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On contractible edges in 3-connected graphs

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Abstract

The existence of contractible edges is a very useful tool in graph theory. For 3-connected graphs with at least six vertices, Ota and Saito (1988) prove that the set of contractible edges cannot be covered by two vertices. Saito (1990) prove that if a three-element vertex set S covers all contractible edges of a 3-connected graph G, then S is a vertex-cut of G provided that G has at least eight vertices. Using Saito's result, Hemminger and Yu (1993) characterize all 3-connected graphs having at least ten vertices which has a 3-element vertex set covering all contractible edges. We give a direct short proof of the last result. Saito's result is a consequence. We also give a short proof of the main results given by Ando, Enomoto and Saito (1987) and McCuaig (1990).

1 Introduction

All graphs considered in this paper are simple. An edge e in a 3-connected graph G is *contractible* if the contraction G/e is still 3-connected. A vertex set S *covers* all contractible edges if each contractible edge is incident to a vertex of S. The existence of contractible edges is a powerful tool in graph theory. For example, Thomassen [6] used it to give a very short proof of Kuratowski's well-known planarity theorem. Therefore it is very natural and important to study contractible edges. Indeed, contractible edges have been studied by numerous authors (see, for example, [1–8]). The following three results study the covering of contractible edges in 3-connected graphs.

Theorem 1 [4] *Let G be a simple 3-connected graph having at least six vertices. Then the set of contractible edges cannot be covered by two vertices.*

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Theorem 2 [5] *Let G be a simple 3-connected graph with at least eight vertices. If a three-element vertex set S covers all contractible edges of G, then S is a vertex-cut of G.*

Using the above result, Hemminger and Yu characterize 3-connected graphs with covers of size 3. They define a class of graphs $\mathcal{K}_3$ based on $K_{3,p}$ (here $p \geq 3$) as follows. Let (A, B) be the bipartition of the vertex set of $K_{3,p}$, where $|A| = 3, |B| = p$. Let B_1 be any subset of B. For each vertex of B_1, delete the vertex first, then add a triangle, then add a 3-matching from each such triangle to A. Moreover, any edge with endvertices of A can be added.

Theorem 3 [2] *Let G be a simple 3-connected graph with at least ten vertices. Then a three-element vertex set covers all contractible edges of G if and only if G is a graph in $\mathcal{K}_3$.*

The next result gives a best-possible bound on the number of contractible edges in a minimally 3-connected graph.

Theorem 4 [3] *Let G be a minimally 3-connected graph with at least five vertices. Then G has at least $\frac{|V(G)|+3t}{2}$ contractible edges, where t is the number of vertices of G having degree great than 3.*

Let G be a 3-connected graph with at least five vertices and H be a minimally 3-connected spanning subgraph. Clearly, any contractible edge of H is also contractible in G. Thus a direct consequence of the last result is following main result of [1, 3].

Theorem 5 [1, 3] *Let G be a 3-connected graph with at least five vertices. Then G has at least $|V(G)|/2$ contractible edges.*

The purpose of this paper is to give short proofs of Theorems 2, 3, 4 (therefore 5). Let $C(G)$ denote the subgraph with vertex set $V(G)$ and edge set $E_c(G)$, the set of contractible edges of G. We use $d_C(u)$ to denote the degree of a vertex u in $C(G)$. We will use the following two lemmas. The first one is the well-known result of Tutte. An edge in a 3-connected graph is *essential* [7] if neither $G\backslash e$ nor G/e is both 3-connected and simple. A *triad* [7] is a set of three edges incident to a vertex of degree three.

Lemma 6 [7] *Let G be a minimally 3-connected graph and e be an essential edge. Then e lies in a triad of G.*

Lemma 7 [1, 3] *Let G be a 3-connected graph with at least five vertices and x be a vertex of degree three with neighbors u, v, w. If neither xu nor xv is contractible, then*

(a) both u and v have degree three, and $uv \in E(G)$, and

(b) each of x, u, v is incident to exactly one contractible edge, these edges form a matching of G.

Corollary 8 *Let v be a vertex in a minimally 3-connected graph G with $d(v) \geq 4$ and u be a neighbor of v. If uv is not contractible, then u has degree three and is incident to exactly two contractible edges.*

Proof. Clearly uv is essential. By Lemma 6, $d(u) = 3$. If u is incident to at most one contractible edge, then by Lemma 7, $d(v) = 3$, a contradiction. □

2 Proofs

In this section, we give proofs of Theorem 3 and 4 (and therefore Theorem 5). We do not use Theorem 2 in our proof of Theorem 3. Instead, Theorem 2 is a consequence.

Proof of Theorem 3. Clearly, all contractible edges of any graph in the class $\mathcal{K}_3$ are covered by three vertices. Next we prove the converse. Let G_1 be a 3-connected graph with at least ten vertices. Suppose that $S = \{u, v, w\}$ covers all contractible edges of G_1. Let G be a minimally 3-connected spanning subgraph of G. We will first show that G is in $\mathcal{K}_3$. Let x be a vertex not in S. If all neighbors of x are in S, then x is adjacent to all vertices of S. Assume that there is a $y \notin S$ such that xy is an edge of G. Then xy is essential thus is in a triad by Lemma 6. By relabelling if necessary, suppose $d(x) = 3$. If x is incident to exactly one contractible edge, by Lemma 7, the edge xy is in a triangle xyz. If $z \notin S$, then the triangle is connected to S by a 3-edge matching. If $z \in S$, then each of x, y, z has degree three and x and y are connected to $S \backslash z$ by a 2-element matching.

Next we suppose that x is adjacent to exactly two contractible edges. Then x is incident to two elements of S, say u and v. If uy or vy is an edge, then it is clearly non-contractible. Therefore y is incident to at most one contractible edge. If $d(y) = 3$, by Lemma 7, either uy or vy is an edge and thus one of the contractible edges incident to x becomes non-contractible, a contradiction. Hence $d(y) \geq 4$. For each $z \in N(y) \backslash S$, the edge yz is not contractible. By Corollary 8, $d(z) = 3$ and z is incident to exactly two contractible edges. Thus $N(z) \subseteq \{u, v, w, y\}$. The subgraph induced by $S \cup \{y\} \cup N(y)$ is called of type $\mathcal{P}$. Suppose there is such a subgraph P. From the above argument, we have

(2.1) G can be obtained by sticking subgraphs along S of the following types: (1) $\mathcal{P}$, (2) a vertex joined to u, v and w (a triad), (3) a triangle connected to S by a 3-edge matching, or (4) a triangle with one degree-3 vertex, say, t, in S, and two other vertices not in S connected to $S \backslash t$ by a 2-element matching.

(2.2) Suppose z is a degree-3 vertex in $P - S$ and $N(z) = \{y, s, t\}$, where $s, t \in S$. Then $G - \{z, y\}$ does not have a block containing both s and t. In particular, there are no three degree-3 vertices in $V(P) - S$ adjacent to the same two vertices of S.

Proof. Suppose (2.2) is false. As zy is not incident to S thus is non-contractible, there is a 3-element vertex-cut $\{y, z, p\}$. As $d(z) = 3$, it is clear that $\{y, p\}$ is a vertex-cut of G, a contradiction. □

(2.3) $V(G) \neq V(P)$. Moreover S is a vertex-cut of G.

Proof. Suppose $V(G) = V(P)$. As $|V(G)| \geq 10$, we conclude that $P - S$ has at least six degree-3 vertices, each of which joins y and two of $\{u, v, w\}$. Then either there are at least three degree-3 vertices in $V(P) - S$ adjacent to the same two vertices of S, or there exists a degree-3 vertex in $V(P) - S$ such that after deleting which there is still a 6-cycle containing S and three degree-3 vertices in $V(P) - S$. Neither cases is possible by (2.2). Therefore $G \neq P$. By (2.1), we conclude that S is a vertex-cut. □

(2.4) There is at most one degree-3 vertex in $P - S$ adjacent to the same two vertices of $\{u, v, w\}$. Moreover, $P - S$ contains at most two degree-3 vertices and $|P| \leq 6$.

Proof. Suppose there are at least two degree-3 vertices, $a_1, a_2 \in N(y)\backslash S$, adjacent to, say u, v in P. By (2.1), as $G \neq P$, there is a path joining u and v in $(G\backslash V(P)) \cup S$. This is a contradiction by (2.2) as now $G - \{a_1, y\}$ has a block containing both u and v. Using this, by a similar argument, we can show that $P - S$ contains at most two degree-3 vertices. Hence $|V(P)| \leq 6$. □

By (2.4), there are at most two degree-3 vertices in $P - S$. As $d(y) \geq 4$, two or three of uy, vy, wy are in $E(G)$. Furthermore, by (2.4), there is no more degree-3 vertex other than x joining both u and v. As $d(y) \geq 4$, there are always two of $\{u, v, w\}$, say s, t, such that sy, ty are in $E(G)$, and each of s and t is adjacent to a degree-3 vertex. Thus sy, ty are non-contractible, therefore $d(s) = d(t) = 3$ by Lemma 6. As S is a 3-element vertex cut of G and G is 3-connected, S is an independent set. Therefore,

(2.5) For each P, there are $\{s, t\} \subseteq S$, such that $d_P(s) = 2$, $d_G(s) = 3$ and $d_P(t) = 2$, $d_G(t) = 3$.

By (2.1), G contains at most one subgraph of type $\mathcal{P}$. If G contains such a subgraph P, as both s and t have degree three in G, exactly one of cases (2), (3), or (4) happens in (2.1). Then G has at most nine vertices, a contradiction. Hence (1) does not happen in (2.1). Suppose (4) happens in (2.1) and H is a subgraph of type (4) in (2.1). Then as one of the vertices in S has degree three in G, by (2.1), we conclude that G has at most three vertices not in H. Hence G has at most eight vertices, a contradiction. By (2.1), we conclude that $G \in \mathcal{K}_3$.

Now G_1 is obtained from G by adding edges. Suppose that pq is an added edge. If $p, q \notin S$, then it is easy to verify that pq is contractible. If $p \in S$, but $q \notin S$, then q is in a triangle $\{q, a, b\}$ which is connected to S by a 3-element matching in G. By Lemma 7, either ab or aq is contractible, a contradiction as neither is incident to S. Thus $G_1 \in \mathcal{K}_3$. □

We also note that one can determine all 3-connected graphs with at least eight vertices having a 3-element vertex set covering all contractible edges with a small modification in the above proof. Next we show that Theorem 2 is an easy consequence.

Proof of Theorem 2 For graphs with at least ten vertices, Theorem 3 (or (2.3)) shows that the theorem is true. Thus we need only consider the case when $8 \leq |V(G)| \leq 9$. By (2.1), we need only show that $V(G) \neq V(P)$ for some subgraph P of type $\mathcal{P}$. Suppose $V(G) = V(P)$. Thus $V(G) - S$ has at least four degree-3 vertices, each such vertex is adjacent to y and has two neighbors in S. By (2.2), there are no three degree-3 vertices in $V(P) - S$ adjacent to the same two vertices of S. Using this fact, it is routine to check that S cannot cover all contractible edges, a contradiction. □

Next we give a short proof of Theorem 4.

Proof of Theorem 4 Let G be a minimally 3-connected graph with at least five vertices. Divide the vertex set $V(G)$ into classes as follows. (1) For each vertex x with $d(x) \geq 4$, let the class containing x be $P_x = \{x\} \cup \{u | ux \in E(G), ux$ is not contractible$\}$. Let S be the set of vertices not in any of the above classes. (2) For each vertex $y \in S$, let the class containing y be $Q_y = \{y\}$. Clearly $d(y) = 3$. Thus each class contains at most one vertex of degree greater than 3. Next we show that the above classes induce a partition of $V(G)$. It suffices to show that for any two distinct vertices x and y with $d(x) \geq 4, d(y) \geq 4$, $P_x \cap P_y = \emptyset$. Suppose that $u \in P_x \cap P_y$. Then $u \neq x, y$. By Corollary 8, each vertex in P_x other than x has degree three and is incident to exactly two contractible edges. But u is incident to two non-contractible edges, a contradiction.

Suppose that P_x is a class with $d(x) \geq 4$. By Corollary 8, each vertex of $P_x - x$ is incident to two contractible edges. Moreover, x is incident to $d(x) - (|P_x| - 1)$ contractible edges. Thus $d_C(P_x) = 2(|P_x| - 1) + [d(x) - (|P_x| - 1)] = |P_x| + d(x) - 1 \geq |P_x| + 3$. If $y \in S$, then $Q_y = \{y\}$ and $d_C(Q_y) \geq 1 = |Q_y|$. Summing the degree of $V(G)$ in the subgraph $C(G)$, we get

$$\begin{aligned} 2 \cdot |E_c(G)| &= \sum_{d(x) \geq 4} d_C(P_x) + \sum_{y \in S} d_C(Q_y) \\ &\geq 3t + \sum_{d(x) \geq 4} |P_x| + \sum_{y \in S} |Q_y| \\ &= 3t + |V(G)|. \end{aligned}$$

Thus G has at least $(|V(G)| + 3t)/2$ contractible edges. □

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References

[1] K. Ando, H. Enomoto and A. Saito, Contractible edges in 3–connected graphs, *J. Combin. Theory Ser. B* **42** (1987), 87–93.

[2] R. E. Hemminger and X. Yu, Covering contractible edges in 3–connected graphs. II. Characterizing those with covers of size three, *J. Graph Theory* **14** (1993), 247–252.

[3] W. McCuaig, Edge contractions in 3–connected graphs, *Ars Combin.* **29** (1990), 299–308.

[4] K. Ota and A. Saito, Non-separating induced cycles in 3–connected graphs, *Scientia Ser. A* **2** (1988) 101–105.

[5] A. Saito, Covering contractible edges in 3–connected graphs, I: Covers of size three are cutsets, *J. Graph Theory* **14** (1990), 635–643.

[6] C. Thomassen, Planarity and duality of finite and infinite graphs, *J. Combin. Theory Ser. B* **29** (1980), 244–271.

[7] W. T. Tutte, A theory of 3–connected graphs, Nederl. Akad. Wetensch. Proc. Ser. A **64** (1961), 441–455.

[8] H. Wu, Contractible elements in graphs and matroids, *Combinatorics, Probability and Computing*, **12** (2003) 457–465.

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