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Concentric Bilinski diagrams

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Abstract

A Bilinski diagram (respectively, B^*-diagram) is a labeling of a planar map with respect to the regional distance of its vertices and faces from a central vertex (respectively, face). Such diagrams are concentric if for each $k \geq 1$, the set of vertices at regional distance k from the central vertex or face induces a circuit. The class $\mathcal{G}_{a,b}$ consists of all 1-ended, 3-connected planar maps with the property that every valence is finite and at least a and every covalence is finite and at least b. A map in the subclass $\mathcal{G}_{a,b+}$ of $\mathcal{G}_{a,b}$ contains no adjacent b-covalent faces, and dually a map in $\mathcal{G}_{a+,b}$ contains no adjacent a-valent vertices. It is shown that all Bilinski diagrams and all B^*-diagrams of all maps in $\mathcal{G}_{6,3}$, $\mathcal{G}_{4,4}$, $\mathcal{G}_{3,6}$, $\mathcal{G}_{5,3+}$ and $\mathcal{G}_{3+,5}$ are concentric.

1 Introduction

A Bilinski diagram is a labeling of the sets of vertices and faces of a planar map with respect to a given vertex x, called its *center*, and the regional distance of the other vertices of the map from x. Bilinski diagrams are of particular interest when this labeling corresponds to a planar embedding wherein the sets of vertices at the same regional distance from the center induce a sequence of concentric circuits about the center. A *concentric* Bilinski diagram is one with this property. If all Bilinski diagrams of a map, i.e., for every choice of center, are concentric, then we say that the map is *uniformly vertex-concentric.* This manner of labeling infinite planar maps has been utilized by Bilinski [1], Grünbaum and Shephard [4], Niemeyer and Watkins [6], and Brand, Morton, and Vertigan [2]. Such a labeling may also be performed with respect to a central face, in which case we also seek maps that are *uniformly face-concentric.*

Bilinski diagrams with such concentric properties are not only aesthetically appealing but are also an essential tool in constructing geodetic paths, rays, and double rays in planar maps (as in [6] and [3]). The purpose of this article is to establish tight sufficient conditions for an infinite, locally finite planar map to be both uniformly vertex-concentric and uniformly face-concentric. In the final section some necessary conditions for uniform concentricity are considered.

Portions of this article appear in the first author's doctoral dissertation [3] written under the supervision of the second author.

2 Definitions and Notation

It will be assumed in this article that a *map* is a planar map whose underlying graph is connected and locally finite (i.e., all valences are finite) without loops or multiple edges. The valence of a vertex v with respect to a map X will be denoted by $\rho_X(v)$, or simply $\rho(v)$ if the context is clear, and the covalence of any face f of the map will be denoted by $\rho^*_X(f)$ or $\rho^*(f)$. To assure uniqueness of the embedding of this underlying graph, it suffices to assume that it is 3-connected (cf. [8] and [5]). Thus $\rho(v) \geq 3$ for all v in the set VX of vertices of a map X, and $\rho^*(f) \geq 3$ for all f in the set FX of faces of X.

Following [6], a *Bilinski diagram* (or briefly, *B-diagram*) B_x with center $x \in VX$ of a map X is defined inductively as follows:

$U_0 = \{x\}$.

F_1 is the set of faces incident with x.

For $r \geq 1$, U_r is the set of those vertices not in U_{r-1} that are incident with a face in F_r.

For $r \geq 1$, F_{r+1} is the set of those faces not in F_r that are incident with a vertex in U_r.

A B-diagram of a map is *concentric* if each of the induced subgraphs $\langle U_r \rangle$, $r \geq 1$, is a circuit. A map is *uniformly vertex-concentric* if all of its B-diagrams are concentric.

Proposition 2.1. *If a map is uniformly vertex-concentric, then its underlying graph is* 3*-connected.*

Proof. Let X be the underlying graph of a uniformly vertex-concentric map. Then X is connected by definition. Let S be a smallest separating set of X and let $v \in S$. Let Y_1 and Y_2 be distinct components of $X - S$. By the minimality of S, the vertex v has neighbors $y_1 \in VY_1$ and $y_2 \in VY_2$. With respect to the Bilinski diagram B_v, we have $y_1, y_2 \in U_1$. If $|S| < 3$, then $|S \cap U_1| < 2$. Since $\langle U_1 \rangle$ is 2-connected, there would exist a $y_1 y_2$-path in $X - S$, which is impossible. ∎

Bilinski and Grünbaum and Shephard also considered the dual situation of labeling maps with a central face rather than a central vertex. We thus have the following dual definition.

A map X may be labeled as a *B^*-diagram* B_f^* with center $f \in FX$ as follows:

$F_1^* = \{f\}$.

U_1^* is the set of vertices incident with f.

For $r \geq 1$, F_{r+1}^* is the set of faces not in F_r^* that are incident with a vertex in U_r^*.

For $r \geq 1$, U_{r+1}^* is the set of those vertices not in U_r^* that are incident with a face in F_{r+1}^*.

With respect to this definition, the terms *concentric* B^*-diagram and *uniformly face-concentric* map are defined analogously. A map X with planar dual X^* is clearly uniformly vertex-concentric if and only if X^* is uniformly face-concentric. It is not immediate whether X^* is also uniformly vertex-concentric; we will partially address this question in Corollary 4.2. A map will be called *uniformly concentric* if it is both uniformly vertex-concentric and uniformly face-concentric.

Some aspects of B^*-diagrams arise in the study of nonperiodic tilings and their relationships to quasicrystals (cf. [7]). In this context, the set F_1^* of faces in the definition of a B^*-diagram corresponds to the "first corona" of the tile f. Similarly, F_2^* is the "second corona" of f.

In a B-diagram (respectively B^*-diagram), let $v \in U_r$ (respectively U_r^*). The neighbors of v in U_{r-1} (respectively U_{r-1}^*) will be called the *lower neighbors* of v; the neighbors of v in U_r (respectively U_r^*) will be called the *level neighbors* of v; and the neighbors of v in U_{r+1} (respectively U_{r+1}^*) will be called the *upper neighbors* of v. Given any vertex in a concentric B-diagram or B^*-diagram, its lower neighbors lie on a circuit closer to the center, it's level neighbors lie on the same circuit as the given vertex (and there are exactly two of them), and its upper neighbors lie on a circuit farther from the center.

For a locally finite graph X, the number of ends of X may be regarded as the supremum of the number of infinite components of $X - W$ as W ranges over all finite subgraphs of X. In particular, an infinite, locally finite graph X is *1-ended* if for every finite subgraph W of X, the subgraph $X - W$ has exactly one infinite component. It is not hard to see that if the underlying graph of a connected map has more than one end, then none of its B-diagrams is concentric. In fact, in this case, for all but finitely many r, the subgraph $\langle U_r \rangle$ is not even connected, let alone an elementary circuit.

Let $\mathcal{G}_{a,b}$ denote the class of all 1-ended, 3-connected maps X such that $a \leq \rho(v) < \infty$ for all $v \in VX$ and $b \leq \rho^*(f) < \infty$ for all $f \in FX$. We define $\mathcal{G}_{a,b+}$ to be the subclass of $\mathcal{G}_{a,b}$ of maps with no adjacent b-covalent faces. Similarly, $\mathcal{G}_{a+,b}$ denotes the subclass of $\mathcal{G}_{a,b}$ of maps with no adjacent a-valent vertices.

Grünbaum and Shephard showed [4, 4.7.1] that the tiling $[j^k]$ (the map with constant valence j and constant covalence k) is uniformly vertex-concentric whenever $\frac{1}{j} + \frac{1}{k} \leq \frac{1}{2}$. (The essentially unique B-diagram of $[6^3]$ is presented in Figure 1.) Note that for j, $k \geq 3$, equality holds here if and only if $(j, k) = (3, 6), (4, 4)$ or $(6, 3)$, corresponding to the regular tilings of the Euclidean plane by hexagons, squares, or triangles, respectively. These three regular tilings are examples of uniformly concentric maps.

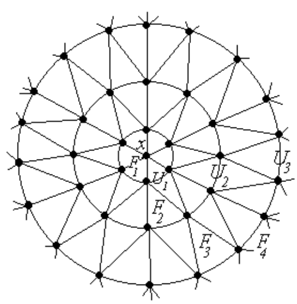

Figure 1: Bilinski diagram of $[6^3]$.

In [6, Theorem 3.2] it was shown that all maps in $\mathcal{G}_{4,4}$ are uniformly vertex-concentric. Brand et al. [2, Theorem 12] proved that maps in which all faces are exactly 3-covalent and all vertices are at least 6-valent are uniformly vertex-concentric. Our **main result** is that every map in $\mathcal{G}_{6,3}$, $\mathcal{G}_{3,6}$, $\mathcal{G}_{5,3+}$ and $\mathcal{G}_{3+,5}$ is uniformly concentric. Moreover, in certain respects, these conclusions are best possible.

3 Preliminary Results

The five statements of the following lemma describe certain combinations of properties that no finite map can possess. This lemma will be used to prove that certain maps are uniformly concentric by showing that the assumption to the contrary produces one of these "forbidden submaps". The proofs of the five statements are similar; assuming the map to exist as described, we count its edges in two ways and apply Euler's formula to arrive at a contradiction.

Lemma 3.1. *No finite map Y with at least four vertices and no pendant vertex has any of the following five sets of properties:*

1. *All vertices incident with the exterior face are at least* 4*-valent with at most three exceptions, at most two of which are* 2*-valent. Furthermore, all vertices not incident with the exterior face are at least* 6*-valent.*

2. *All vertices not incident with the exterior face and at least two vertices incident with the exterior face are at least* 6*-valent. All other vertices incident with the exterior face are at least* 4*-valent with at most four exceptions.*

3. *All vertices incident with the exterior face are at least* 4*-valent with at most three exceptions. All vertices not incident with the exterior face are at least* 5*-valent. Every interior edge is incident with at most one* 3*-covalent face.*

4. *All vertices incident with the exterior face are at least* 3*-valent with* $d \leq 2$ *exceptions (*2*-valent vertices). The number of* 3*-valent vertices incident with the exterior face can exceed the number of* (≥ 5)*-valent vertices incident with the exterior face by at most* $4 - d$*. All vertices not incident with the exterior face are at least* 5*-valent. In addition, every interior edge is incident with at most one* 3*-covalent face.*

5. *All vertices not incident with the exterior face and at least two vertices incident with the exterior face are at least* 5*-valent. All vertices incident with the exterior face are at least* 3*-valent with* $d \leq 4$ *exceptions (*2*-valent vertices). The number of* 3*-valent vertices incident with the exterior face can exceed the number of* (≥ 5)*-valent vertices incident with the exterior face by at most* $3 - d$*. In addition, every interior edge is incident with at most one* 3*-covalent face.*

Proof. Let ν_0, ν_1 and ν_2 denote the numbers of vertices, edges and faces, respectively, of a finite planar map. Let p denote the covalence of the exterior face of Y and let t denote the number of 3-covalent faces of Y. In statements (3), (4), and (5) of the lemma we assume that each interior edge is incident with at most one 3-covalent face; thus $t \leq \frac{\nu_1 - p}{3} + 1 \leq \frac{\nu_1}{3}$.

Only the proof of statement (4) is provided, as the proof of (5) is very similar and the proofs of the first three statements are simpler. Assume that the conditions in (4) hold. Let a, b, and c denote respectively the numbers of 3-valent, 4-valent, and (≥ 5)-valent vertices incident with the exterior face. Counting ν_1 in two ways, we obtain:

$$2\nu_1 \geq 5(\nu_0 - p + c) + 4b + 3a + 2d;$$

$$2\nu_1 \geq 4(\nu_2 - 1 - t) + p + 3t.$$

Since $p = a + b + c + d$, after adding the two inequalities we have

$$0 \geq 5(\nu_0 - \nu_1 + \nu_2) + \nu_1 - \nu_2 - a + c - 2d - t - 4.$$

Applying Euler's Formula and the inequality $t \leq \nu_1/3$, we have:

$$0 \geq 6 + \frac{2}{3}\nu_1 - \nu_2 - a + c - 2d.$$

By assumption, $a - c \leq 4 - d$ and $d \leq 2$, and therefore

$$0 \geq 2 - d + \frac{2}{3}\nu_1 - \nu_2 \geq \frac{2}{3}\nu_1 - \nu_2. \tag{1}$$

Since $\nu_0 \geq 4$ and no edge is incident with more than one 3-covalent face, there exists a face of covalence at least 4. Thus

$$3\nu_2 < \sum_{f \in FY} \rho^*(f) = 2\nu_1,$$

contrary to inequality (1). ∎

Notation. Let B_x be a concentric B-diagram of a map X. Given any two vertices v_1 and v_2 of U_r, $r \geq 1$, $d_r(v_1, v_2)$ will denote the distance in $\langle U_r \rangle$ between v_1 and v_2 and $U_r[v_1, v_2]$ will denote a v_1v_2-path in $\langle U_r \rangle$ of length $d_r(v_1, v_2)$.

One of our main results will also require the following technical lemma.

Lemma 3.2. *Let $X \in \mathcal{G}_{5,3+}$ and suppose that B_x is a concentric B-diagram of X with center x. Suppose also that each vertex of B_x has at most two lower neighbors. Then on every $\langle U_j \rangle$, between any pair of vertices with exactly two lower neighbors each lies a vertex with no lower neighbors.*

Proof. Let $X \in \mathcal{G}_{5,3+}$ and assume B_x satisfies the hypotheses of the lemma. (It is necessary at this point to postulate that each vertex has at most two lower neighbors. However, we will prove in Theorem 4.1 that this condition always holds for maps in $\mathcal{G}_{5,3+}$.) Note that this implies that every vertex has at least one upper neighbor. We first show that if a vertex $v \in U_j$ has exactly two lower neighbors w and z, then these three vertices induce the boundary of a 3-covalent face of X. Were this not the case, then consider the circuit C consisting of $[v, w]$,$[v, z]$ and the wz-path in $\langle U_{j-1} \rangle$ chosen so that x is exterior to C . By assumption, the interior of C must contain at least one vertex and both $\langle U_{j-1} \rangle$ and $\langle U_j \rangle$ are circuits. Thus this interior vertex cannot be in U_{j-1} or U_j. However, if this vertex is in U_{j+1} then $\langle U_{j+1} \rangle$ is not connected, also a contradiction.

We proceed by double induction on r and $k := d_r(u, v)$, where both u and v are in U_r and each has exactly two lower neighbors. Since U_1 contains no vertices with two lower neighbors, our claim holds vacuously when $r = 1$.

Suppose that s is the least integer such that U_s contains a pair u, v of adjacent vertices (i.e., $k = 1$) such that each has exactly two lower neighbors. Thus $s \geq 2$. We may assume that u and v have no common lower neighbor v'; otherwise the edge $[u, v']$ would be incident with two 3-covalent faces, violating the definition of $\mathcal{G}_{5,3+}$. Thus u and v have adjacent lower neighbors, u' and v' , respectively. Since u' and v' each have exactly one upper neighbor, each has exactly two lower neighbors, contrary to the minimality of s. Hence there are no adjacent vertices on U_r both of which have two lower neighbors, proving the claim for $k = 1$ and all $r \geq 1$.

To continue the inductive proof, assume that for some integer $k \geq 1$ and any $r \geq 2$, if u and v are both vertices with exactly two lower neighbors satisfying $d_r(u,v) \leq k$, then $U_r[u,v]$ contains a vertex with no lower neighbors.

Suppose for some $r \geq 2$ that $d_r(u,v) = k+1$. Let u' and v' be the lower-neighbors of u and v, respectively, such that the path $U_{r-1}[u',v']$ contains no other lower neighbors of u or v.

Suppose that $u' = v'$, and let w be the level neighbor of u in $U_r[u,v]$. By the induction hypothesis, w has at most one lower neighbor. If indeed w had a lower neighbor, then its lower neighbor would be u' and the edge $[u,u']$ would be incident with two 3-covalent faces, contradicting the definition of $\mathcal{G}_{5,3+}$. Thus w has no lower neighbor.

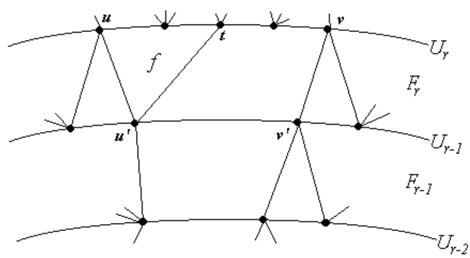

Figure 2: The inductive step in the proof of Lemma 3.2.

If $d(u',v') = 1$, then by the foregoing argument, we may assume without loss of generality that u' has at most one lower neighbor. Thus u' has at least two upper neighbors u and t such that no other upper-neighbor of u' lies in $U_r[u,t]$ (see Figure 2). Consider the unique face f in F_r incident with $[u,u']$ and $[u',t]$. If f were 3-covalent, then $[u,u']$ would be incident with two 3-covalent faces. Thus f must be at least 4-covalent, and therefore u has a level-neighbor in $U_r[u,t]$ which has no lower neighbor.

Now suppose that $d_{r-1}(u',v') \geq 2$ and that every vertex of $U_r[u,v]$ has at least one lower neighbor. By the induction hypothesis, each nonterminal vertex of $U_r[u,v]$ has exactly one lower neighbor. There are two possibilities: either $d_{r-1}(u',v') = k+1$ and the set of edges with one endpoint in each of $U_r[u,v]$ and $U_{r-1}[u',v']$ determines a perfect matching between the sets of vertices in these paths, or $d_{r-1}(u',v') < k+1$.

Suppose that the first possibility holds. Consider w, the level neighbor of u in $U_r[u,v]$. Its unique lower neighbor w' is a level neighbor of u'. Since w' has only one upper neighbor, it must have exactly two lower neighbors. However, u' also has exactly one upper neighbor and so it, too, has two lower neighbors, a contradiction.

Finally suppose $d_{r-1}(u',v') < k+1$. By assumption, all vertices of $U_r[u,v]$ have lower neighbors. Thus if u' had more than one upper neighbor, $[u',u]$ would be incident with two 3-covalent faces, and similarly for v'. Thus both u' and v' must have two lower neighbors, and by the induction hypothesis, $U_{r-1}[u',v']$ contains a vertex z with no lower neighbors. Since $\rho(z) \geq 5$, z must have at least three upper neighbors in $U_r[u, v]$. However, if all vertices of $U_r[u, v]$ had lower neighbors, then z and three of its upper neighbors would induce a pair of adjacent 3-covalent faces. Hence $U_r[u, v]$ must contain a vertex with no lower neighbors. ∎

4 Main Results

We are now ready to state and prove the first of our two main results, the second being obtainable from the first via a duality argument.

Theorem 4.1. *All B-diagrams and all B^*-diagrams of a map in $\mathcal{G}_{6,3} \cup \mathcal{G}_{5,3+}$ satisfy the following two conditions:*

1. *They are concentric: thus every map in $\mathcal{G}_{6,3} \cup \mathcal{G}_{5,3+}$ is uniformly concentric.*
2. *Every vertex in U_r or U_r^* has at most two lower neighbors.*

Proof. We will use induction to prove that properties (1) and (2) hold simultaneously. Intuitively, B-diagrams and B^*-diagrams of the same map should differ only locally (near the center), and should have many of the same properties elsewhere. Thus we investigate separately what happens near the centers of the two kinds of diagrams, and then examine them simultaneously when considering what happens away from the center.

Let $X \in \mathcal{G}_{6,3} \cup \mathcal{G}_{5,3+}$ and let B_x be a B-diagram of X with center x. For $r \geq 1$, let $A(r)$ denote the two-fold proposition that $\langle U_r \rangle$ is a circuit and that every vertex in U_r has at most two lower neighbors.

Clearly every vertex in U_1 is incident with at most one vertex (namely x) in U_0. Since all valences and covalences are finite and at least 3, $\langle U_1 \rangle$ contains a circuit C such that x is the only vertex interior to C and the edges interior to C are exactly those incident with x. Suppose $\langle U_1 \rangle$ also contains an edge $[v, w]$ not in C. Let S denote the vw-subpath of C such that the circuit $S \cup [v, w]$ does not contain x in its interior. S has length at least 2. Let Y denote the submap consisting of the circuit $S \cup [v, w]$ together with its interior. We then have $\rho_Y(v) \geq 2$ and $\rho_Y(w) \geq 2$.

If $X \in \mathcal{G}_{6,3}$, then all nonterminal vertices of S are at least 5-valent in Y. Since S contains at least one nonterminal vertex, $|VY| \geq 6$. By Lemma 3.1(1), Y cannot exist.

If $X \in \mathcal{G}_{5,3+}$, then all nonterminal vertices of S are at least 4-valent in Y. All interior vertices of Y are at least 5-valent. Since S contains at least one non-terminal vertex, it follows that $|VY| \geq 5$. By Lemma 3.1(3), such a map cannot exist. Thus A(1) holds.

Let $X \in \mathcal{G}_{6,3} \cup \mathcal{G}_{5,3+}$ and let B_f^* be a B^*-diagram of X with center f. For $r \geq 1$, let $B(r)$ denote the two-fold proposition that $\langle U_r^* \rangle$ is a circuit and that every vertex in U_r^* has at most two lower neighbors.

Clearly, the vertices in U_1^* have no lower neighbors. Since all covalences are finite, the boundary ∂f of f is a circuit and is contained in $\langle U_1^* \rangle$. Suppose $\langle U_1^* \rangle$ also contains an edge $[t, u]$ not in ∂f. Let R denote the tu -subpath of ∂f such that the circuit $R \cup [t, u]$ does not contain f in its interior. R has length at least 2. Let Z denote the submap consisting of the circuit $R \cup [t, u]$ together with its interior. We then have $\rho_Z(t) \geq 2$ and $\rho_Z(u) \geq 2$.

If $X \in \mathcal{G}_{6,3}$, then Lemma 3.1(1) implies as above that Z cannot exist. If $X \in \mathcal{G}_{5,3+}$, then Lemma 3.1(3) again implies that Z cannot exist, establishing $B(1)$.

The remainder of the proof is identical for B-diagrams and B^*-diagrams. To avoid repetitious notation, each time we refer to U_r, F_r, etc., one may substitute the corresponding notation for B^* -diagrams (U_r^*, F_r^*, etc.).

For some $r \geq 1$ assume $A(j)$ (respectively $B(j)$) holds for $1 \leq j \leq r$. Suppose that $z \in U_{r+1}$ has more than two lower neighbors. Since $\langle U_r \rangle$ is a circuit, it is possible to choose three lower neighbors v, u and w of z such that the following two conditions are satisfied: (i) $\langle U_r \rangle$ contains a vw-path T which includes u, v and w but no other neighbor of z, and (ii) the center (x or f, respectively) of the diagram lies outside of the circuit $[v, z] \cup [z, w] \cup T$. Let L be the submap consisting of this circuit together with its interior. Clearly $\rho_L(v) \geq 2$, $\rho_L(w) \geq 2$, and $\rho_L(z) \geq 3$.

If $X \in \mathcal{G}_{6,3}$, then by the induction hypothesis, any vertex of T other than v or w must be at least 4-valent in L. By Lemma 3.1(1) this submap cannot exist.

If $X \in \mathcal{G}_{5,3+}$, then by the induction hypothesis, any nonterminal vertex of T must be at least 3-valent in L. Let T' denote the set of non-terminal vertices of T. By Lemma 3.2, the number of vertices in T' with exactly two lower neighbors may exceed the number of vertices in T' with no lower neighbors by at most one. The latter type of vertex is at least 5-valent in L and the former is at least 3-valent in L, contrary to Lemma 3.1(4). We conclude that every vertex in U_{r+1} has at most two lower neighbors.

It remains to show that $\langle U_{r+1} \rangle$ is a circuit. Once again let $z \in U_{r+1}$; thus z has at most two lower neighbors.

If z has zero lower neighbors, then z is incident with a unique face $f \in F_{r+1}$ and two edges $e_1, e_2 \in E\langle U_{r+1} \rangle$ which are also incident with f. Otherwise z is incident with distinct faces $f_1, f_2 \in F_{r+1}$ and distinct edges $e_1, e_2 \in E\langle U_{r+1} \rangle$ incident with f_1 and f_2, respectively. Let z' be any other vertex in U_{r+1} which is not incident with e_1 or e_2. Define e_1', e_2', f', f_1', and f_2' in the same way with respect to z' as the unprimed symbols are defined with respect to z.

We suppose that an edge e_3 joins z to z' and derive a contradiction from this assumption.

Let w and w' denote the endvertices of e_i and e_j' other than z and z', respectively, for some i and some j in $\{1, 2\}$. Suppose f_i and f_j' are incident with a common vertex $v \in U_{r+1}$. Let W denote the submap consisting of the circuit $U_{r+1}[z, z'] \cup e_3$ and its interior (where v lies on $U_{r+1}[z, z']$). Clearly $\rho_W(z) \geq 2$ and $\rho_W(z') \geq 2$. If $X \in \mathcal{G}_{6,3}$, then $\rho_W(v) \geq 4$ and all other vertices of $U_{r+1}[z,z']$ are at least 6-valent in W. If $X \in \mathcal{G}_{5,3+}$ then $\rho_W(v) \geq 3$ and all other vertices of $U_{r+1}[z,z']$ are at least 5-valent in W. By Lemma 3.1(1,3), W cannot exist. Thus f_i and f'_j are not incident with a common vertex of U_{r+1}.

(In the following construction, in the case where z or z' has no lower neighbors, respectively substitute f for f_i or f_j, or substitute f' for f'_i or f'_j.)

Construct a zz'-path P in $\langle U_r \cup U_{r+1}\rangle$ as follows (see Figure 3):

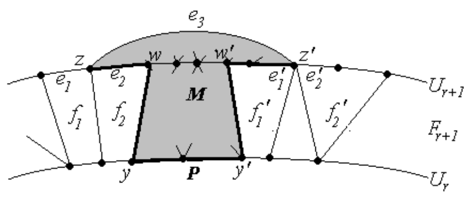

Figure 3: A zz'-path in $\langle U_r \cup U_{r+1}\rangle$.

Exit z via e_i and continue along the boundary ∂f_i of f_i to U_r. By Lemma 3.1(1,3) z and z' are not incident with the same face in F_{r+1}; therefore z' is not encountered before U_r. Leaving ∂f_i, continue along $\langle U_r\rangle$ to $\partial f'_j$, then, leaving $\langle U_r\rangle$, continue along the boundary of f'_j to e'_j to z'. It is possible to have chosen $i, j \in \{1,2\}$ and the directions around the boundaries of the faces f_i and f'_j so that x, f_1, f_2, f'_1, and f'_2 all lie outside of the circuit $P \cup e_3$. Let M be the submap of X consisting of this circuit together with its interior, and let y and y' (not necessarily distinct) be the vertices in ∂f_i and $\partial f'_j$ which lie on $P \cap \langle U_r\rangle$.

Since f_i and f'_j are not incident with a common vertex in U_{r+1}, their boundaries are disjoint except possibly for the single vertex $y = y'$. We see that z, z', y and y' are at least 2-valent in M, and $w \neq w'$. If $X \in \mathcal{G}_{6,3}$ then $\rho_M(w), \rho_M(w') \geq 6$. By the induction hypothesis and Lemma 3.1(2), such a map M cannot exist. If $X \in \mathcal{G}_{5,3+}$ then $\rho_M(w), \rho_M(w') \geq 5$ and we can apply Lemma 3.2 to the nonterminal vertices of $P \cap \langle U_r\rangle$ to conclude that the number of vertices with exactly two lower neighbors (which are at least 3-valent in M) and the number of vertices with no lower neighbors (which are ($\geq$ 5)-valent in M) satisfy the hypotheses of Lemma 3.1(5). Thus M cannot exist and z has exactly two level neighbors.

It follows that the underlying graph of $\langle U_{r+1}\rangle$ is the union of one or more disjoint circuits. It suffices to show that $\langle U_{r+1}\rangle$ is connected. If not, then $\langle U_{r+1}\rangle$ would consist of at least two disjoint circuits which partition the plane into at least three regions. Since X is 1-ended, exactly one of these regions contains an infinite submap of X. By assumption, all covalences are finite, and by definition, $\langle U_{r+1}\rangle$ separates x from U_j for all $j > r+1$. Thus there is a region containing the finite connected submap $\langle \bigcup_{i=0}^{r} U_i\rangle$. Hence there is another region which, together with its boundary (one of the disjoint circuits making up $\langle U_{r+1}\rangle$), is a finite component W of $\langle \bigcup_{i=r+1}^{\infty} U_i\rangle$. Since W is finite, there exists a largest value $\hat{r}$ of r such that $VW \cap U_{\hat{r}} \neq \emptyset$. However, then no vertex in $VW \cap U_{\hat{r}}$ has any upper neighbors (because $VW \cap U_{\hat{r}}$ is separated from $U_{\hat{r}+1}$ by $\langle U_{r+1} \rangle$), giving a contradiction, and $\langle U_{r+1} \rangle$ is connected. By this inductive argument, both B_x and B_f^* are concentric. ∎

We use the dual relationship between B-diagrams of maps and B^*-diagrams of their planar duals to show that the maps in $\mathcal{G}_{3,6}$ and $\mathcal{G}_{3+,5}$ are also uniformly concentric.

Corollary 4.2. *All B-diagrams and all B^*-diagrams of a map in $\mathcal{G}_{3,6} \cup \mathcal{G}_{3+,5}$ satisfy the following:*

1. *They are concentric: thus every map in $\mathcal{G}_{3,6} \cup \mathcal{G}_{3+,5}$ is uniformly concentric.*
2. *Every vertex in U_r or U_r^* has at most one lower neighbor.*
3. *Every face in F_r or F_r^* is incident with at most two edges of $\langle U_{r-1} \rangle$ or $\langle U_{r-1}^* \rangle$.*

Proof. Any map $Z \in \mathcal{G}_{3,6} \cup \mathcal{G}_{3+,5}$ is the dual of a map $X \in \mathcal{G}_{6,3} \cup \mathcal{G}_{5,3+}$. By Theorem 4.1, a labeling of a concentric B-diagram B_x of X corresponds to a labeling of a concentric B^*-diagram B_f of Z. In particular, x corresponds to the face f, the faces in F_r become the vertices in U_r^*, and the vertices in U_r become the faces in F_{r+1}^*. Since X is uniformly vertex-concentric and these diagrams are uniquely determined by their centers, every B^*-diagram of Z is concentric, and so Z is uniformly face-concentric.

Similarly, since X is uniformly face-concentric, a labeling of a concentric B^*-diagram B_g^* of X corresponds to a labeling of a concentric B-diagram B_w of Z where g corresponds to w, the vertices in U_r^* correspond to the faces in F_r, and the faces in F_r^* correspond to the vertices in U_{r-1}. It follows similarly that Z is uniformly concentric.

Since, by Theorem 4.1, all B-diagrams and all B^*-diagrams of X have the property that each vertex has at most two lower neighbors, every such vertex has at least one upper neighbor. Hence no face in F_r or F_r^* can be incident with more than one edge of $\langle U_{r-1} \rangle$ or $\langle U_{r-1}^* \rangle$, respectively. Thus, given a B-diagram of $X \in \mathcal{G}_{6,3} \cup \mathcal{G}_{5,3+}$, if $g \in F_r$ then the corresponding vertex in the dual map (labeled as a B^*-diagram) has at most one lower neighbor. The argument proceeds similarly given a B^*-diagram of X. ∎

Niemeyer and Watkins [6] have shown that every map in $\mathcal{G}_{4,4}$ is uniformly vertex-concentric. However, since $\mathcal{G}_{4,4}$ is closed with respect to planar duality, we have the following result:

Corollary 4.3. *Every map in $\mathcal{G}_{4,4}$ is uniformly concentric.*

5 Concluding Remarks

Are the foregoing results best-possible? Theorem 4.1 fails if the map is assumed to be in $\mathcal{G}_{5,3} \backslash \mathcal{G}_{5,3+}$. For example, the 5-valent tessellation of the plane shown in Figure 4 is neither uniformly vertex-concentric nor uniformly face-concentric. A B^*-diagram of this map is concentric if the central face is 4-covalent but not if the central face is 3-covalent.

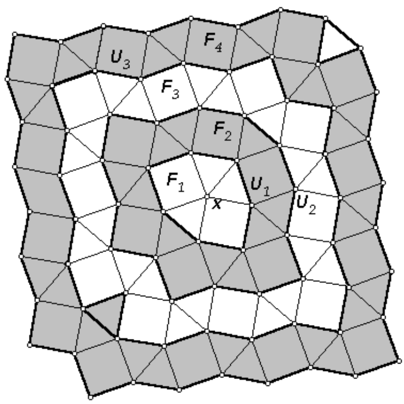

Figure 4: A map in $\mathcal{G}_{5,3}$ which is not uniformly concentric.

What can be said about the infinite maps in $\mathcal{G}_{4,3}$, $\mathcal{G}_{3,4}$, or $\mathcal{G}_{3,3}$? Some necessary conditions for uniform concentricity of maps in these classes are presented below.

Theorem 5.1. *If an infinite planar map X admits any of the following configurations, then it is not uniformly concentric:*

1. *a 3-valent vertex incident with a 3-covalent face;*
2. *a 4-valent vertex incident with two nonadjacent 3-covalent faces;*
3. *an edge incident with two 3-valent vertices and two 4-covalent faces;*
4. *a 4-covalent face incident with two nonadjacent 3-valent vertices;*
5. *an edge incident with two 4-valent vertices and two 3-covalent faces.*

Proof. Suppose that X is an infinite uniformly concentric map. By Proposition 2.1, X is 3-connected.

(1) Let z be a 3-covalent vertex incident with a 3-covalent face g. Let u and v be the other two vertices incident with g, and let f be the other face incident with edge $[u, v]$. We show that B_f^* is not concentric.

Clearly $u, v \in U_1^*$ and $g \in F_2^*$. If $z \in U_1^*$, then $\langle U_1^* \rangle$ contains edges $[u, z]$ and $[v, z]$ not in ∂f. Hence $z \in U_2^*$. But then z has at most one neighbor in U_2^*, and so $\langle U_2^* \rangle$

is not a circuit.

(2) Let z be a 4-valent vertex, and let g_1 and g_2 be nonadjacent 3-covalent faces incident with z. For $i = 1, 2$, let u_i and v_i be the other two vertices incident with g_i. Let f be the other face incident with edge $[u_1, v_1]$. We again show that B^*_f is not concentric.

As in (1), we have $u_1, v_1 \in U^*_1$, $g_1 \in F^*_2$, and similarly we obtain that $z \in U^*_2$. Since $\langle U^*_2 \rangle$ is a circuit, it must contain the neighbors u_2, v_2 of z. But since g_2 is 3-covalent, the edge $[u_2, v_2]$ exists and lies in $\langle U^*_2 \rangle$, which therefore cannot be a circuit.

(3) Let X contain an edge whose incident vertices v_1 and v_2 are 3-valent and whose incident faces g and h are 4-covalent. Let u_1 and u_2 be the other vertices incident with g, and let f be the other face incident with edge $[u_1, u_2]$. If B^*_f were concentric, then we would have $g \in F^*_2$ and $[v_1, v_2] \in \langle U^*_2 \rangle$. Since v_1 and v_2 are 3-valent and $\langle U^*_2 \rangle$ must be a circuit, two other edges incident with h, say $[v_1, w_1]$ and $[v_2, w_2]$ must also be in $\langle U^*_2 \rangle$. But then the fourth edge $[w_1, w_2]$ incident with h is in $\langle U^*_2 \rangle$ as well, and $\langle U^*_2 \rangle$ is exactly the boundary of h. It follows that either X is finite or $\{v_1, v_2\}$ is a separating set, giving a contradiction.

(4) If X satisfies the conditions of (4) then X^* satisfies the conditions of (2). Hence X^* is not uniformly face-concentric, and so X is not uniformly vertex-concentric.

(5) This is the dual situation of (3). ∎

If X satisfies any of the first three conditions of this theorem and if y is any vertex incident with f other than the vertices already named, then it is easy to verify that B_y is not concentric, i.e., X is neither uniformly vertex-concentric nor uniformly face-concentric. We do not know whether X is uniformly face-concentric when X satisfies conditions (4) or (5), even though they are dual to conditions (2) and (3), respectively.

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