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Decomposition of lambda-fold complete graphs into a certain five-vertex graph

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Abstract

A $(\lambda K_v, G)$-design is a partition of the edges of λK_v into subgraphs each of which is isomorphic to G. In this paper, we completely solve the case when $G = G_{18}$ (G_{18} is notation from Bermond, Huang, Rosa and Sotteau, Ars Combin. 10 (1980), 211–254) and prove that the necessary condition $\lambda v(v-1) \equiv 0 \pmod{14}$ for the existence of a $(\lambda K_v, G_{18})$-design with any positive integer λ is also sufficient except for $(v, \lambda) = (8, 1), (14, 1)$.

1 Introduction

A complete multigraph λK_v is a complete graph K_v in which every edge is taken λ times. Let $G = (V(G), E(G))$ be a simple graph without isolated vertices. A $(\lambda K_v, G)$-*design* is a partition of the edges of λK_v into subgraphs (G-*blocks*) each of which is isomorphic to G. When the graph G is itself a complete graph K_k, the $(\lambda K_v, K_k)$-design is known as a (v, k, λ)-BIBD. If there exists a $(\lambda K_v, G)$-design, then

(1) $\lambda v(v-1) \equiv 0 \pmod{2|E(G)|}$, and

(2) $\lambda(v-1) \equiv 0 \bmod d$, where d is the greatest common divisor of the degrees of the vertices of G.

It was proved in [10] that the necessary conditions (1) and (2) for the existence of a $(\lambda K_v, G)$-design are asymptotically sufficient, that is, there exists an integer $N(G, \lambda)$ such that there is a $(\lambda K_v, G)$-design for all $v \geq N(G, \lambda)$ and λ satisfying the necessary conditions (1) and (2).

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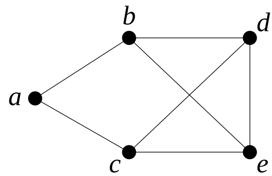
The existence of a $(\lambda K_v, G)$-design for various graphs G has been studied in the literature (see [3, 5, 6, 11]). The case where G is a graph with at most four vertices has been solved completely in [2]. If G has no isolated vertices and $|V(G)| = 5$, the known existence of a (K_v, G)-design has been very nearly solved in [1, 7, 8, 9], which is also summarized in [5]. The second author of this paper, in [4], removed the open cases for the two classes of graph designs (i.e. $G = G_{16}, G_{20}$; the notation is borrowed from [1]). There remain several graphs for which there are a few values of v for which it is not known whether or not decompositions of K_v exist. In this article, we will deal with the graph $G_{18} = (V, E)$ (with notation the same as in [1]), where $V = \{a, b, c, d, e\}$, and $E = \{ab, ac, bd, de, ec, be, cd\}$. We usually denote G_{18} as $[a, b, c, d, e]$.



G_{18}

It is known, see [5], that:

Lemma 1.1 *The necessary condition* $v \equiv 0, 1 \pmod 7$, $v \geq 7$ *and* $v \neq 8,\ 14$ *for the existence of a* (K_v, G_{18})*-design is sufficient except for the possible exceptions of* $v = 36,\ 56,\ 92,\ 98,\ 120$.

In this paper, we first remove the unsolved cases in Lemma 1.1, and then we will give the existence spectrum of a $(\lambda K_v, G_{18})$-design with $\lambda > 1$.

2 Preliminaries

For the sake of convenience, sometimes we denote a $(\lambda K_v, G)$-design as $\lambda K_v \to G$. Let $\lambda K_{m_1, m_2, \ldots, m_n}$ be the complete multipartite multigraph with vertex set $V = \bigcup_{i=1}^{n} V_i$, where V_i $(1 \leq i \leq n)$ are disjoint sets with $|V_i| = m_i$ $(i = 1,\ 2, ..., n)$ and where two vertices x and y from different sets V_i and V_j are joined by exactly λ edges. We denote a $(\lambda K_{m_1, m_2, \ldots, m_n}, G)$-design as $\lambda K_{m_1, m_2, \ldots, m_n} \to G$.

For $\lambda = 1$, the following three lemmas are well illustrated in [1]. The development for $\lambda \geq 1$ is natural.

Lemma 2.1 *If* $\lambda K_{n_i} \to G$ *for* $1 \leq i \leq h$, $\lambda \geq 1$ *and* $\lambda K_{n_1, n_2, \ldots, n_h} \to G$, *then* $\lambda K_n \to G$, *where* $n = \sum_{i=1}^{h} n_i$.

Lemma 2.2 *If* $\lambda K_{n_i+1} \rightarrow G$ *for* $1 \leq i \leq h$, $\lambda \geq 1$ *and* $\lambda K_{n_1,n_2,\ldots,n_h} \rightarrow G$, *then* $\lambda K_n \rightarrow G$, *where* $n = 1 + \sum_{i=1}^{h} n_i$.

Lemma 2.3 *If* $\lambda K_{r_1,r_2,r_3} \rightarrow G$ *and* $\lambda K_{r_1,r_2,r_3'} \rightarrow G$, *then* $\lambda K_{ar_1,ar_2,(a-b)r_3+br_3'} \rightarrow G$ *for integers* a, b *with* $0 \leq b \leq a$.

The following two lemmas are simple but useful.

Lemma 2.4 *Let* m, λ *be positive integers. If* $\lambda K_v \rightarrow G$, *then* $m\lambda K_v \rightarrow G$.

Lemma 2.5 *If* $2K_v \rightarrow G$, *and there exists an odd integer* $q > 0$ *such that* $qK_v \rightarrow G$, *then for any positive integer* $\lambda \geq q$, $\lambda K_v \rightarrow G$.

3 Constructions of (K_v, G_{18})-designs for unsolved v

In this section, we will first give direct constructions for the cases $v = 36, 56$, and then decompositions when $v = 92,\ 98,\ 120$ can be obtained by recursive constructions.

Lemma 3.1 [1] *A* $(K_{7,7,7}, G_{18})$*-design and a* $(K_{7,7,14}, G_{18})$*-design exist.*

Lemma 3.2 *There exists a* (K_v, G_{18})*-design for* $v = 36, 56$.

Proof With the aid of a computer, we find a (K_v, G_{18})-design for $v = 36, 56$ by listing the base blocks as follows.

$K_{36} \rightarrow G_{18}$: Let $V(K_{36}) = Z_9 \times I_4$ where $I_4 = \{0, 1, 2, 3\}$. The base blocks are:

$$\begin{array}{lll}
[0_0, 1_0, 2_0, 5_0, 0_1], & [0_0, 0_1, 1_1, 4_0, 6_1], & [0_0, 3_1, 0_2, 1_1, 1_2], \\
[0_0, 1_2, 2_2, 3_0, 7_2], & [0_0, 3_2, 5_2, 0_1, 1_1], & [0_0, 6_2, 0_3, 0_1, 1_3], \\
[0_0, 1_3, 2_3, 3_0, 7_3], & [0_1, 2_3, 7_3, 3_1, 4_2], & [0_2, 5_3, 6_3, 4_2, 6_2], \\
[0_3, 3_0, 3_1, 6_3, 8_3] & \text{(cycled mod 9).} &
\end{array}$$

$K_{56} \rightarrow G_{18}$: Let $V(K_{56}) = (Z_{11} \times I_5) \cup \{\infty\}$ where $I_5 = \{0, 1, 2, 3, 4\}$. The base blocks are:

$$\begin{array}{lll}
[0_0, 1_0, 2_0, 5_0, 0_1], & [0_0, 0_1, 5_0, 1_1, 8_1], & [0_0, 1_1, 5_1, 8_0, 10_1], \\
[0_0, 0_2, 1_2, 2_0, 4_2], & [0_0, 3_2, 4_2, 7_0, 2_2], & [0_0, 5_2, 0_3, 0_1, 1_3], \\
[0_0, 1_3, 2_3, 3_0, 8_3], & [0_0, 7_3, 8_3, 4_0, 10_3], & [0_0, 0_4, 1_4, 4_0, 3_4], \\
[0_0, 2_4, 4_4, 0_1, 0_2], & [0_0, 5_4, 6_4, 6_1, 5_2], & [0_1, 1_2, 2_2, 5_1, 0_3], \\
[0_1, 3_2, 6_2, 0_3, 0_4], & [0_2, 0_3, 1_3, 2_1, 5_4], & [0_2, 2_3, 3_3, 0_1, 9_4], \\
[0_3, 6_1, 7_2, 2_4, 3_4], & [0_3, 4_1, 2_4, 1_3, 9_4], & [0_4, 1_2, 1_3, 4_4, 10_4], \\
[\infty, 0_2, 0_3, 7_1, 5_2], & [2_4, 1_1, 4_0, \infty, 7_4] & \text{(cycled mod 11).}
\end{array}$$

□

Lemma 3.3 *There exists a (K_v, G_{18})-design for $v = 92,\ 98,\ 120$.*

Proof **Case** $v = 92$: $K_{28,28,35} \to G_{18}$ can be obtained by applying Lemma 2.3 and Lemma 3.1 with $\lambda = 1,\ r_1 = r_2 = r_3 = 7,\ r_3' = 14,\ a = 4,\ b = 1$. There exist $K_{29} \to G_{18}$ by Lemma 1.1 and $K_{36} \to G_{18}$ by Lemma 3.2. Then $K_{92} \to G_{18}$ follows by Lemma 2.2.

Case $v = 98$: $K_{28,28,42} \to G_{18}$ can be obtained by applying Lemma 2.3 and Lemma 3.1 with $\lambda = 1,\ r_1 = r_2 = r_3 = 7,\ r_3' = 14,\ a = 4,\ b = 2$. Since $K_{28} \to G_{18}$ and $K_{42} \to G_{18}$ by Lemma 1.1, $K_{98} \to G_{18}$ follows by Lemma 2.1.

Case $v = 120$: $K_{35,35,49} \to G_{18}$ can be obtained by applying Lemma 2.3 and Lemma 3.1 with $\lambda = 1,\ r_1 = r_2 = r_3 = 7,\ r_3' = 14,\ a = 5,\ b = 2$. Since $K_{50} \to G_{18}$ by Lemma 1.1 and $K_{36} \to G_{18}$ by Lemma 3.2, $K_{120} \to G_{18}$ follows by Lemma 2.2. □

Theorem 3.4 *The necessary condition $v \equiv 0, 1 \pmod 7$, $v \geq 7$ and $v \neq 8,\ 14$ for the existence of (K_v, G_{18})-design is also sufficient.*

Proof This follows from Lemmas 1.1, 3.2 and 3.3. □

4 Decompositions of λK_v with $\lambda \geq 2$ into G_{18}

In this section, we investigate the existence of $(\lambda K_v, G_{18})$-designs with $\lambda \geq 2$, and prove that the necessary conditions for the existence of a $(\lambda K_v, G_{18})$-design are also sufficient. We know that if there exists a $(\lambda K_v, G_{18})$-design then $\lambda v(v-1) \equiv 0 \pmod{14}$, which produces two cases:

(1) $v \equiv 0, 1 \pmod 7$ and $\gcd(\lambda, 7) = 1$;

(2) $v \geq 5$ and $\gcd(\lambda, 7) = 7$.

In Case (1), the existence of $\lambda K_v \to G_{18}$ has been nearly solved by applying Theorem 3.4 and Lemma 2.4 except for $v = 8, 14$. We will consider $\lambda K_v \to G_{18}$ for $v = 8, 14$.

Theorem 4.1 *If $v \equiv 0, 1 \pmod 7$ and $v \geq 7$, then a $(\lambda K_v, G_{18})$-design exists for every integer $\lambda \geq 1$ except for $(v, \lambda) = (8, 1), (14, 1)$.*

Proof By Theorem 3.4 and Lemmas 2.4–2.5, we only need to construct $2K_v \to G_{18}$ and $3K_v \to G_{18}$ for $v = 8, 14$.

$2K_8 \to G_{18}$: $[0, 1, 2, 3, 6]$ (cycled mod 8).

$2K_{14} \to G_{18}$: Let $V(2K_{14}) = Z_{13} \cup \{\infty\}$. The base blocks are:

$$[0,1,2,4,10], \quad [\infty,3,1,7,6] \quad \text{(cycled mod 13)}.$$

$3K_8 \to G_{18}$: Let $V(3K_8) = (Z_3 \times I_2) \cup \{\infty_1, \infty_2\}$. The base blocks are:

$$[\infty_1,\infty_2,0_1,0_0,1_0], \quad [\infty_2,0_1,1_0,2_1,\infty_1], \quad [\infty_1,0_0,1_0,0_1,2_0],$$
$$[\infty_2,0_1,2_1,1_1,2_0] \quad \text{(cycled mod 3)}.$$

$3K_{14} \to G_{18}$: Let $V(3K_{14}) = Z_{13} \cup \{\infty\}$. The base blocks are:

$$[1,\infty,3,7,2], \quad [0,1,2,3,6], \quad [0,5,6,2,12] \quad \text{(cycled mod 13)}.$$

□

Next we consider a $(7K_v, G_{18})$-design for any integer $v \geq 5$. We need the following lemmas.

Lemma 4.2 *Let v be an odd integer such that $5 \leq v \leq 17$; then there exists a $(7K_v, G_{18})$-design.*

Proof The conclusion follows from Lemma 2.4 and Theorem 3.4 when $v = 7, 15$. The other cases are constructed by listing the base blocks of a $(7K_v, G_{18})$-design as follows (where $V(7K_v) = Z_v$).

$7K_5 \to G_{18}$:

$$[0,1,2,3,4], \quad [0,1,3,2,4] \quad \text{(cycled mod 5)}.$$

$7K_9 \to G_{18}$:

$$[0,1,2,3,4], \quad [0,1,2,3,4], \quad [0,3,4,7,8], \quad [0,4,5,1,7]$$
$$\text{(cycled mod 9)}.$$

$7K_{11} \to G_{18}$:

$$[0,1,2,3,4], \quad [0,1,2,3,4], \quad [0,2,3,6,9], \quad [0,4,5,1,9],$$
$$[0,4,5,9,10] \quad \text{(cycled mod 11)}.$$

$7K_{13} \to G_{18}$:

$$[0,1,2,3,4], \quad [0,1,2,3,4], \quad [0,1,2,5,8], \quad [0,3,4,8,11],$$
$$[0,5,6,1,10], \quad [0,6,7,1,11] \quad \text{(cycled mod 13)}.$$

$7K_{17} \to G_{18}$:

$$\begin{array}{llll} [0,1,2,3,4], & [0,1,2,3,4], & [0,1,2,5,8], & [0,3,4,7,10], \\ [0,4,5,9,13], & [0,6,8,1,13], & [0,8,9,2,14], & [0,8,9,2,15] \\ \text{(cycled mod 17).} & & & \end{array}$$

□

Lemma 4.3 *Let v be even such that $5 \le v \le 14$; then there exists a $(7K_v, G_{18})$-design.*

Proof The conclusion follows from Theorem 4.1 when $v = 8, 14$. Next we construct $7K_v \to G_{18}$ for $v = 6, 10, 12$.

$7K_6 \to G_{18}$: Let $V(7K_6) = Z_3 \times I_2$ where $I_2 = \{0, 1\}$. The base blocks are:

$$\begin{array}{lll} [0_0,1_0,2_0,0_1,1_1], & [0_0,1_0,2_0,0_1,1_1], & [0_0,1_0,2_0,0_1,1_1], \\ [0_0,0_1,1_1,1_0,2_1], & [0_1,0_0,2_1,1_0,1_1] & \text{(cycled mod 3).} \end{array}$$

$7K_{10} \to G_{18}$: Let $V(7K_{10}) = Z_5 \times I_2$ where $I_2 = \{0, 1\}$. The base blocks are:

$$\begin{array}{lll} [0_0,1_0,2_0,3_0,4_0], & [0_0,1_0,2_0,3_0,0_1], & [0_0,1_0,2_0,0_1,1_1], \\ [0_0,1_0,0_1,1_1,2_1], & [0_0,0_1,1_1,1_0,2_1], & [0_0,0_1,1_1,1_0,2_1], \\ [0_0,2_1,3_1,1_0,0_1], & [0_0,2_1,3_1,1_0,4_1], & [0_0,2_1,3_1,4_0,1_1] \\ \text{(cycled mod 5).} & & \end{array}$$

$7K_{12} \to G_{18}$: Let $V(7K_{12}) = Z_3 \times I_4$ where $I_4 = \{0, 1, 2, 3\}$. The base blocks are:

$$\begin{array}{lll} [0_0,1_0,2_0,0_1,1_1], & [0_0,0_1,1_0,2_0,1_1], & [0_0,0_1,1_0,2_0,1_1], \\ [0_0,1_1,1_0,0_1,2_1], & [0_0,0_1,1_1,1_0,0_2], & [0_0,1_1,2_1,0_2,1_2], \\ [0_0,0_2,1_2,1_0,2_2], & [0_0,0_2,1_2,1_0,2_2], & [0_0,0_2,1_2,1_0,2_2], \\ [0_0,0_2,2_2,1_0,0_3], & [0_0,2_2,0_3,0_1,1_1], & [0_0,0_3,1_3,1_0,2_3], \\ [0_0,0_3,1_3,1_0,2_3], & [0_0,0_3,2_3,1_0,1_3], & [0_0,1_3,2_3,0_1,0_2], \\ [0_0,1_3,2_3,0_1,0_2], & [0_1,0_2,1_2,1_1,0_3], & [0_1,0_2,1_2,2_1,0_3], \\ [0_1,0_3,1_3,1_1,0_2], & [0_1,0_3,1_3,1_1,2_2], & [0_2,0_3,1_3,0_1,1_2], \\ [0_3,0_1,0_2,1_3,2_3] & \text{(cycled mod 3).} & \end{array}$$

□

Lemma 4.4 *There exists a $(7K_v, G_{18})$-design for $v = 16$.*

Proof Let $V(7K_{5,5,5}) = Z_{15} = X_1 \cup X_2 \cup X_3$ where $X_i = \{3j + i : j = 0, 1, 2, 3, 4\}$. $7K_{5,5,5} \to G_{18}$ is constructed by listing the base blocks as follows:

$$\begin{array}{lll}[0,1,4,2,3], & [0,1,4,2,6], & [0,1,4,9,11], \\ [0,2,5,10,12], & [0,5,8,1,12] & \text{(cycled mod 15).}\end{array}$$

By Lemmas 2.2 and 4.2, there exists a $(7K_{16}, G_{18})$-design. □

To complete the existence of a $(7K_v, G_{18})$-design, we also need the following designs.

Lemma 4.5 *There exist a $(7K_{2,2,2}, G_{18})$-design and a $(7K_{2,2,4}, G_{18})$-design.*

Proof $7K_{2,2,2} \to G_{18}$: Let $V(7K_{2,2,2}) = Z_6 = X_1 \cup X_2 \cup X_3$ where $X_i = \{i, i+3\}$, $i = 0, 1, 2$. The base blocks are: $[0,1,4,2,3]$, $[0,1,4,3,5]$ (cycled mod 6).

$7K_{2,2,4} \to G_{18}$: Let $V(7K_{2,2,4}) = Z_4 \times I_2 = X_1 \cup X_2 \cup X_3$ where $X_1 = \{0_0, 1_0, 2_0, 3_0\}$, $X_2 = \{0_1, 2_1\}$, $X_2 = \{1_1, 3_1\}$. The base blocks are:

$$\begin{array}{lll}[0_0,0_1,2_1,1_0,1_1], & [0_0,0_1,2_1,1_0,1_1], & [0_1,0_0,1_0,2_1,3_1], \\ [0_1,0_0,3_0,1_1,2_1], & [0_1,1_0,3_0,1_1,2_1] & \text{(cycled mod 4).}\end{array}$$

□

Theorem 4.6 *If* $\gcd(\lambda, 7) = 7$*, then there exists a $(\lambda K_v, G_{18})$-design for any integer $v \geq 5$.*

Proof Use induction on v to prove that there exists a $7K_v \to G_{18}$ for any integer $v \geq 5$. The result follows by Lemmas 4.2–4.4 for $5 \leq v \leq 17$. Next we consider the case $v \geq 18$. Let $v = 6t + w$ where $0 \leq w \leq 5$. Then $t \geq 3$. We divide the problem into two cases:

Case 1: $w = 0, 2, 4$**.** Since both $7K_{2,2,2} \to G_{18}$ and $7K_{2,2,4} \to G_{18}$ exist by Lemma 4.5, there exists a $7K_{2t,2t,2t+w} \to G_{18}$ by Lemma 2.3. By induction there is a $7K_{2t} \to G_{18}$ and $7K_{2t+w} \to G_{18}$. Thus, $7K_{6t+w} \to G_{18}$ by Lemma 2.1.

Case 2: $w = 1, 3, 5$**.** A similar argument shows that there is a $7K_{2t,2t,2t+w-1} \to G_{18}$. By induction and Lemma 2.2, there is a $7K_{6t+w} \to G_{18}$.

By induction, we know that there exists a $7K_v \to G_{18}$ for any integer $v \geq 5$. Since $7|\lambda$, there exists a $(\lambda K_v, G_{18})$-design for any integer $v \geq 5$ by Lemma 2.4. This completes the proof. □

5 Conclusion

Theorem 5.1 *The necessary condition $\lambda v(v-1) \equiv 0 \pmod{14}$ for the existence of a $(\lambda K_v, G_{18})$-design is sufficient except for $(v, \lambda) = (8, 1), (14, 1)$.*

Proof This follows from Lemma 2.4, Theorem 4.1 and Theorem 4.6. □

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