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Magic labellings of infinite graphs

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Abstract

There are many results on edge-magic, and vertex-magic, labellings of finite graphs. Here we consider magic labellings of countably infinite graphs over abelian groups. We also give an example of a finite connected graph that is edge-magic over one, but not over all, abelian groups of the appropriate order.

1 Introduction

Throughout this paper, G is a graph, and V and E are the sets of vertices and edges of G, with cardinalities $|V|$ and $|E|$, respectively. We shall assume throughout that $E \neq \emptyset$, and we let $|G| = |E|+|V|$. Briefly, a labelling λ of a finite graph is a function that attaches distinct integers to the edges and (or) the vertices of the graph, and different algebraic constraints on λ correspond to different types of labelling. There are many ways to label graphs; indeed, according to Gallian [2], "over the past three decades in excess of 500 papers have spawned a bewildering array of graph labeling methods".

Let $E(v)$ be the set of edges that have a vertex v as an end-point, and $V(e)$ the pair of distinct vertices that are the end-points of the edge e (for simplicity, we shall exclude graphs with loops, that is with edges whose two end-points coincide). Then an *edge-magic total labelling* of a graph G is a bijection λ from $E\cup V$ to $\{1, 2, \dots, |G|\}$ such that the *weight*

$$\omega(e) = \lambda(e) + \sum_{v\in V(e)} \lambda(v) \tag{1.1}$$

of an edge e is the same for all edges. Likewise, λ is a *vertex-magic total labelling* if the *weight*

$$\omega(v) = \lambda(v) + \sum_{e\in E(v)} \lambda(e) \tag{1.2}$$

of a vertex v is the same for all vertices. In each case, the constant value of the weight is called the *magic constant* of the labelling. The notion of an edge-magic total labelling was introduced by Kotzig and Rosa in 1970 [4], and the vertex-magic total labellings were introduced in 1999 by MacDougall, Miller, Slamin and Wallis [5]. For a recent survey on edge-magic, and vertex-magic, finite graphs, see [8].

A labelling of a graph G attaches values to the vertices and edges of G, and these values lie in some set $\mathcal{L}$ which we call the set of *labels*. It is clear that in order to define any of the standard labellings of graphs it is only necessary to assume that $\mathcal{L}$ supports some associative and commutative binary operation, and the insistence that the set of labels should be $\{1, \ldots, |G|\}$ seems to be more in recognition of their origins in magic squares than for any deeper mathematical reason. Moreover, any attempt to generalize these ideas to the labelling of infinite graphs by the positive integers will necessarily fail. For this reason alone, it seems preferable to seek other sets of labels. Diana Combe has recently introduced and studied labellings over finite abelian groups [1], and once this step has been taken it is natural to seek labellings of infinite graphs with labels in an infinite group. As far as I know, such a study has not been made before, and in this paper we shall undertake a preliminary investigation into magic labellings of infinite graphs over abelian groups. The natural starting point seems to be to try to label countably infinite graphs by the additive group $\mathbb{Z}$ of integers, and most of this paper is on this topic. It is worth noting that if a finite graph G has an edge-magic labelling with labels $1, \ldots, |G|$, then it automatically has an edge-magic labelling with labels in the cyclic group of order $|G|$; thus, in this sense, the earlier ideas of edge-magic labellings are subordinate to the labellings over abelian groups. However, many new questions arise, and the labelling of a graph over a group seems to depend on some type of compatibility between the combinatorics of the graph and the relations within the group. For these (and other) reasons, we suggest that the labelling of graphs by abelian groups is more natural than the more traditional forms of labelling. Moreover, by studying labellings over abelian groups we may possibly gain more insight into the number theoretic criteria that occur in certain results in this topic.

In Section 2 we introduce some terminology, assumptions and definitions. In order to give the reader some familiarity with these ideas we devote Section 3 to some examples of labellings (and non-labellings) of finite graphs over finite groups. In Section 4 we discuss infinite graphs. We are going to construct edge-magic labellings of infinite graphs over the additive group $\mathbb{Z}$ of the integers, and the technique that we use is (very roughly) analogous to the construction of harmonic functions on a non-compact surface. This construction is based on an exhaustion of a non-compact surface by an increasing sequence of compact sets; in our analogy, finite graphs correspond to compact sets, and we shall view an infinite graph as the limit of an increasing sequence of finite graphs. If we pursue this analogy still further, it suggests that the magic-labelling of infinite graphs may actually be easier than that of finite graphs, and this is consistent with our results. For example, there is an unsolved conjecture that all finite trees can be given an edge-magic total labelling [2], [6]. We shall show (among other things) that *every countably infinite tree that contains an infinite path supports an edge-magic total labelling over* $\mathbb{Z}$.

Section 5 contains two examples. In the first of these we show that the semi-infinite path supports uncountably many bijective edge-magic $\mathbb{Z}$-labellings, and this example embodies (in its simplest form) the main idea in this paper. The proof of our main result on edge-magic infinite graphs is given in Section 6. This paper presents a technique rather than a complete, formal result, and in Section 7 we illustrate this technique by giving a proof of the existence of edge-magic labellings of certain countably infinite trees. We extend these ideas still further in Section 8, and our investigations raise several open questions which we discuss in Section 9. Finally, in Section 10, we end with a brief discussion of vertex-magic countably infinite graphs. I am grateful to Diana Combe for introducing me to this subject.

2 Definitions and assumptions

Throughout this paper we shall restrict ourselves to those graphs that have at most one edge joining any two vertices of G (that is, graphs with no multi-edges), and we have already excluded graphs with loops. If the vertices u and v are the end-points of a single (unique) edge, this edge is denoted by $[u, v]$ or $[v, u]$. It is convenient to modify the usual terminology for labellings. Our definitions are based on the standard definitions, but we find it more convenient to address each of the properties of the labelling map λ separately. Let G be a graph (possibly infinite, and not necessarily connected), and Γ an abelian group. A *Γ-labelling of G*, or a *labelling of G over Γ*, is simply a map $\lambda : E \cup V \to \Gamma$, so it is clear what is meant by an *injective*, *surjective*, or *bijective*, Γ-labelling of G. The definitions of *edge-magic*, and *vertex-magic*, Γ-labellings of a graph are exactly as described above in (1.1) and (1.2), and these apply whether or not λ is injective, surjective or bijective. Here, we are considering Γ as an additive group, but we can (and shall) also consider Γ to be a multiplicative group if we make the obvious modifications to (1.1) and (1.2). Clearly, the existence (or otherwise) of an edge-magic, or a vertex-magic, labelling of a graph depends only on the isomorphism class of the group.

Given a general graph G, the sets V or E may be uncountable. Naturally, we say that *G is finite* if and only if $E \cup V$ is finite, and that *G is countably infinite* if and only if $E \cup V$ is countably infinite. Clearly, for a bijective Γ-labelling of a graph G to exist, Γ and G must have the same cardinality. The main thrust of this paper is to find those countable graphs that have a bijective edge-magic $\mathbb{Z}$-labelling.

The degree of a vertex v is $|E(v)|$, and we say that the vertex has degree $\aleph_0$ when this set is countably infinite. If G is countable then E is countable, so that the degree of each vertex is at most $\aleph_0$. Conversely, if each degree is at most $\aleph_0$, and if G is connected, then, as every vertex can be reached from any given vertex by a finite chain of edges, we see that V is countable. Thus *a connected graph G with no multi-edges is countable if and only if the degree of each vertex is at most $\aleph_0$*. It is known that a connected graph G is infinite if and only if it contains a semi-infinite path, a doubly-infinite path, or an infinite star as a subgraph ([7], p.130). In particular, an infinite connected graph in which every vertex has finite degree contains a semi-infinite path.

3 Finite graphs

Throughout this section we shall only consider finite graphs. We shall use Γ_n for the generic cyclic group of order n. If a finite graph G has an edge-magic total labelling, then this labelling induces an edge-magic labelling over the cyclic group Γ_n, where $|G| = n$. In particular, if G does not have a Γ_n-labelling, then it does not have an edge-magic total labelling. We now give some examples.

Example 3.1 This example is taken from [1]. For every n, $n \geq 1$, the graph that is *the star with n spokes is edge-magic over any group of order* $2n+1$. Indeed, a group of odd order cannot contain an element of order two (by Lagrange's Theorem) so a group of order $2n+1$ can be written as $\{e, g_1, g_1^{-1}, \ldots, g_n, g_n^{-1}\}$. We may take G to have vertices at 0 and at the n-th roots of unity ω^k, $k = 1, \ldots, n$ (in the complex plane), and edges $[0, \omega^k]$, and the labelling is defined by $\lambda(0) = e$, $\lambda(\omega_k) = g_k$ and $\lambda([0, \omega^k]) = g_k^{-1}$. □

Example 3.2 The complete graph K_4 (the skeleton of a tetrahedron) *is not edge-magic over the cyclic group of order ten.* Let Γ be the cyclic group of order ten, and suppose that K_4 has a labelling over Γ_{10} that is generated, say, by g. We let the magic constant be g^t, and we suppose that the product of the vertex labels is g^s. The product over all group elements is g^5 so, by considering the product of the weights of all six edges, we see that $g^{6t} = g^{2s+5}$. As $6t - (2s+5)$ is odd, $6t$ is not congruent to $2s+5$ modulo 10, so that such a labelling is not possible. This shows, incidentally, that K_4 has no edge-magic total labelling in the earlier sense (see [8], p.19). □

Example 3.3 The graph $P_2 \cup P_2$ (with $V = \{0, 1, 2, 3\}$ and $E = \{[0, 1], [2, 3]\}$) *is not edge-magic over the cyclic group of order six.* Indeed, if this graph is edge-magic over Γ_6, then there are choices of a_j such that $\{a_1, \ldots, a_6\} = \{0, 1, 2, 3, 4, 5\}$ and $a_1 + a_2 + a_3 = a_4 + a_5 + a_6$ modulo 6. This cannot be so as one side of this equation is odd, and the other even. □

Example 3.4 The complete graph K_3 with $V = \{u, v, w\}$ and $E = \{[u, v], [v, w], [w, u]\}$ has an edge-magic total labelling λ with $\lambda(u) = 1$, $\lambda(v) = 2$, $\lambda(w) = 3$ and magic constant 9. □

Example 3.5 This is an interesting example of a graph G that is edge-magic over some, but not all, abelian groups of order eight. The graph G is illustrated in Figure 1 with the vertices labelled a, b, c and d, and the edges labelled p, q, r and s.

Now there are three abelian groups of order eight, namely Γ_8, $\Gamma_2 \times \Gamma_4$, and $\Gamma_2 \times \Gamma_2 \times \Gamma_2$, and we shall show that G is edge-magic over the first two of these groups, but not over the third. First, G has an edge-magic total labelling with magic constant 11, namely $(a, b, c, d) = (3, 1, 2, 4)$ and $(p, q, r, s) = (7, 8, 5, 6)$. It follows from this that G has an edge-magic labelling over Γ_8.

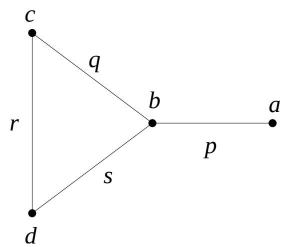

Figure 1

Next, we write $\Gamma_2 \times \Gamma_4$ in the form $\{(-1^u, i^v : u = 0, 1;\ v = 0, 1, 2, 3\}$. Then there is an edge-magic labelling over this group with labels given by $a = (1, -1)$, $b = (1, 1)$, $c = (1, i)$, $d = (1, -i)$, and edge-magic constant $(-1, 1)$. Finally, we show that G does not have an edge-magic labelling $\Gamma_2 \times \Gamma_2 \times \Gamma_2$. Suppose that such a labelling exists with magic constant k. Then, by considering the product of the four edge weights, and using the labels in Figure 1, we have

$$k^4 = (apb)(bqc)(crd)(dsb) = (abcdpqrs)b^2cd.$$

Now every element in $\Gamma_2 \times \Gamma_2 \times \Gamma_2$ has order two, and the product of all group elements is the identity e. Thus $cd = e$ and hence $c = d$, which is a contradiction. $\square$

It would be particularly interesting to find a finite graph that has an edge-magic labelling over a non-cyclic group, but not over the cyclic group, of the appropriate order (and hence no edge-magic total labelling). We have noted that if G has an edge-magic total labelling then it has an edge-magic labelling over the appropriate cyclic group. Is the converse true?

Some of the standard results on edge-magic total labellings hold in the context of labelling by groups (see [1]), and we give just one (new) example of this here. Let G be a finite graph that is edge-magic over the cyclic group Γ_n, where $|V| = v$, $|E| = e$ and $e + v = n$. Suppose that g generates Γ_n, and let the magic constant k be g^t. Next, let the j-th vertex have degree d_j and label g^{a_j}. Then, by considering the product of the weights of all edges, we have

$$(g^t)^e = \prod_j g_j^{(d_j-1)a_j} g^{0+1+\cdots+(n-1)}.$$

This implies that

$$te \equiv \sum_j (d_j - 1)a_j + \tfrac{1}{2}n(n-1)$$

modulo n. Now suppose that each d_j is odd and that e and v are even. Then n $(= e + v)$ is even, and $\frac{1}{2}n(n-1)$ is even. If we let $n = 2q$ we see that q must be even and we have proved the following result (see [8], Theorem 2.1, for a result of this type with labels $1, \dots, n$).

Theorem 3.6. *Suppose that G is a finite graph, that $|E|$ and $|V|$ are even, and that every vertex has odd degree. If G has an edge-magic labelling over a cyclic group then $|G|$ is divisible by 4.*

This result provides many examples of graphs that have no edge-magic labellings over cyclic groups. It shows, for example, that the union of a p-star and a q-star has no such labelling if $p+q$ is even. It also shows that the skeletal graph of a pyramid whose base is a q-gon, where $q = 4m+3$ for some m, has no such labellings. These applications generalize Examples 3.2 and 3.3.

Finally, let G be a graph with $|G| = 2^m$, where $m \geq 3$. We shall suppose that $|E|$ is even, and that among the vertices of G there are exactly two, say v_1 and v_2, of even degree. Now suppose that G has an edge-magic labelling, with magic constant k, over the m-fold product $\Gamma = \Gamma_2 \times \cdots \times \Gamma_2$ of cyclic groups of order two. Then, by considering the product of the weights of all edges, and using the fact that every element of Γ has order two, we see that

$$k^{|E|} = \Big(\prod_{g\in\Gamma} g\Big)\lambda(v_1)\lambda(v_2).$$

As $|E|$ is even, $k^{|E|}$ is the identity e of Γ. Next, if write the elements of Γ in the form $(\varepsilon_1, \ldots, \varepsilon_m)$, where each ε_j is ± 1, it is evident that the product of all elements in Γ is e. Thus

$$\lambda(v_1) = \lambda(v_2)^{-1} = \lambda(v_2),$$

which is false. Thus we have proved the following result.

Theorem 3.7. *Suppose that G is a finite graph with $|G| = 2^m$, where $m \geq 3$, and that $|E|$ even. Suppose also that among the vertices of G, exactly two have even degree. Then G does not support an edge-magic labelling over the m-fold product of cyclic groups of order two.*

It is easy to construct, for each $m \geq 4$, a graph of the type described in Theorem 3.7. Let $r = 2^{m-2}$, so that $r \geq 4$, and form a regular polygon with r sides, and r vertices, say $v_1, \ldots, v_r$. At each vertex v_j we add a side, say $[v_j, v_j']$; this gives a 'sun' with r rays $[v_j, v_j']$. Now remove the ray $[v_1, v_1']$, and adjoin it to the vertex v_2' and let the resulting graph be G. Then $|G| = 2^m$, $|E| = 2r$, and v_1 and v_2' are the only vertices of even degree so that Theorem 3.7 is applicable to G.

In conclusion, we remark that the existence, or non-existence, of edge-magic labellings of a graph G over a group Γ seems somehow to reflect the compatibility (or otherwise) of the combinatorics of G with the relations that exist in Γ.

4 Infinite graphs

We want to consider an infinite graph as the limit of an increasing sequence of finite graphs. Consider any sequence G_n of graphs, and let V_n and E_n be the sets of vertices and edges, respectively, of G_n. The sequence G_n is *increasing* if, for each n, $V_n \subset V_{n+1}$ and $E_n \subset E_{n+1}$. If G_n is increasing, we let G_∞ be the graph whose sets of vertices and edges are $\cup_n V_n$, and $\cup_n E_n$, respectively, and we write $\lim_n G_n$ for G_∞. It is clear that if each G_n is countable, connected and without multi-edges, then so is $\lim_n G_n$. The essence of our construction is to start with an infinite sequence $G_1', G_2', \ldots$, of finite graphs, join them sequentially to form an increasing sequence

$\{G_n\}$ of finite graphs, and then obtain a bijective edge-magic $\mathbb{Z}$-labelling of the limit graph $\lim_n G_n$.

The process of joining two graphs together will be called *amalgamation*, and is as follows. Let G and G' be any two graphs. We may assume that G and G' have no common vertices or edges (for if they are not disjoint, we replace G' by an isomorphic copy G'' that is disjoint from G, form the amalgamation of G and G'', and then revert back to G'). Select a vertex v of G and a vertex v' of G'. Then the *amalgamation* $G\#G'$ of G and G' is formed by taking the disjoint union of G and G' (the union of the two sets of vertices, and the two sets of edges) and then identifying v with v'. If $[v, v]$ is an edge of G and $[v', v']$ is an edge of G', we must also identify these two edges (as otherwise, $G\#G'$ will have multi-edges contrary to our global assumptions). The graph $G\#G'$ is the amalgamation of G and G' *across v and v'*. This process is a special case of the general construction of a 'quotient' graph which mimics the construction of quotient (and covering) spaces in topology (see [3], Chapter 6). Given a graph G, suppose that R is an equivalence relation on the set $V(G)$ of vertices of G. Then the quotient graph G/R has vertex set $V(G)/R$ (so each that vertex of G/R is an equivalence class of vertices of G), and two vertices of G/R are adjacent (that is, joined by an edge) if and only if each equivalence class contains a vertex where these two vertices are adjacent in G. If G and G' are disjoint graphs, containing vertices v and v', respectively, we may take their union and then define the equivalence relation R on $V(G) \cup V(G')$ by saying that each equivalence class contains exactly one vertex except for the equivalence class $\{v, v'\}$.

Now let $G'_1, G'_2, \ldots$ be an infinite sequence of graphs. We shall define a new sequence G_n inductively by $G_1 = G'_1$, and $G_{n+1} = G_n\#G'_{n+1}$ (the details of these amalgamations will be given later). Then the sequence G_n is increasing, and, under certain assumptions, we can construct, by induction, a bijective edge-magic $\mathbb{Z}$-labelling λ of $\lim_n G_n$. The key step here is to show that any injective edge-magic $\mathbb{Z}$-labelling λ_n of G_n can be extended to an injective edge-magic $\mathbb{Z}$-labelling λ_{n+1} of G_{n+1}. Given this, we define a $\mathbb{Z}$-labelling λ of $\lim_n G_n$ by $\lambda = \lambda_n$ on G_n, and the injectivity of λ follows directly from the injectivity of the λ_n. A quite different technique is needed to ensure that $\lambda : \lim_n G_n \to \mathbb{Z}$ is surjective, and this will be achieved (essentially) by making a completely unrestricted choice of one value of λ_n for each n.

5 Two examples

In this section we give two examples of edge-magic labellings of infinite graphs. The first example is a tree; the second is not.

Example 5.1: *the semi-infinite path*

Let G be the graph (in the complex plane $\mathbb{C}$) whose vertices are at $0, 1, 2, \ldots$, and whose edges are $[0, 1]$, $[1, 2]$ and so on. We shall now construct a bijective edge-magic $\mathbb{Z}$-labelling of G with magic constant zero. We need to list the integers in some way, and we choose the listing

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}. \tag{5.1}$$

Let $\lambda(0) = 0$. Now let $\lambda(2) = t$, where t is to be determined shortly. To achieve our objective we must define (for some s) $\lambda(1) = s$, $\lambda([0,1]) = -s$ and $\lambda([1,2]) = -t - s$. First, we select t to be the first integer in the list (5.1) that has not already been allocated; thus $\lambda(2) = 1$. Next, we choose s so that the values taken by the functions s, $-s$ and $-s-1$ are distinct from each other, and from any of the labels allocated so far (namely, 0 and 2). The value $s = 2$ will suffice, so we have now defined

$$\lambda(0) = 0, \quad \lambda([0,1]) = -2, \quad \lambda(1) = 2, \quad \lambda([1,2]) = -3, \quad \lambda(2) = 1.$$

Next, we define $\lambda(4) = p$, say and $\lambda(3) = q$. We choose p to be the first integer in the list (5.1) that has not yet been allocated; thus $p = -1$. We now choose q so that the labels q, $-q-1$ and $1-q$ are distinct from each other and from the labels that have been allocated so far. For example, $q = -4$ will suffice, and we have now achieved the labelling illustrated in Figure 2.

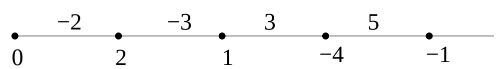

Figure 2

This process can now be continued indefinitely, where next we define the label on 6, then (simultaneously) on 5, $[4,5]$, $[5,6]$; then on 8, and so on. The labels $\lambda(2n)$ are always chosen to be the first integer in the list (5.1) that has not yet been allocated, and this guarantees that the resulting map $\lambda : G \to \mathbb{Z}$ is surjective. By construction, λ is also edge-magic and injective. Note that in this construction (and in many others) we have to make an infinite number of choices; thus there are *uncountably many ways* to construct a bijective edge-magic $\mathbb{Z}$-labelling of this graph. This suggests that not all such labellings can be described by a simple algebraic rule, nor need they have any symmetry. □

It is worthwhile to isolate the key ideas here. First, the edge-magic properties of the labelling are guaranteed by the construction. In order to guarantee surjectivity of the labelling we need an infinite supply of vertices that we can label without constraint (except, of course, that the labels must not have already been allocated); here these vertices are $0, 2, 4, \ldots$. The injectivity of the labelling depends on the fact that given two linear distinct polynomials $p_1(s)$ and $p_2(s)$, and a finite set, say Z_0, of integers (the labels that have already been allocated), there is always a choice of s such that the values $p_1(s)$ and $p_2(s)$ are distinct from each other, and from the integers in $\mathbb{Z}_0$. We shall see that this part of the argument holds more generally, and this will enable us to construct many edge-magic labellings of many other countably infinite graphs.

Example 5.2
Let G be the infinite graph illustrated in Figure 3 and lying in $\mathbb{C}$. The vertices are at $0, 2, 4, \ldots$, and $1 \pm i, 3 \pm i, \ldots$, and the edges are as illustrated. We shall construct a bijective edge-magic $\mathbb{Z}$-labelling λ of G with edge-magic constant zero, and again we use the listing (5.1).

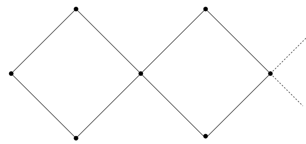

Figure 3

Let $\lambda(0) = 0$; this is the first integer in the list (5.1). Next, let $\lambda(2)$ be the first integer in (5.1) that has not already been allocated; thus $\lambda(2) = 1$. We now let $\lambda(1-i) = t$ and $\lambda(1+i) = s$, where the parameters t and s are to be defined shortly. In order to have edge-magic constant zero, we must label the first four edges as follows:

$$\lambda([0,1+i]) = -s, \quad \lambda([1+i,2]) = -s-1, \quad \lambda([0,1-i]) = -t, \quad \lambda([1-i,2]) = -t-1. \tag{5.2}$$

Now any value of s and t will suffice here provided only that the values t, $-t$, $-1-t$, s, $-s$ and $-s-1$ are distinct, and have not already been allocated. This restriction means that (s,t) in $\mathbb{Z} \times \mathbb{Z}$ has only to avoid some finite set of lines, and so any one of an infinite number of values of t and s are available here. We now make any admissible choice of s and t; in this case we take $t = 2$ and $s = 4$, and we have now allocated labels as in Figure 4.

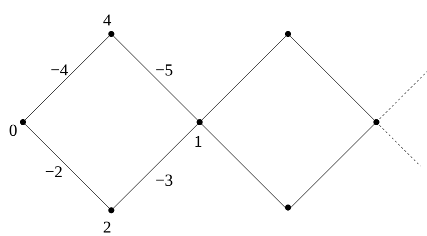

Figure 4

The process continues by induction. Next, we let $\lambda(4)$ be the first integer in (5.1) that has not yet been allocated; thus $\lambda(4) = -1$. Then we let $\lambda(3-i) = p$ and $\lambda(3+i) = q$, and assign integers to the edges from the vertices $3 \pm i$ so that these edges have weight zero. We then choose p and q appropriately, and move on to define $\lambda(6)$. The process continues, and in this way we construct an injective $\mathbb{Z}$-labelling λ of G. Because at each stage we choose $\lambda(2n)$ to be the first integer in the list (5.1) that has not yet been allocated, $\lambda : G \to \mathbb{Z}$ is surjective. $\square$

The method that we are using here is more flexible than these two examples might suggest. Consider, for example, the graph which begins in a manner illustrated in Figure 5.

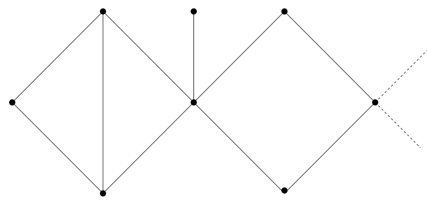

Figure 5

In this case, we define $\lambda(0) = 0$, $\lambda(2) = 1$, $\lambda(1-i) = t$ and $\lambda(1+i) = s$ as before, and then use the same values as in (5.2). In addition, we now need $\lambda([1-i, 1+i]) = -s-t$. Once again, there are infinitely many choices of s and t available, and we can now choose s and t so that these polynomials in s and t take values that are distinct from each other, and from all integers that have been allocated so far. Now consider the vertex $2+i$ and the edge $[2, 2+i]$. We allocate a label m to the vertex $2+i$, and the label $-1-m$ to the edge $[2, 2+i]$ and then choose m so that these values have not yet been allocated. Clearly this process can be continued as before. □

There are two features of this extension of Example 5.2 that are worth noting. First, this example provides an example of a polynomial attached to an edge, namely $-s-t$, in *more than one variable*. Second, the inclusion of *vertices of degree one* (for example, the vertex $2+i$) does not invalidate the technique.

6 The main result

We shall now exploit the ideas discussed above to obtain a general result on edge-magic labellings. The main part of this argument is contained in the following lemma.

Lemma 6.1. *Let λ_0 be an injective edge-magic $\mathbb{Z}$-labelling of a finite graph G_0 with magic constant zero. Let $\mathbb{Z}_0$ be the set of (integer) values taken by λ_0 on G_0, and suppose that $m \in \mathbb{Z}\backslash\mathbb{Z}_0$. Now let G be any finite connected graph, and form an amalgamated graph $G_0\#G$ by identifying a vertex v of G with a vertex v_0 of G_0. If G has a vertex u that is not joined by an edge in G to v, then λ_0 extends to an injective edge-magic $\mathbb{Z}$-labelling λ of $G_0\#G$ with magic constant zero and $\lambda(u) = m$.*

Proof We have to construct an injective edge-magic map $\lambda : G_0\#G \to \mathbb{Z}$, with magic constant zero, that satisfies $\lambda = \lambda_0$ on G, and $\lambda(u) = m$. We therefore define λ to have these values on $G_0 \cup \{v, u\}$ (where v is now identified with v_0). Now let $v_1, \ldots, v_k$ be the vertices in G other than u and v (there is at least one such v_j as u and v are not joined by an edge in G, and G is connected). We define $\lambda(v_i) = m_i$, where the m_i are to be regarded as parameters (to be chosen shortly) and, finally, we define λ on the unlabelled edges of G in such a way that the λ-weight of each edge is zero. Now, for any choice of the parameters m_i, this defines an edge-magic map λ of $G_0\#G$ into $\mathbb{Z}$ with magic constant zero, and with $\lambda = \lambda_0$ on G, and $\lambda(u) = m$.

As there is no edge $[u, v_0]$ in G, each edge in G is of one of the forms $[v_0, v_i]$, $[v_i, v_j]$ (where $i \neq j$) or $[v_i, u]$. As these edges have been allocated weights $-\lambda(v_0) - m_i$,

$-m_i - m_j$ and $-m_i - m$, respectively, we see that each edge e in G is labelled by a *non-constant* linear polynomial $p_e(m_1, \ldots, m_k)$ in the variables m_i. Moreover, it is clear that if $e \neq e'$ then (as polynomials) $p_e(m_1, \ldots, m_k) \neq p_{e'}(m_1, \ldots, m_k)$. We want to ensure that λ is injective and for this it suffices to choose the m_i so that
(i) for each e, $p_e(m_1, \ldots, m_k) \notin \mathbb{Z}_0 \cup \{m\}$, and
(ii) for each e and e', $p_e(m_1, \ldots, m_k) \neq p_{e'}(m_1, \ldots, m_k)$, when $e \neq e'$.
Now the condition $p_e(m_1, \ldots, m_k) \in \mathbb{Z}_0 \cup \{m\}$ is equivalent to the statement that the lattice-point $(m_1, \ldots, m_k)$ in $\mathbb{Z}^k$ lies on one of a finite number of given hyperplanes in $\mathbb{R}^k$. Thus (i) holds if $(m_1, \ldots, m_k)$ is chosen to be in the complement of these hyperplanes. A similar argument holds for (ii); thus there is some choice (in fact, infinitely many choices) of $(m_1, \ldots, m_k)$ that makes the function $\lambda : G_0 \# G \to \mathbb{Z}$ injective. □

We note that the ideas in this proof of Lemma 6.1 show that any finite graph G (with no loops or multi-edges) supports an injective edge-magic $\mathbb{Z}$-labelling λ with magic constant zero. Indeed, for each vertex v_i we let $\lambda(v_i) = m_i$, $i = 1, \ldots, |V|$, and we define λ on the edges of G so as to ensure that it is edge-magic with magic constant zero. Thus each edge has a label of the form $-m_i - m_j$, and each label (on an edge or a vertex) is a polynomial in the variables m_i. Clearly these $|G|$ polynomials, say $p_1, \ldots, p_{|G|}$, are distinct (as polynomials); thus each polynomial in the finite set of polynomials of the form $p_i - p_j$ is non-constant. We now select $m_1, \ldots, m_{|V|}$ so that for $i \neq j$,

$$p_i(m_1, \ldots, m_{|V|}) - p_j(m_1, \ldots, m_{|V|}) \neq 0,$$

and the resulting labelling is injective, and edge-magic with magic constant zero.

We come now to the main theorem on edge-magic labellings. The proof of this is by induction (using Lemma 6.1 together with the argument used in Example 5.1) and is omitted.

Theorem 6.2. *Let G'_n be an infinite sequence of finite, connected graphs (with no loops or multi-edges). Let $G_1 = G'_1$ and, for each n, let $G_{n+1} = G_n \# G'_{n+1}$, where this is an amalgamation of the form described in Lemma 6.1. Then $\lim_n G_n$ has a bijective edge-magic $\mathbb{Z}$-labelling.*

Theorem 6.2 can be generalized in the following way. We recall that after discussing Example 5.2, we showed how this could be modified by (in our new terminology) amalgamating some graph G_n with a graph that consists of a single edge. We can even make this modification infinitely often provided that there is an infinite sequence of amalgamations of the type described in Lemma 6.1 (for it is this infinite sequence that guarantees surjectivity). Thus we have the following result.

Theorem 6.3. *Let G'_n be an infinite sequence of finite, connected graphs (with no loops or multi-edges). Let $G_1 = G'_1$ and, for each n, let $G_{n+1} = G_n \# G'_{n+1}$. Suppose that for each n, G'_{n+1} is a single edge, or that the amalgamation $G_n \# G'_{n+1}$ is of the form described in Lemma 6.1, and that this latter case occurs infinitely often. Then $\lim_n G_n$ has a bijective edge-magic $\mathbb{Z}$-labelling.*

We end this section by showing that *the results we have proved so far apply equally well to any countably infinite abelian group Γ as they do to $\mathbb{Z}$*. Indeed, the argument holds in this more general case without change except for the last paragraph in the proof of Lemma 6.1, and this can be generalized as follows.

Lemma 6.4. *Let* Γ *be an infinite abelian group (written additively), and let* Γ_0 *be a finite subset of* Γ. *Consider, for* $m_1, \ldots, m_k$ *in* Γ, *the* $3^k - 1$ *non-trivial expressions of the form*

$$L_j(m_1, \ldots, m_k) = a_{j1}m_1 + \cdots + a_{jk}m_k,$$

where each a_{ij} *is* 1, 0 *or* -1. *Then there exists a choice of* $m_1, \ldots, m_k$ *in* Γ *such that no* $L_j(m_1, \ldots, m_k)$ *is in* Γ_0.

Proof It is convenient to define $-\Gamma_0$ so that $x \in -\Gamma_0$ if and only if $-x \in \Gamma_0$. The proof is by induction on k (for all choices of Γ_0). First, the conclusion is obvious if $k = 1$ for, regardless of Γ_0, we need only select m_1 to be outside of $\Gamma_0 \cup (-\Gamma_0)$.

Suppose now that the conclusion holds, for every finite subset Γ_0, for $k = 1, \ldots, s$. Now consider the variables $m_1, \ldots, m_k, m_{k+1}$ and any finite subset, say Γ_1, of Γ. Choose any value of m_{s+1} that lies outside the finite set $\Gamma_1 \cup (-\Gamma_1)$. By the induction hypothesis we can now choose $m_1, \ldots, m_s$ such that, for each j, $L_j(m_1, \ldots, m_s) \notin \Sigma$, where

$$\Sigma = \Gamma_1 \cup \big(m_{s+1} + \Gamma_1\big) \cup \big(- m_{s+1} + \Gamma_1\big).$$

Now any linear form in the variables $m_1, \ldots, m_{s+1}$ is of the form $L_r(m_1, \ldots, m_s) + am_{s+1}$, where a is 1, 0 or -1. Obviously, in each of these cases, $L_r(m_1, \ldots, m_s) + am_{s+1} \notin \Gamma_1$, for otherwise, $L_r(m_1, \ldots, m_s) \in \Sigma$ contrary to our choice of $m_1, \ldots, m_s$. The proof is complete. □

We return to the more general version of Theorem 6.2. First, we consider the situation described in Lemma 6.1, except that we are now seeking injective edge-magic Γ-labellings. As before, we assign to each vertex v_j a label m_j, and to each edge e of G an edge polynomial $p_e(m_1, \ldots, m_k)$, where each such polynomial is of the form $-m_i - m_j$, or of the form $-m_i - c$, where c is in Γ. These polynomials are of the form described in Lemma 6.4, so that we can (as before) choose the m_i so that the extended labelling is injective and edge-magic with magic constant zero.

7 Countably infinite trees

A *tree* is a connected graph that has no cycles of length three or more, and it is clear that each tree is one (and only one) of the following four types:
(a) a finite tree;
(b) a countably infinite tree with each vertex of finite degree;
(c) a countably infinite tree with some vertex of degree $\aleph_0$;
(d) a tree with a vertex of uncountable degree.
It is conjectured that each finite tree is edge-magic (in the sense described in Section 1), and we can also ask whether each finite tree supports an edge-magic bijective Γ-labelling for some finite abelian group Γ. We shall show that every tree of type (b), and some trees of type (c), support a bijective edge-magic $\mathbb{Z}$-labelling.

Theorem 7.1. *Let G be a countably infinite tree that contains a semi-infinite simple path. Then G supports a bijective edge-magic $\mathbb{Z}$-labelling.*

Proof Let G be a countably infinite tree that contains a semi-infinite simple path $\mathcal{P}$. We can think of G as $\lim_n G_n$ where the G_n are constructed by amalgamation from a sequence G'_n as described in Theorem 6.3. In fact, we can take the G'_n here to be either a single edge (the graph P_2), or the graph formed by joining two edges together at a vertex (the graph P_3). We use the copies of P_3 to build the semi-infinite path $\mathcal{P}$ (much as in Example 5.1), and all other edges of $\lim_n G_n$ are obtained by an amalgamation with a single edge. This completes the proof of Theorem 7.1 subject to showing that even in the case when some, or all, all vertices of G have degree $\aleph_0$, we can still construct $\lim_n G_n$ by a sequence of amalgamations. In effect, this means that we have to define the countable sequence of amalgamations in such a way that when we wish to adjoin a P_2-edge, say e, to the partially constructed graph, this partially constructed graph already includes one vertex of e. We can do this as follows. At each vertex of G we label the incident edges either by a sequence $1, 2, \ldots, N$ (for the appropriate N) or by the sequence $1, 2, \ldots$. Choose a root vertex v_0. The graph G is connected, and for each vertex v there is a unique simple path in G from v_0 to v. Each vertex v (other than v_0) can now be represented by a finite sequence $[n_0, n_1, \ldots, n_{k-1}]$, where n_0 specifies the edge from v_0, n_1 specifies the edge from v_1, and so on, and n_{k-1} specifies the edge from v_{k-1} to v. We now build the graph $\lim_n G_n$ inductively from a sequence of amalgamations as follows. We start with the P_3 graph from the root vertex that constitutes the initial segment of $\mathcal{P}$. Assuming we have constructed the graph at the n-th stage we consider each vertex that lies in this partially constructed graph. Let v be such a vertex. If v is the vertex at the end of a finite sequence of abutting P_3-graphs that constitute an initial segment of $\mathcal{P}$, then we add the next P_3-graph that lies in $\mathcal{P}$. For all other possibilities of v, we add the next edge according to the listing of the edges incident to v by the integers n_j as described above. This completes the proof of Theorem 7.1. □

Theorem 7.1 has the following corollary.

Corollary 7.2. *Let G be a countably infinite tree in which every vertex has finite degree. Then G supports a bijective edge-magic $\mathbb{Z}$-labelling.*

This follows directly from Theorem 7.1 and the following result.

Lemma 7.3. *Let G be a tree in which every vertex has finite degree. Then either G is finite, or G has an infinite simple path.*

Proof We choose any vertex, say v_0, of G. Suppose that G is infinite; thus there exist infinitely many simple paths from v_0 (namely one to each vertex). As there are only finitely many edges leaving v_0, there is one such edge, say $[v_0, v_1]$, with the property that there exist infinitely many simple paths that start with v_0, v_1. As there are only finitely many edges leaving v_1, the same reasoning shows that there is a vertex v_2 with the property that infinitely many simple paths that start with v_0, v_1, v_2. We can define a sequence v_n inductively in this way, and this sequence defines an infinite simple path from v_0. □

8 Further extensions

The technique described earlier depends on the notion of the amalgamation of two graphs, and clearly this can be generalized in the following way. Let G and G' be two graphs, and let θ be a map of some set of vertices of G' into the set of vertices of G. We can then form the sum $G + G'$, and then identify each vertex in the domain of θ with its image in G. It is not necessary for θ to be injective here, so that distinct vertices of G' may be identified with the same vertex in G (and hence with each other). If two vertices, say u and v, of G' are in the domain of θ then (in order to avoid multi-edges) it is necessary to identify the edge $[u, v]$ in G' with the edge $[\theta(u), \theta(v)]$ in G should both of these edges exist. With this more general notion of amalgamation available, it is possible to prove a still more general version of Theorem 5.3. However, we shall be content with the following example.

Example 8.1 Consider the graph G (in $\mathbb{R}^2$) whose vertices are the points in the set $\mathbb{Z} \times \mathbb{Z}$, and whose edges are the segments $[a, a \pm 1]$ and $[a, a \pm i]$, where $a \in V$. We claim that G supports a bijective edge-magic $\mathbb{Z}$-labelling with magic constant zero. We shall use complex notation, and we start with the graph G_1 that is the part of G that lies in the square $[-1, 1] \times [-1, 1]$. We let $\lambda(0) = 0$ and $\lambda(1 + i) = 1$; these are arbitrary choices. As these two vertices are not joined by an edge, we can complete the definition of λ on G_1 so that it is an injective edge-magic $\mathbb{Z}$-labelling of G_1. Now let C_n be the cycle that is the boundary of the square $[-n, n] \times [-n, n]$, and let $\mathcal{G}_2$ be the part of G that lies between (and includes) C_1 and C_2. We can form the amalgamated graph $G_2 = G_1 + \mathcal{G}_2$ by identifying the boundary C_1 of G_1 with the inner boundary of $\mathcal{G}_2$, and as the vertex $2 + 2i$ is not joined by an edge to any vertex in C_1, we can extend λ to an injective edge-magic $\mathbb{Z}$-labelling of G_2. The process continues by induction, and ultimately provides a bijective edge-magic $\mathbb{Z}$-labelling of G. It is clear that the same idea will work in any dimension, and for many other tesselations. □

In the next example we consider uncountable graphs.

Example 8.2 The uncountable graph consisting of all edges $[0, z]$, where $|z| = 1$, has a trivial bijective edge-magic $\mathbb{R}$-labelling λ, for we simply take a bijection $\theta : \{z : |z| = 1\} \to \mathbb{R}^+$ and let $\lambda(0) = 0$, and $\lambda(z) = \theta(z)$ and $\lambda([0, z]) = -\theta(z)$, where $|z| = 1$. The graph obtained by adding the additional edge $[1, 2]$ to this graph also supports a bijective edge-magic $\mathbb{R}$-labelling with magic constant zero. To see this, let $\varphi(x) = 2 - x$, where x is real, and note that x and $\varphi(x)$ are symmetric with respect to the fixed point 1 of φ. Now let $\lambda(0) = -2$. Then, for each ray $[0, z]$, we must have $\lambda(z) = x$ and $\lambda([0, z]) = \varphi(x)$ for some real x. This means that we must also have

$$\{\lambda(0), \lambda([1, 2]), \lambda(2)\} = \{-2, 1, \varphi(-2)\} = \{-2, 1, 4\}.$$

We can now construct the map λ as follows. Let $\lambda(0) = -2$, $\lambda([1, 2]) = 1$ and $\lambda(2) = 4$. Next, for each ray $[0, z]$, where $|z| = 1$, we let $\lambda(z) = x$ and $\lambda([0, z]) = \varphi(x)$, where x now ranges over the real numbers excluding $-2, 1, 4$. Finally, we insist that $\lambda(1) = -5$, and $\lambda([0, 1]) = 7$ and the construction is complete. □

9 Some open problems

We raise the following questions.

(1) Are there any countably infinite graphs that do not support a bijective edge-magic $\mathbb{Z}$-labelling? In particular, is the complete graph K_∞ (with $V = \mathbb{Z}$ and and every pair of vertices joined by an edge) edge-magic over $\mathbb{Z}$?

(2) Do all countably infinite trees support a bijective edge-magic $\mathbb{Z}$-labelling?

(3) Which groups provide a bijective edge-magic labelling of some graph, and which do not? The constructions given here can be carried out for the additive group of rationals and indeed for any countably infinite abelian group Γ. To what extent (if any) is the group structure of Γ relevant to the subject of labellings of graphs? More generally, to what extent is any algebraic structure on the set $\mathcal{L}$ of labels relevant? Could it be, for example, that one obtains more (or fewer) labellings if one assumes that $\mathcal{L}$ has a certain algebraic structure (for example, a semi-group), and if so, what are the structures that are best suited to labelling graphs?

(4) Which uncountable graphs support a bijective edge-magic labelling over some abelian group?

10 Vertex-magic labellings

The ideas described above can also be used to construct vertex-magic labellings of countably-infinite graphs, and we shall confine ourselves to just one example of this, namely the binary tree. It will be clear, however, that the same technique works for many other graphs.

Example 10.1: *the binary tree*
We shall construct a bijective vertex-magic $\mathbb{Z}$-labelling of the binary tree (illustrated in Figure 6) with magic constant zero.

First, we give the root vertex v_0 the value zero. The two edges leaving it must be given the values x and $-x$, and we take $x = 1$. Next, we label the two vertices at a distance one from v_0 using the integers 2 and -2 (the first two integers in (4.1) that have not been used so far). We now label the (four) edges leaving these two vertices by the parameters a, b, c, d (see Figure 6) where these must be chosen so that $a + b = -3$ and $c + d = 3$. We can choose any integers satisfying these equations provided only that they have not been used so far. We choose $a = -3$, $b = 6$, $c = -4$ and $d = 7$. Next we move on and label the four vertices of level two; then the edges emanating from these and so on. Using arguments much the same as in the edge-magic case, we can now construct a bijective vertex-magic $\mathbb{Z}$-labelling of G.

□

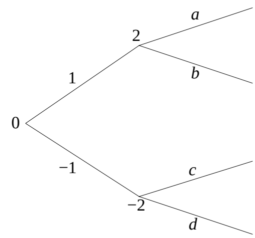

Figure 6

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