AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **30** (2004), Pages 247–259

A new approach for enumerating maps on orientable surfaces

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Abstract

Classifying embeddings of a given graph G on orientable surfaces under the action of its automorphisms, a relation between the genus distribution of rooted maps and embeddings of graph G on orientable surfaces is established. Applying this relation enables us to enumerate rooted maps by automorphism groups or by enumerating labelled graphs with vertex partition and find new formulas for the number of rooted maps with vertex partition.

1 Introduction

All surfaces considered in this paper are 2-dimensional orientable compact closed manifolds and graphs are connected simple graphs, not being trees or circuits.

A *combinatorical map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a permutation $\mathcal{P}$ acting on $\mathcal{X}_{\alpha,\beta}$ of a disjoint union of quadricells Kx of $x \in \mathcal{X}$, where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the Klein group, satisfying the following conditions:

i) $\forall x \in \mathcal{X}_{\alpha,\beta}$, there does not exist an integer k such that $\mathcal{P}^k x = \alpha x$;

ii) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

iii) the group $\Psi_J = \langle \alpha, \beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

According to condition *ii*), the vertices of a map are defined as the pairs of conjugate of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$ and edges the orbits of K on $\mathcal{X}_{\alpha,\beta}$, for example, $\{x, \alpha x, \beta x, \alpha\beta x\}$, an edge of map. Geometrically, any map M is an embedding of a graph Γ on a surface (see also [10], [11]), denoted by $M = M(\Gamma)$ and $\Gamma = \Gamma(M)$. The graph Γ is called the underlying graph of the map M. If $r \in \mathcal{X}_{\alpha,\beta}$ is marked beforehand, then M is called a *rooted map*, denoted by M^r.

For example, the graph B_4 (a bouquet with 4 loops) on the torus shown in the following Fig. 1,

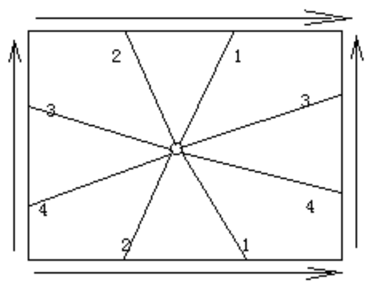

Fig. 1

can be algebraically represented as follows:

A map $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with $\mathcal{X}_{\alpha,\beta} = \{x, y, z, u, \alpha x, \alpha y, \alpha z, \alpha u, \beta x, \beta y, \beta z, \beta u, \alpha\beta x, \alpha\beta y, \alpha\beta z, \alpha\beta u\}$ and

$$\begin{aligned} \mathcal{P} &= (x, y, z, u, \alpha\beta y, \alpha\beta x, \alpha\beta u, \alpha\beta z) \\ &\times (\alpha x, \beta z, \beta u, \beta x, \beta y, \alpha u, \alpha z, \alpha y) \end{aligned}$$

The only one vertex of this map is $\{(x, y, z, u, \alpha\beta y, \alpha\beta x, \alpha\beta u, \alpha\beta z)(\alpha x, \beta z, \beta u, \beta x, \beta y, \alpha u, \alpha z, \alpha y)\}$ and four edges are $\{e, \alpha e, \beta e, \alpha\beta e\}$, where, $e \in \{x, y, z, u\}$.

Two maps $M_1 = (\mathcal{X}^1_{\alpha,\beta}, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}^2_{\alpha,\beta}, \mathcal{P}_2)$ are said to be isomorphic if there exists a bijection $\tau : \mathcal{X}^1_{\alpha,\beta} \to \mathcal{X}^2_{\alpha,\beta}$ such that for $\forall x \in \mathcal{X}^1_{\alpha,\beta}$, $\tau\alpha(x) = \alpha\tau(x)$, $\tau\beta(x) = \beta\tau(x)$, $\tau\mathcal{P}_1(x) = \mathcal{P}_2\tau(x)$ and τ is called an *isomorphism* between them. If $M_1 = M_2 = M$, then an isomorphism between M_1 and M_2 is called an *automorphism* of M. All automorphism of a map M form a group, called the *automorphism group* of M and denoted by $\mathrm{Aut}\, M$. Similarly, two rooted maps M_1^r, M_2^r are said to be isomorphic if there is an isomorphism θ between them such that $\theta(r_1) = r_2$, where r_1, r_2 are the roots of M_1^r, M_2^r, respectively and denote the automorphism group of M^r by $\mathrm{Aut}\, M^r$. It has been known that $\mathrm{Aut}\, M^r$ is the trival group.

The enumerative problem of rooted maps on orientable surfaces was first considered by Walsh and Lehman in 1972. They proved that the enufunction $F_n(x) = \sum_{i=0}^{\infty} F_{n,i}x^i$, where $F_{n,i}$ denotes the number of rooted maps with $n+i$ edges and $i+1$ vertices, satisfies the following equation [13]:

$$F_n(x) = x\sum_{k=0}^{n} F_k(x)F_{n-k}(x) + (2n-1)F_{n-1}(x) + 2x\frac{d}{dx}F_{n-1}(x) \tag{1.1}$$

Liu found the enufunction $f(x) = \sum_{k=0}^{\infty} f_k x^k$, where f_k denotes the number of rooted maps with size k on orientable surfaces, satisfying Riccati's equation [10]:

$$2x^2\frac{df}{dx} = -1 + (1-x)f - xf^2 \tag{1.2}$$

with $f_0 = f(0) = 1$.

Arquès and Bèraud [1] proved the enufunction $M(y,z) = \sum_{s,l} m(s,l)y^s z^l$, where $m(s,l)$ denotes the rooted maps on orientable surfaces with order s and size l, also satisfying Riccati's equation [1]:

$$2z^2\frac{\partial}{\partial z}[M(y,z)] = -y + (1-z)M(y,z) - zM(y,z)^2. \tag{1.3}$$

Using an algebraic approach, Jackson and Visentin obtained $M(y,z)$ as follows [8]:

$$M(y,z) = 2z\frac{\partial}{\partial z}\log\sum_{k=0}^{\infty}\frac{1}{2^n n!}y(y+1)\cdots(y+2n-1)z^n. \tag{1.4}$$

In [1], a continued fraction form of $M(y,z)$ and the number of rooted maps on orientable surfaces with size n are also obtained.

The main purpose of this paper is to enumerate the rooted maps on orientable surfaces with vertex partition and get a formula with the form as (1.4). For this object, solving Riccati's equation does not seem very efficient again even if the enufunction also satisfies this equation. By classifying embeddings of a given graph G on orientable surfaces under the action of their automorphisms, we find a relation between the genus distribution polynomial of rooted maps and embeddings of graph G on orientable surfaces and an invariant (called *map index*) depending only on the valency sequence of maps. This enables us to enumerate rooted maps by automorphism groups or by enumerating labelled (underlying) graphs of these maps with vertex partition and get the number of rooted maps with given vertex partition.

Terminology and notation not defined here can be seen in [10] and [11] for, respectively, maps and graphs, in [6] for labelled graphs and in [3] for permutation groups.

2 Enumerating maps by automorphism groups

For a given graph G with maximum valency ≥ 3, assume that $r_0(G), r_1(G), r_2(G), \cdots$, respectively are the number of rooted maps with underlying graph G on the orientable surface with genus $\gamma(G)$, $\gamma(G)+1$, $\gamma(G)+2, \cdots$, where $\gamma(G)$ denotes the minimum orientable genus of G and define the *rooted map polynomial* on genus by:

$$r[G](x) = \sum_{i\geq 0} r_i(G)x^i \tag{2.1}$$

The total number of orientable embeddings of G is $\prod_{d\in D(G)}(d-1)!$, where $D(G)$ is its valency sequence. Now let $g_0(G)$, $g_1(G)$, $g_2(G), \cdots$, respectively be the number of embeddings of G on the orientable surface with genus $\gamma(G)$, $\gamma(G)+1$, $\gamma(G)+2, \cdots$. Groos and Furst define the *genus polynomial* of G in [4] as follows:

$$g[G](x) = \sum_{i\geq 0} g_i(G)x^i \tag{2.2}$$

For $\forall g \in \mathrm{Aut}G$, there is an extended action $g|^{\mathcal{X}_{\alpha,\beta}}$ acting on $\mathcal{X}_{\alpha,\beta}$ with $\mathcal{X} = E(G)$ defined as follows:

For $\forall x \in \mathcal{X}_{\alpha,\beta}$, if $x^g = y$, then $(\alpha x)^g = \alpha y$, $(\beta x)^g = \beta y$ and $(\alpha\beta x)^g = \alpha\beta y$.

In the first place, we establish a relation between $r[G](x)$ and $g[G](x)$.

Lemma 2.1 [13] *For a given map M, the number of non-isomorphic rooted maps by rooting on M is*

$$\frac{4\varepsilon(M)}{|\mathrm{AutM}|}$$

where $\varepsilon(M)$ is the size of M.

Theorem 2.1 *For any connected graph G,*

$$|\mathrm{AutG}|r[G](x) = 2\varepsilon(G)g[G](x),$$

where AutG *and $\varepsilon(G)$ denote the automorphism group and the size of G, respectively.*

Proof For an integer k, denote by $\mathcal{M}_k(G,S)$ all the non-isomorphic unrooted maps with underlying graph G on orientable surface S with genus $\gamma(G)+k-1$. According to Lemma 2.1, we know that

$$\begin{aligned} r_k(G) &= \sum_{M\in\mathcal{M}_k(G,S)} \frac{4\varepsilon(M)}{|\mathrm{Aut}\, M|} \\ &= \frac{4\varepsilon(G)}{|\mathrm{Aut}\, G\times\langle\alpha\rangle|}\sum_{M\in\mathcal{M}_k(G,S)}\frac{|\mathrm{Aut}\, G\times\langle\alpha\rangle|}{|\mathrm{Aut}\, M|}. \end{aligned}$$

Notice that every element $\xi \in \mathrm{Aut}\, G$ naturally induces an $\xi|^{\mathcal{X}_{\alpha,\beta}}$ action on $\mathcal{X}_{\alpha,\beta}$. Since for an embedding M, $\xi \in$ AutM if and only if $\xi \in (\mathrm{Aut\,G} \times \langle\alpha\rangle)_M$, so $\mathrm{Aut\,M} = (\mathrm{Aut\,G} \times \langle\alpha\rangle)_M$. From $|\mathrm{Aut\,G} \times \langle\alpha\rangle| = |(\mathrm{Aut\,G} \times \langle\alpha\rangle)_M||M^{\mathrm{Aut\,G}\times\langle\alpha\rangle}|$, we get that

$$|M^{\mathrm{Aut\,G}\times\langle\alpha\rangle}| = \frac{|\mathrm{Aut}\, G \times \langle\alpha\rangle|}{|\mathrm{Aut}\, M|}.$$

We have that

$$\begin{aligned} r_k(G) &= \frac{4\varepsilon(G)}{|\mathrm{Aut}\, G \times \langle\alpha\rangle|} \sum_{M\in\mathcal{M}_k(G,S)} |M^{\mathrm{Aut}\, G\times\langle\alpha\rangle}| \\ &= \frac{2\varepsilon(G) g_k(G)}{|\mathrm{Aut}\, G|}. \end{aligned}$$

Therefore, we get that

$$\begin{aligned} |\mathrm{Aut}\, G| r[G](x) &= |\mathrm{Aut}\, G| \sum_{i\geq 0} r_i(G) x^i \\ &= \sum_{i\geq 0} |\mathrm{Aut}\, G| r_i(G) x^i \\ &= \sum_{i\geq 0} 2\varepsilon(G) g_i(G) x^i = 2\varepsilon(G) g[G](x). \end{aligned}$$ ♮

According to Theorem 2.1, we can get $r[G](x)$ if $g[G](x)$ has been known. For example, we know $g[K_4](x)$ from [4]:

$$g[K_4](x) = 2 + 14x.$$

Since $|\mathrm{Aut}\, K_4| = 24$, we have that

$$\begin{aligned} r[K_4](x) &= \frac{2\varepsilon(K_4)}{|\mathrm{Aut}\, K_4|} g[K_4](x) \\ &= \frac{2\times 6}{24} \times (2 + 14x) = 1 + 7x \end{aligned}$$

All rooted maps on orientable surfaces with underlying graph K_4 are shown in Fig. 2.

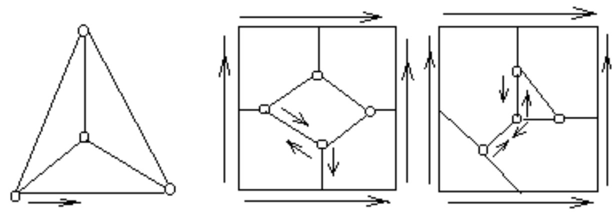

Fig. 2

Similarly, we know $g[K_{3,3}](x)$[4]:

$$g[K_{3,3}](x) = 40x + 24x^2.$$

Since Aut $K_{n,n} = S_2[S_n]$, where S_n denotes the symmetric group of degree n, we have that $|\mathrm{Aut}\, K_{n,n}| = 2(n!)^2$, particularly, $|\mathrm{Aut}\, K_{3,3}| = 72$. Whence, we get that

$$\begin{aligned} r[K_{3,3}](x) &= \frac{2\varepsilon(K_{3,3})}{|\mathrm{AutK_{3,3}}|} g[K_{3,3}](x) \\ &= \frac{2 \times 9}{72} \times (40x + 24x^2) = 10x + 6x^2. \end{aligned}$$

As a by-product, we get the following rather interesting number-theoretic property on the number of embeddings of some special graphs, such as the complete graphs K_n with $n \geq 4$ and complete bipartite graph $K_{m,n}$.

Corollary 2.1 *If* $|\mathrm{Aut\, G}| = 2\mathrm{i(G)}\varepsilon(\mathrm{G})$ *and* $i(G)$ *is an integer, then the number* $g_s(G)$ *of embeddings on the orientable surface* P_s *of genus* s *is a multiple of* $i(G)$, *and especially, for the complete graph* K_n *with* $n \geq 4$ *and complete bipartite graph* $K_{m,n}$, *the number* $g_s(K_n)$, $g_s(K_{m,n})$ *of embeddings of* K_n *and* $K_{m,n}$ *on the orientable surface* P_s *of genus* s *with* $s = 0, 1, 2, \cdots$, *have a common divisor*

$$(n-1)!$$

for K_n *and*

$$\frac{(m-1)!(n-1)!}{2}$$

for $K_{m,n}$ *with* $m \neq n$ *and* $(n-1)!^2$ *for* $K_{n,n}$, *respectively.*

The following theorem gives us an approach for enumerating rooted maps on orientable surfaces with vertex partition by automorphism groups.

Theorem 2.2 *Let $\mathcal{G}(D)$ be a set of graphs with the same valency sequence D. Then the number $n(\mathcal{G}(D))$ of rooted maps on orientable surfaces with underlying graphs in $\mathcal{G}(D)$ is*

$$n(\mathcal{G}(D)) = 2\varepsilon(D)\pi(D)\sum_{G\in\mathcal{G}(D)}\frac{1}{|\mathrm{Aut}\, G|}$$

where $2\varepsilon(D) = \sum_{d\in D} d$ *, and* $\pi(D) = \prod_{d\in D}(d-1)!$.

Proof Notice that $n(\mathcal{G}(D)) = \sum_{G\in\mathcal{G}(D)} n(\{G\})$. By Theorem 2.1, we get that

$$\begin{aligned} n(\{G\}) &= r[G](1) \\ &= \frac{2\varepsilon(G)g[G](1)}{|\mathrm{Aut}\, G|} = \frac{2\varepsilon(G)}{|\mathrm{Aut}\, G|}\sum_{i\geq 0} g_i(G) \\ &= \frac{2\varepsilon(G)\prod_{d\in D}(d-1)!}{|\mathrm{Aut}\, G|} = \frac{2\varepsilon(D)\pi(D)}{|\mathrm{Aut}\, G|}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} n(\mathcal{G}(D)) &= \sum_{G\in\mathcal{G}(D)} n(\{G\}) = \sum_{G\in\mathcal{G}(D)}\frac{2\varepsilon(D)\pi(D)}{|\mathrm{Aut}\, G|} \\ &= 2\varepsilon(D)\pi(D)\sum_{G\in\mathcal{G}(D)}\frac{1}{|\mathrm{Aut}\, G|}. \end{aligned}$$ ♮

The *map index* $\theta(D)$ of a valency sequence D is defined as follows:

$$\theta(D) = \sum_{G\in\mathcal{G}(D)}\sum_{M\in\mathcal{M}(G)}\frac{1}{|\mathrm{Aut}\, M|},$$

where $\mathcal{G}(D)$, $\mathcal{M}(G)$ denote the set of non-isomorphic graphs with valency sequence D, the set of non-isomorphic maps with underlying graph G, respectively.

Corollary 2.2 *The map index $\theta(D)$ has the following expression:*

$$\theta(D) = \frac{\pi(D)}{2}\sum_{G\in\mathcal{G}(D)}\frac{1}{|\mathrm{Aut}\, G|}.$$

Proof From Lemma 2.1, and the definition of map index, we know that

$$n(\mathcal{G}(D)) = 4\varepsilon(D)\theta(D).$$

According to Theorem 2.2, we have the following identity

$$2\varepsilon(D)\pi(D)\sum_{G\in\mathcal{G}(D)}\frac{1}{|\mathrm{Aut}\,G|} = 4\varepsilon(D)\theta(D).$$

Whence,

$$\theta(D) = \frac{\pi(D)}{2}\sum_{G\in\mathcal{G}(D)}\frac{1}{|\mathrm{Aut}\,G|}. \qquad \natural$$

Theorem 2.2 also enables us to count the rooted maps on orientable surfaces if all $|\mathrm{Aut}\,G|$, $G \in \mathcal{G}(D)$, are known. For example, since $|\mathrm{Aut}\,K_n| = n!$, $|\mathrm{Aut}\,W_n| = (n-1)!$ and

$$|\mathrm{Aut}\,(C_n \times K_2)| = \begin{cases} 4n, & \text{if} \quad n \neq 4, \\ 48, & \text{if} \quad n = 4, \end{cases}$$

where C_n denotes the circuit of length n and W_n the wheel $K_1 \times C_{n-1}$. We know the number of rooted maps of order $2n$ on orientable surfaces with underlying graphs in $\mathcal{G} = \{K_n, C_n \times K_2, W_{2n-1}\}$ is

$$n(\mathcal{G}) = (2n-2)!^{2n-1} + 2^{n+1} + \begin{cases} 3 \times 2^{2n-1}, & \text{if} \quad n \neq 4, \\ 128, & \text{if} \quad n = 4. \end{cases}$$

3 Enumerating maps via labelled graphs with vertex partition

We have known many results for the number of labelled graphs in references such as [5]–[7], [12]. Combining the already known results with Theorem 2.2, we get a new form for the number of rooted maps on orientable surfaces with vertex partition. The following result is well-known.

Lemma 3.1 [6] *Let $n^{\mathrm{lab}}(G)$ denote the number of labelled graphs of order p. Then*

$$n^{\mathrm{lab}}(G) = \frac{p!}{|\mathrm{Aut}\,G|}.$$

Theorem 3.1 *Let $n^{\mathrm{lab}}(D)$ denote the number of labelled graphs with valency sequence D. Then*

$$n(\mathcal{G}(D)) = \frac{2\varepsilon(D)\pi(D)n^{\mathrm{lab}}(D)}{p(D)!},$$

where $\mathcal{G}(D)$ denotes the set of non-isomorphic graphs with valency sequence D, and $p(D)$ denotes the number of elements in D.

Proof From the definition of $n^{\rm lab}(D)$ and Lemma 3.1, we know that

$$\begin{aligned} n^{\rm lab}(D) &= \sum_{G\in\mathcal{G}(D)} n^{\rm lab}(G) \\ &= \sum_{G\in\mathcal{G}(D)} \frac{|G|!}{|\mathrm{Aut}\, G|} = p(D)! \sum_{G\in\mathcal{G}(D)} \frac{1}{|\mathrm{Aut}\, G|}. \end{aligned}$$

Whence, we get that

$$\sum_{G\in\mathcal{G}(D)} \frac{1}{|\mathrm{Aut}\, G|} = \frac{n^{\rm lab}(D)}{p(D)!}.$$

According to Theorem 2.2, we get that

$$\begin{aligned} n(\mathcal{G}(D)) &= 2\varepsilon(D)\pi(D) \sum_{G\in\mathcal{G}(D)} \frac{1}{|\mathrm{Aut}\, G|} \\ &= \frac{2\varepsilon(D)\pi(D)n^{\rm lab}(D)}{p(D)!}. \end{aligned}$$

♮

Let $D(G)$ be the valency sequence of a connected G. We get the following result for maps on orientable surfaces with given underlying graph.

Corollary 3.1 *Let G be a connected graph. Then the number $n(G)$ of rooted maps on orientable surfaces with underlying graph G is*

$$n(G) = \frac{2\varepsilon(G)\pi(D(G))n^{\rm lab}(D(G))}{|G|!}.$$

For the complete graph K_n, it is obvious that $n^{\rm lab}(D(K_n)) = 1$. Therefore, we know the number of rooted maps on orientable surfaces with underlying graph K_n is

$$n(K_n) = \frac{2\times\frac{n(n-1)}{2}\times(n-2)!^n\times 1}{n!} = (n-2)!^{n-1},$$

which is the same as got by Theorem 2.1 and Theorem 2.2.

Now let $\underline{\mathrm{j}}_n = (j_1, j_2, \cdots, j_k, \cdots)$, $\underline{\mathrm{j}}_n! = j_1!j_2!\cdots j_k!\cdots$ and $\underline{\mathrm{x}}^{\underline{\mathrm{j}}_n} = x_1^{j_1}x_2^{j_2}\cdots x_k^{j_k}\cdots$. According to Theorem 3.1, we can immediately get the enufunction $f_{\mathcal{T}}(\underline{\mathrm{x}})$ of trees with vertex partition as follows.

Corollary 3.2 *The enufunction $f_{\mathcal{T}}(\underline{\mathrm{x}})$ of trees with vertex partition is*

$$f_{\mathcal{T}}(\underline{\mathrm{x}}) = \sum_{n=1}^{+\infty}\sum_{\underline{\mathrm{j}}_n} \frac{2(n-1)!}{\underline{\mathrm{j}}_n!}\underline{\mathrm{x}}^{\underline{\mathrm{j}}_n}.$$

Proof According to the references [6],[7], we know the number of labelled trees of order n in which the vertex labelled k has degree d_k is

$$\binom{n-2}{d_1-1, d_2-1, \cdots, d_n-1}$$

Now let $\underline{\mathrm{j}}_n$ be the vertex partition vector of $D = (d_1, d_2, \cdots, d_n)$, then we have that

$$n^{lab}(D) = \frac{n!}{\underline{\mathrm{j}}_n!} \times \binom{n-2}{d_1-1, d_2-1, \cdots, d_n-1}$$

Therefore, by Theorem 3.1, we get the number of rooted trees with valency sequence D is

$$\begin{aligned} n(\mathcal{G}(D) &= \frac{2\varepsilon(D)\pi(D)n^{\mathrm{lab}}(D)}{p(D)!} \\ &= \frac{2(n-1)\prod_{i=1}^{n}(d_i-1)!}{n!} \times \frac{n!}{\underline{\mathrm{j}}_n!} \times \frac{(n-2)!}{\prod_{i=1}^{n}(d_i-1)!} = \frac{2(n-1)!}{\underline{\mathrm{j}}_n!}. \end{aligned}$$

Whence, the enufunction $f_{\mathcal{T}}(\underline{\mathrm{x}})$ of rooted trees with vertex partition is

$$f_{\mathcal{T}}(\underline{\mathrm{x}}) = \sum_{n=1}^{+\infty} \sum_{\underline{\mathrm{j}}_n} \frac{2(n-1)!}{\underline{\mathrm{j}}_n!} \underline{\mathrm{x}}^{\underline{\mathrm{j}}_n}. \qquad \natural$$

Now we recall some facts in the enumerative theory of labelled graphs (see also [5] and [12]). Consider the expression

$$f_p(\underline{\mathrm{x}}) = f(x_1, x_2, \cdots, x_p) = \prod_{1 \le i \le j \le p} (1 + x_i x_j + x_i^2 x_j^2 + \cdots + x_i^s x_j^s).$$

When we expanded it, each term is obtained by choosing a term $x_i^k x_j^k$ from each factor and then multiplying them. This corresponds to placing k edges between the vertices labelled i and j of a graph being constructed, where s is the maximum multiple number of edges. If $i = j$, then $x_i x_j$ corresponds to a loop on vertices i. So $f_p(\underline{\mathrm{x}})$ is the ordinary generating function of labelled graphs of order p. Two extremal cases of $f_p(\underline{\mathrm{x}})$ are as follows:

Case 1 $\qquad s = 1$ and $i \neq j$

$$f_p(\underline{\mathrm{x}}) = \prod_{1 \le i < j \le p} (1 + x_i x_j),$$

which corresponds to labelled simple graphs of order p.

Case 2 $s = +\infty$

$$f_p(\underline{x}) = \prod_{1\le i\le j\le p} \frac{1}{1 - x_i x_j},$$

which corresponds to labelled general graphs of order p.

Define the exponential generating functions $\widetilde{f(\underline{x}^*)}$ and $\widetilde{c(\underline{x}^*)}$ of labelled graphs and labelled connected graphs, respectively, by

$$\widetilde{f(\underline{x}^*)} = \sum_{n=1}^{+\infty} \frac{f_n(\underline{x})}{n!} x^n,$$

and

$$\widetilde{c(\underline{x}^*)} = \sum_{n=1}^{+\infty} \sum_{\underline{j}_n} \frac{c(\underline{j}_n)}{n!} \underline{x}^{\underline{j}_n} x^n,$$

where $\underline{x}^* = (x, x_1, x_2, x_3, \cdots)$, and $\underline{j}_n$, $c(\underline{j}_n)$ denote the vertex partition vector, the number of labelled connected graphs with vertex partition vector $\underline{j}_n$, respectively.

Lemma 3.2[6] *$\widetilde{f(\underline{x}^*)}$ and $\widetilde{c(\underline{x}^*)}$ are related by*

$$1 + \widetilde{f(\underline{x}^*)} = e^{\widetilde{c(\underline{x}^*)}}.$$

Theorem 3.2 *For a given vertex partition vector $\underline{j}_n$, let*

$$D = (\underbrace{1, 1, \cdots, 1}_{j_1}, \underbrace{2, 2, \cdots, 2}_{j_2}, \cdots, \underbrace{k, k, \cdots, k}_{j_k}, \cdots)$$

be the corresponding valency sequence. Then

$$n(\mathcal{G}(D) = \frac{2\varepsilon(D)\pi(D)}{n!\underline{j}_n!^2} \times (\frac{\partial^n}{\partial x^n} \frac{\partial^{\underline{j}_n}}{\partial \underline{x}^{\underline{j}_n}} \log(1 + \widetilde{f(\underline{x}^*)}))|_{\underline{x}^*=\underline{0}}.$$

Proof According to Lemma 3.2, we get that

$$\widetilde{c(\underline{x}^*)} = \log(1 + \widetilde{f(\underline{x}^*)}).$$

Since

$$\begin{aligned} n^{\text{lab}}(D) &= \frac{p(D)!}{\underline{j}_n!} \times c(\underline{j}_n) \\ &= \frac{p(D)!}{\underline{j}_n!} \times (\frac{1}{n!} \frac{\partial^n}{\partial x^n} \frac{1}{\underline{j}_n!} \frac{\partial^{\underline{j}_n}}{\partial \underline{x}^{\underline{j}_n}} \log \widetilde{c(\underline{x}^*)})|_{\underline{x}^*=\underline{0}} \\ &= \frac{p(D)!}{n!\underline{j}_n!^2} \times (\frac{\partial^n}{\partial x^n} \frac{\partial^{\underline{j}_n}}{\partial \underline{x}^{\underline{j}_n}} \log \widetilde{c(\underline{x}^*)})|_{\underline{x}^*=\underline{0}} \end{aligned}$$

from Theorem 3.1, we have that

$$\begin{aligned} n(\mathcal{G}(D) &= \frac{2\varepsilon(D)\pi(D)n^{\text{lab}}(D)}{p(D)!} \\ &= \frac{2\varepsilon(D)\pi(D)}{n!\underline{\mathrm{j}}_n!^2} \times (\frac{\partial^n}{\partial x^n}\frac{\partial^{\underline{\mathrm{j}}_n}}{\partial \underline{\mathrm{x}}^{\underline{\mathrm{j}}_n}} \log(1 + \widetilde{f(\underline{\mathrm{x}}^*)}))|_{\underline{\mathrm{x}}^*=\underline{0}} \end{aligned}$$

This completes the proof. ♮

4 Future discussions

4.1 Many enumeration results for maps on surface appeared in the past two decades, especially, for maps on the sphere, torus, projective plane and Klein bottle [10]. As the genus increases, the difficulty in enumeration grows rapidly. For maps on orientabe surface with genus g, Bender and Canfield proved that the enufunction $f(z, g)$ with size as the parameter has the following form [2]:

$$f(z, g) = \frac{z^{2g}P_g(\rho)}{\rho^a(\rho + 1)^2(\rho + 2)^b(\rho + 5)^c},$$

where, $\rho = \sqrt{1 - 12z}$, and a, b, c are integers and $P_g(\rho)$ is a polynomial of ρ. Whence, it is very natural to ask whether we can choose new parameters such that the enufunction is rational. Theorem 2.2 gives an example for this object. We think that there are new other parameters, not being the automorphism groups, can be used for this target. For example, in [1], a new forms enufunction (continued fractions) is used for enumerating maps on orientable surfaces and in [8],the enufunction is represented by characters of a symmetry group.

4.2 The relation of the number of rooted maps on orientable surfaces and the number of labelled underlying graphs is established in Theorem 3.1. Corollary 3.1 is an interesting example for getting the enufunction of plane trees by our approach. Using the results in the enumerative theory of labelled graphs, especially, the regular graphs, the symmetry graphs, new enumeration results can be obtained. Therefore, enumerating labelled graphs with vertex partition in first is also value for enumeration of rooted maps on surfaces.

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(Received 3 Mar 2003; revised 25 Oct 2003)