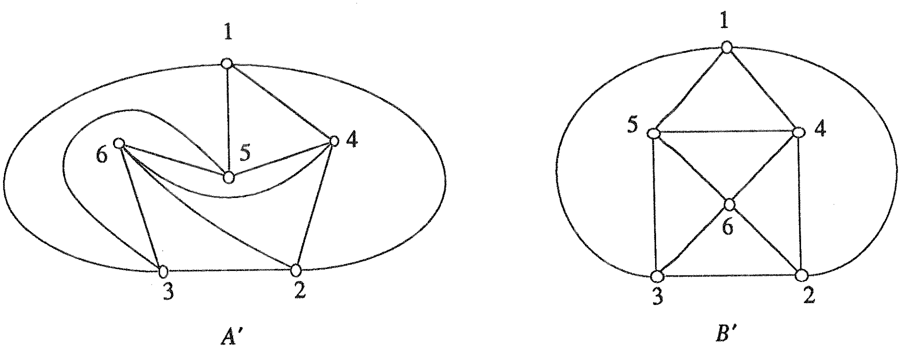

FIGURE 6. *Two equivalent plane triangulations.*

In order to prove Theorem 2 we need to develop some machinery. This is done in the next two sections, and the proof of Theorem 2 is then given in Section 7.

5. Cycles with Interior Vertices

Let T be any plane triangulation of order $n \geq 4$, and let $C := a_1a_2...a_ka_1$ be a k–cycle in T, that is, a simple closed curve comprising k vertices and k edges of T. The elements of T (that is, the vertices, edges and faces) which lie in the interior of C are the *C–interior* elements of T; we define the *C–exterior* elements of T similarly. A C–interior face is of *type t* ($t = 0,1,2$ or 3) if precisely t of its edges are contained in C. We further distinguish two kinds of type 1 face: such a face is of *type 1a* if it is incident with a C–interior vertex, and otherwise it is of *type 1b* (in which case it is incident with 3 vertices of C).

Let C be a k–cycle in T, and let u,v,w be three consecutive vertices of C. The *C–interior fan* at v is the set of all C–interior faces incident with v. Let $u_1,u_2,...,u_m$ be the sequence of vertices of T such that $u_1 := u$, $u_m := w$ and $vu_iu_{i+1}v$ (where $1 \leq i \leq m-1$) is the boundary of one of the faces in the C–interior fan at v. The *rim* of the fan is the path $u_1u_2...u_m$.

LEMMA 1. *Let C be a cycle in a plane triangulation T with at least one C–interior vertex. There is a sequence of diagonal operations which transforms only C–interior edges and faces and yields a triangulation T′ in which some C–interior face is of type 1a.*

Proof. A plane triangulation is necessarily connected, so at least one of the C–interior vertices must be adjacent to some vertex v of C. Let u,v,w be consecutive vertices of C and let $u_1u_2...u_m$ be the rim of the C–interior fan at v, with $u_1 := u$ and $u_m := w$. Since u_1 and u_m are vertices of C, there is a C–interior vertex u_k, with $1 < k < m$, such that each u_i with $1 \leq i < k$ is a vertex of C. Since u_{k-1} and v are distinct vertices of C, there are two paths in C with endpoints u_{k-1} and v: let P be the path which contains the vertex u (and not w). Let C' be the cycle derived from C by replacing the path P by the edge vu_{k-1}, that is, $C' := (C \setminus P) \cup vu_{k-1}$. Then u_k is a C'–interior vertex and u_{k-2} is a C'–exterior vertex, so u_k and u_{k-2} cannot be adjacent in T (Figure 7). Hence the edge vu_{k-1} is unbraced and we can perform the diagonal operation $\delta(vu_{k-1})$. It transforms two C–interior faces, deletes the C–interior edge vu_{k-1} and inserts a new C–interior edge u_ku_{k-2}. In the resulting triangulation the C–interior fan at v has rim $u_1u_2...u_{k-2}u_k...u_m$ and u_k is its first C–interior vertex. Iterating, we can perform the diagonal operations

$\delta(vu_{k-2})$, $\delta(vu_{k-3})$,..., $\delta(vu_2)$, resulting in a triangulation T'. This is identical with T outside and on C, and has a C–interior face bounded by uvu_ku, so is of type 1a, as required. ◻

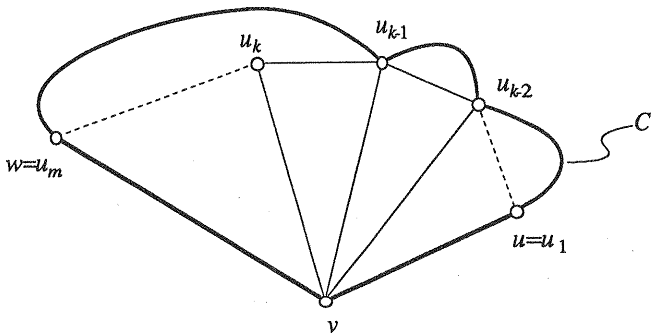

FIGURE 7. *The C–interior fan at v.*

LEMMA 2. *Let C be a cycle in a plane triangulation T with at least one C–interior face of type 1a. There is a sequence of diagonal operations which transforms only C–interior edges and faces and yields a triangulation T' in which a C–interior face of type 1a is incident with a prescribed edge of C.*

Proof. Let u,v,w be consecutive vertices of C such that the edge vw is incident with a C–interior face F of type 1a, with boundary $\partial F:=vwxv$. Then the C–interior fan at v has a C–interior vertex x in its rim, and the proof of Lemma 1 shows that a sequence of diagonal operations on C–interior edges of this fan will produce a triangulation T^* in which the C–interior face incident with uv is of type 1a. (Note that x is not necessarily one of its vertices.) Moreover T and T^* are identical outside and on C. In transforming T into T^*, we have arranged for the property of being incident with a C–interior face of type 1a to "migrate" from the edge vw of C to the adjacent edge uv of C. Iterating, we can arrive at a triangulation T' in which any chosen edge of C has this property. ◻

Lemmas 1 and 2 will be useful successively to modify two given plane triangulations by diagonal operations so that they correspond outside and on cycles of increasing order, with fewer and fewer interior vertices. In the next section we develop similar machinery for the case of cycles with no interior vertices.

6. Cycles without Interior Vertices

LEMMA 3. *Let C be a cycle of order $c>3$ in a plane triangulation T with no C–exterior vertices. If uvw is any path of two consecutive edges of C, and uw is not a C–exterior edge of T, there is a sequence of diagonal operations which transforms only C–interior edges and faces and yields a triangulation T' in which a C–interior face of type 2 is incident with uvw.*

Proof. Let $u_1u_2...u_m$ be the rim of the C–interior fan at v, with $u_1:=u$ and $u_m:=w$. (In this case all the rim vertices are vertices of C, though consecutive rim vertices are not necessarily consecutive vertices of C.) If $m=2$ then T already has the desired property, because the fan at v comprises just one face, which is necessarily of type 2 and incident with uvw. So suppose $m>2$.

If vu_{m-1} is unbraced, the diagonal operation $\delta(vu_{m-1})$ transforms two C–interior faces, deletes the C–interior edge vu_{m-1} and inserts a new C–interior edge u_mu_{m-2}. In the resulting triangulation the C–interior fan at v has a rim of lower order, $u_1u_2...u_{m-2}u_m$. We can iterate this procedure, performing the diagonal operations $\delta(vu_{m-2}),..., \delta(vu_2)$, provided none of the edges vu_i $(1 < i < m)$ is braced, and the resulting triangulation T' has a C–interior face of type 2 incident with uvw.

The only possible obstacle to these diagonal operations is that one of the radial edges of the fan is braced. So, without loss of generality, suppose vu_{m-1} is braced (Figure 8). Then wu_{m-2} must be a C–exterior edge; by hypothesis, this cannot be uw, so $m>3$. Consider the cycle $C':=vwu_{m-2}v$ in T. The vertex u_{m-1} is separated from all of $u_1,u_2,..., u_{m-3}$ by C' (though whether u_{m-1} is a C'–interior or C'–exterior vertex depends on the location of the edge wu_{m-2}). It follows that u_{m-1} is not adjacent in T to any of $u_1,u_2,..., u_{m-3}$. Hence vu_{m-2} is unbraced, and we can perform successively the diagonal operations $\delta(vu_{m-2}), \delta(vu_{m-3}),..., \delta(vu_2)$. In the resultant triangulation T^* the faces incident with vu_{m-1} are the C–interior faces with boundaries $vwu_{m-1}v$ and $uvu_{m-1}u$. The edge vu_{m-1} is unbraced in T^*, because by hypothesis there is no C–exterior edge uw, and a C–interior edge uw would have to cross vu_{m-1}. The diagonal operation $\delta(vu_{m-1})$ yields a triangulation T' containing a C–interior face with boundary

$uvwu$: this is the required face of type 2. Moreover, each diagonal operation has only modified C–interior faces and edges. ¤

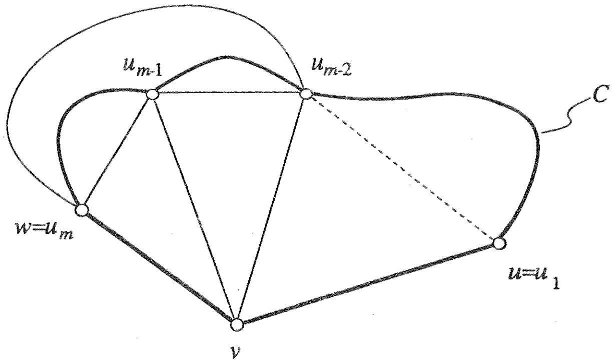

FIGURE 8. *The C–interior fan at v with vu_{m-1} braced.*

Lemma 3 will be useful successively to modify two given plane triangulations by diagonal operations so that they correspond outside and on cycles of decreasing order, with no interior vertices.

7. Proof of the Main Result

We now turn to the proof of the main result, as formulated in Section 4.

THEOREM 2. *Let A and B be any two plane triangulations of order $n \geq 5$. There exists a sequence of diagonal operations which transforms A into a triangulation equivalent to B.*

Proof. Any plane triangulation of order n has $2n$–4 faces. We shall carry out a sequence of $2n$–4 steps, in each of which we select one new face from a triangulation derived from A by a (possibly empty) sequence of diagonal operations, and a corresponding face from a triangulation derived from B by a (possibly empty) sequence of diagonal operations.

As first step, let F_1 be the external face of A, with (clockwise) boundary $\partial F_1 := C_1 := a_{11}a_{12}a_{13}a_{11}$, and correspondingly let F_1' be the external face of B, with (clockwise) boundary $\partial F_1' := C_1' := b_{11}b_{12}b_{13}b_{11}$. Clearly there is a homeomorphism θ_1: $F_1 \rightarrow F_1'$ such that $\theta_1(F_1)=F_1'$, $\theta_1(\partial F_1)=\partial F_1'$ and $\theta_1(a_{1s})=b_{1s}$ $(s=1,2,3)$.

As second step, choose F_2 to be the C_1–interior face of A incident with $a_{11}a_{12}$. Its (clockwise) boundary is $\partial F_2 := a_{21}a_{22}a_{23}a_{21}$, where $a_{21}=a_{11}$, $a_{22}=a_{12}$ and a_{23} is necessarily a C_1–interior vertex because $n > 3$ (Figure 9). Let F_2' be the corresponding face of B, with (clockwise) boundary $\partial F_2' := b_{21}b_{22}b_{23}b_{21}$, where $b_{21}=b_{11}$, $b_{22}=b_{12}$ and b_{23} is a C_1'–interior vertex. It is clear there is a homeomorphism $\theta_2 : F_2 \to F_2'$ such that

$\theta_2(F_2)=F_2'$, $\theta_2(\partial F_2)=\partial F_2'$ and $\theta_2(a_{2s})=b_{2s}$ $(s=1,2,3)$.

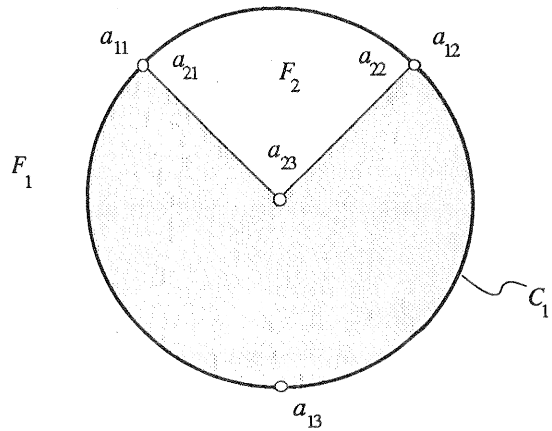

FIGURE 9. *Faces F_1 and F_2 of triangulation A.*

Moreover, θ_2 can be chosen to agree with θ_1 on the edge $a_{11}a_{12}$, that is $\theta_1|F_1 \cap F_2 = \theta_2|F_1 \cap F_2$.

Let $C_2 := a_{11}a_{23}a_{12}a_{13}a_{11}$. The (clockwise) boundary of the region $R_2 := F_1 \cup F_2$ is the 4–cycle $\partial R_2 = C_2$, and the C_2–exterior faces of A are F_1 and F_2. Similarly the region $R_2' := F_1' \cup F_2'$ has boundary $C_2' := \partial R_2'$ in B, and the C_2'–exterior faces of B are F_1' and F_2'. There is a homeomorphism $\Theta_2 : R_2 \to R_2'$, coinciding with θ_1 on F_1 and with θ_2 on F_2, which maps the C_2–exterior elements of A onto the C_2'–exterior elements of B, and the elements of C_2 onto the elements of C_2'.

Now suppose we have just completed $m \geq 2$ steps. Then we have specified (possibly empty) sequences $\delta_1, \delta_2, \ldots, \delta_{r(m)}$ and $\delta_1', \delta_2', \ldots, \delta'_{s(m)}$ of diagonal operations, and corresponding order n plane triangulations $A_0 := A, A_1, A_2, \ldots, A_{r(m)}$ and $B_0 := B, B_1, B_2, \ldots, B_{s(m)}$, such that $A_i := \delta_i(A_{i-1})$ for $1 \leq i < r(m)$, and $B_j := \delta_j'(B_{j-1})$ for $1 \leq j < s(m)$. (Here $r(m)$ and $s(m)$ are appropriate nonnegative increasing integer-valued functions of m.) Further, we may suppose that for $1 \leq k \leq m$ we have specified a sequence of m faces F_k, where each F_k belongs to some A_i, and has (clockwise) boundary $\partial F_k := a_{k1}a_{k2}a_{k3}a_{k1}$; a sequence of m faces F_k', where each F_k' belongs to some B_j, and has (clockwise) boundary $\partial F_k' := b_{k1}b_{k2}b_{k3}b_{k1}$; and a sequence of m homeomorphisms $\theta_k: F_k \rightarrow F_k'$. We also suppose that these sequences have the following properties:

(1) The union of the faces F_k $(1 \leq k \leq m)$ is a region R_m with boundary $C_m := \partial R_m$ which is a cycle in $A_{r(m)}$, and the m faces F_k are all the C_m–exterior faces of $A_{r(m)}$. Correspondingly, the union of the faces F_k' is a region R_m' with boundary C_m' which is a cycle in $B_{s(m)}$; the m faces F_k' are all the C_m'–exterior faces of $B_{s(m)}$.

(2) The homeomorphisms θ_k $(1 \leq k \leq m)$ satisfy $\theta_k(F_k) = F_k'$, $\theta_k(\partial F_k) = \partial F_k'$, $\theta_k(a_{ks}) = b_{ks}$ $(s=1,2,3)$ and $\theta_i | F_i \cap F_j = \theta_j | F_i \cap F_j$ for $1 \leq i < j \leq m$.

It follows that there exists a homeomorphism $\Theta_m: R_m \rightarrow R_m'$ such that $\Theta_m | F_k = \theta_k$ $(1 \leq k \leq m)$ and Θ_m maps the elements of $A_{r(m)}$ comprising C_m and its exterior onto the corresponding elements of $B_{s(m)}$, which comprise C_m' and its exterior.

Assuming $m < 2n-4$, we now show how to take the next step. (For notational convenience, let $M := m+1$.) If C_m has any interior vertices, so also does C_m'. Let uv be any edge of C_m, and let $u'v' := \Theta_m(uv)$ be the corresponding edge of C_m'. By Lemmas 1 and 2, there is a sequence $\delta_{r(m)+1}, \ldots, \delta_{r(M)}$ of diagonal operations which transforms

$A_{r(m)}$ into a plane triangulation $A_{r(M)}$ which is identical with $A_{r(m)}$ outside and on C_m, and has C_m–interior face of type 1a incident with uv. Select this face as F_M; its boundary is $\partial F_M := a_{M1}a_{M2}a_{M3}a_{M1}$, where $a_{M1}=u$, $a_{M2}=v$ and a_{M3} is a C_m–interior vertex. Similarly, there is a sequence $\delta'_{s(m)+1},\ldots, \delta'_{s(M)}$ of diagonal operations which transforms $B_{s(m)}$ into a plane triangulation $B_{s(M)}$, differing from $B_{s(m)}$ only on C_m'–internal elements, and having a C_m'–interior face of type 1a incident with $u'v'$. Select this face as F_M'; its boundary is $\partial F_M' := b_{M1}b_{M2}b_{M3}b_{M1}$, where $b_{M1}=u'$, $b_{M2}=v'$ and b_{M3} is a C_m'–interior vertex. Clearly there exists a homeomorphism θ_M: $F_M \rightarrow F_M'$ with the properties $\theta_M(F_M) = F_M'$, $\theta_M(\partial F_M) = \partial F_M'$, $\theta_M(a_{M3}) = b_{M3}$ and $\theta_M|uv = \Theta_m|uv$. Define $R_M := R_m \cup F_M$ and $R_M' := R_m' \cup F_M'$. These regions have boundaries $C_M := \partial R_M = (C_m \setminus uv) \cup ua_{M3}v$ and $C_M' := \partial R_M' = (C_m' \setminus u'v') \cup u'b_{M3}v'$ which are cycles. Moreover, C_M has fewer interior vertices than C_m. We let Θ_M: $R_M \rightarrow R_M'$ be the homeomorphism defined by $\Theta_M|R_m = \Theta_m$ and $\Theta_M|F_M = \theta_M$.

If C_m has no interior vertices, neither does C_m'. All C_m'–interior faces of $B_{s(m)}$ are triangles, incident with three vertices of C_m'. Let k be the order of C_m'. Euler's Polyhedral Formula shows that such a configuration comprises k–2 triangles. If k>3, none of these triangles is of type 3, so at least two of them must be of type 2, by the Pigeonhole Principle. Let uvw be a path of two consecutive edges of C_m, such that $\Theta_m(uvw) = u'v'w'$ is incident with a C_m'–interior face of type 2. Select this face as F_M'. Its boundary is $\partial F_M' = u'v'w'u'$. Since $u'w'$ is a C_m'–interior edge, and $\Theta_m(R_m) = R_m'$, it follows that uw is not a C_m–exterior edge of $A_{r(m)}$. Hence Lemma 3 applies. There is a sequence $\delta_{r(m)+1},\ldots, \delta_{r(M)}$ of diagonal operations which yields a triangulation $A_{r(M)}$ that differs from $A_{r(m)}$ only on C_m–interior elements and has a C_m–interior face of type 2 incident with uvw. Select this face as F_M; its boundary is $\partial F_M = uvwu$. No diagonal operations were required by $B_{s(m)}$, so we define $s(M) := s(m)$. Also let $R_M := R_m \cup F_M$ and $R_M' := R_m' \cup F_M'$. These regions have boundaries $C_M := \partial R_M = (C_m \setminus uvw) \cup uw$ and $C_M' := \partial R_M' = (C_m' \setminus u'v'w') \cup u'w'$ which are cycles. Moreover, these cycles have no interior vertices and have smaller order than C_m and C_m'. It is clear that there is a homeomorphism $\theta_M : F_M \to F_M'$ such that $\theta_M(F_M) = F_M'$, $\theta_M(\partial F_M) = \partial F_M'$, and $\theta_M|uvw = \Theta_m|uvw$. Define $\Theta_M : R_M \to R_M'$ as before.

Finally, if C_m has no interior vertices and has order k=3, say C_m:= $uvwu$, there can remain only one C_m–interior face of $A_{r(m)}$, which we select as F_M, and one C_m'–interior face of $B_{s(m)}$, which we select as F_M'. Hence $M = 2n–4$. These two faces are of type 3, and $\partial F_M = C_m$, $\partial F_M' = C_m'$. It is clear that there is a homeomorphism $\theta_M : F_M'$ such that $\theta_M(F_M) = F_M'$ and $\theta_M|uvwu = \Theta_m|uvwu$. Now $R_M := R_m \cup F_M$ is the whole euclidean plane E^2, as is $R_M' := R_m' \cup F_M'$. Define $\Theta_M : E^2 \to E^2$ as before. Now $\Theta_M = \Theta_{2n-4}$ is a homeomorphism of E^2 which shows that the triangulations $A' := A_{r(2n-4)}$ and $B' := B_{s(2n-4)}$ are equivalent. Hence, if we perform any sequence of corresponding diagonal operations on A' and B', the resulting triangulations will still be equivalent. In particular, if $\delta_{r(2n-4)+i}$ is the inverse of $\delta'_{s(2n-4)-i+1}$, performing the operations for i=1,2,..., $s(2n–4)$ in turn transforms B' back into B; let A^* result from A' by the same sequence of operations. Then the sequence δ_i, for $1 \le i \le r(2n-4) + s(2n-4)$, transforms A into A^*, and we have just noted that this is equivalent to B, as required. ¤

8. Ning's Paper

Foulds and Robinson [3] essentially showed that, up to equivalence, any plane triangulation can be transformed into any other of the same order by a sequence of α–operations and β–operations. The β–operation can be regarded as triangulating one triangular face and deleting the triangulation of another triangulated face. Ning [8] notes that if the two triangular faces are adjacent, the β–operation can be achieved as a second case α–operation; furthermore, the dual graph of a plane triangulation is connected, so a suitable sequence of α–operations will achieve the same transformation as any

β–operation. This idea fills the gap in the arguments of Foulds and Robinson, so completes the proof of their Conjecture 2.

9. Lehel's Paper

Lehel published a Communication [7] which claimed to prove Conjecture 2 of Foulds and Robinson [3]. His definition of an α–operation precisely corresponds to our diagonal operation, and corresponds to what Foulds and Robinson call the first case of an α–operation. (A careful reading of his definition reveals several minor mistakes, but these are easily corrected.)

Lehel's paper proceeds to outline a proof that a sequence of α–operations suffices to transform any deltahedron into any other on the same vertex set. However, the first assertion of the outlined proof is false, and this invalidates the whole proof. To explain the error, we recast the assertion in terms of the terminology and viewpoint of the present paper. In a plane triangulation T of order n, let C be the cycle spanning the vertices adjacent to the vertex u, and assume that there is at least one vertex v which is not adjacent to u. Lehel asserts that for a suitable choice of v there is an edge ab which is common to a face incident with u and a face incident with v, that is, there is an unbraced edge ab with $\varepsilon(ab) = \{u,v\}$. (The intention is to perform the diagonal operation $\delta(ab)$ to produce a plane triangulation in which v is adjacent to u, so that iteration would yield a triangulation in which all vertices are adjacent to u.) However, Figure 10 shows that such an edge ab need not exist. In fact, one diagonal operation applied to Figure 10(i) suffices to convert one of the edges of C to the status required for ab, but Figure 10(ii) requires a minimum of two diagonal operations to achieve an appropriate transformation, and clearly the examples can be extended to show that any finite number of diagonal operations may be required to achieve a plane triangulation with an appropriate edge ab. Our results in Section 5 show that in fact a sequence of diagonal operations will always

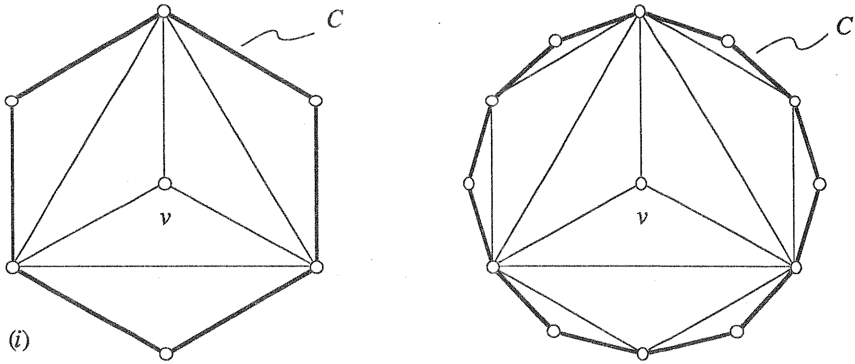

FIGURE 10. *Two triangulated cycles with one interior vertex which do not satisfy Lehel's requirement.*

yield a plane triangulation with an appropriate edge *ab*, so Lehel's ideas *can* be rescued with additional machinery from Section 5.

One of the attractive ideas in Lehel's program is the transformation of any plane triangulation of order $n \geq 5$ to a canonical triangulation, with two vertices of degree $n-1$, two vertices of degree 3, and all others of degree 4. In our proof we have avoided passing through a prescribed canonical form. This sacrifices some conceptual simplicity for the sake of a constructive argument which readily adapts to a practical algorithm with probably fewer steps needed to complete the transformation.

10. Closing Remarks

In a practical context it would be important to transform one plane triangulation into another by as few diagonal operations as possible. We hope to discuss in a subsequent paper the problem of choosing efficient sequences of diagonal operations.

Another matter which arises naturally from the present paper concerns the detailed structure of braced and unbraced edges in a plane triangulation. For example: How high a proportion of the edges can be braced? What restrictions exist on the relative locations of braced edges? We discuss such questions in a forthcoming paper [2].

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