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Orthogonal designs from negacyclic matrices

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Abstract

We study the use of negacyclic matrices to form orthogonal designs and hence Hadamard matrices. We give results for all possible tuple for order 12, all but 3 for order 20 and all but 3 for order 28.

1 Introduction

An *orthogonal design* of order n and type $(s_1, s_2, \ldots, s_k)$ in variables $x_1, x_2, \ldots, x_k$, denoted $OD(n; s_1, s_2, \ldots, s_k)$, is a matrix A of order n with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_k\}$ satisfying

$$AA^T = (\sum_{i=1}^{k} s_i x_i^2) I_n,$$

where I_n is the identity matrix of order n. Alternatively, the rows of A are formally orthogonal and each row has precisely s_i entries of the type $\pm x_i$. In [1], where this was first defined, it was mentioned that

$$A^T A = (\sum_{i=1}^{k} s_i x_i^2) I_n$$

and so our alternative description of A applies equally well to the columns of A.

An *Hadamard matrix* H of order n is a square $(1, -1)$ matrix having inner product of distinct rows zero. Hence $HH^T = nI_n$. We note that $n = 1, 2$ or $n \equiv 0 \pmod 4$.

A matrix $A \pm I$ is *skew-type* if A has zero diagonal and $A^T = -A$. A skew-type Hadamard matrix is said to be *skew-Hadamard*.

Circulant matrices of order n are polynomials in the shift matrix

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & & 0 \end{pmatrix}.$$

Negacyclic matrices of order n are polynomials in the negashift matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & & 1 \\ -1 & 0 & 0 & & 0 \end{pmatrix}.$$

The *back-diagonal matrix* R of order n is the matrix whose elements r_{ij} are given by

$$r_{ij} = \begin{cases} 1 & \text{if } i + j = n + 1, \\ 0 & \text{otherwise} \end{cases}$$

where $i, j = 1, \ldots, n$. We note that if A, B are polynomial in S then $A(BR)^T = (BR)A^T$. We show below that the same is true if A, B are polynomial in N.

Lastly, we define the *nonperiodic autocorrelation function*. Let

$$X = \{\{a_{11}, \ldots, a_{1n}\}, \{a_{21}, \ldots, a_{2n}\}, \ldots, \{a_{m1}, \ldots, a_{mn}\}\}$$

be m sequences of commuting variables of length n. Then the nonperiodic autocorrelation function of the family of sequences X (denoted N_X) is the function defined by

$$N_X(j) = \sum_{i=1}^{n-j} (a_{1,i}a_{1,i+j} + a_{2,i}a_{2,i+j} + \cdots + a_{m,i}a_{m,i+j}).$$

2 Preliminary results

We note some properties of the negashift matrix N given above:

$$(N^i)^T = -N^{n-i} \quad \text{and} \quad N^i R = -RN^{n-i}.$$

Hence we have:

Lemma 1 $N^i(N^jR)^T = (N^jR)(N^i)^T$.

Proof. $N^i(N^jR)^T = N^iR(N^j)^T = -N^iRN^{n-j} = N^iN^jR = N^jN^iR = -N^jRN^{n-i} = (N^jR)(N^i)^T$. □

Lemma 2 *Suppose A, B are polynomial in S or N then $A(BR)^T = (BR)A^T$.*

Proof. The result for S, circulant, can be found in Wallis [6]. For N, negacyclic, we note A and B are polynomials in N so by repeated applications of Lemma 1 we have the result. □

We now develop some properties of negacyclic matrices.

Theorem 1 *Let R_i, R_j be two rows of a negacyclic matrix of dimension n, where $1 < i < j \leq n$. Then $R_iR_j^T = R_1R_{1+j-i}^T$.*

Proof. Let $i = 1 + s$ and $j = 1 + s + t$. If we write $R_1 = (x_1, \ldots, x_n)$, we have

$$\begin{aligned} R_{1+s} &= (-x_{n-s+1}, \ldots, -x_n, x_1, \ldots, x_{n-s}) \\ R_{1+t} &= (-x_{n-t+1}, \ldots, -x_n, x_1, \ldots, x_{n-t}) \\ R_{1+s+t} &= (-x_{n-s-t+1}, \ldots, -x_n, x_1, \ldots, x_{n-s-t}) \end{aligned}$$

We note that

$$R_{1+s} = (\overbrace{-x_{n-s}, \ldots, -x_n}^{s}, \overbrace{x_1, \ldots, x_t}^{t}, x_{t+1}, \ldots, x_{n-s})$$

$$R_{1+s+t} = (\underbrace{-x_{n-s-t+1}, \ldots, -x_{n-t}}_{s}, \underbrace{-x_{n-t+1}, \ldots, -x_n}_{t}, x_1, \ldots, x_{n-s-t})$$

Then

$$\begin{aligned} R_i R_j^T &= R_{1+s} R_{1+s+t}^T \\ &= x_{n-s+1} x_{n-s-t+1} + \cdots + x_n x_{n-t} - x_1 x_{n-t+1} - \cdots - x_t x_n \\ &\quad + x_{t+1} x_1 + \cdots + x_{n-s} x_{n-s-t} \\ &= -x_1 x_{n-t+1} - \cdots - x_t x_n + x_{t+1} x_1 + \cdots + x_{n-s} x_{n-s-t} \\ &\quad + x_{n-s+1} x_{n-s-t+1} + \cdots + x_n x_{n-t} \\ &= -x_1 x_{n-t+1} - \cdots - x_t x_n + x_{t+1} x_1 + \cdots + x_n x_{n-t} \\ &= R_1 R_{1+t}^T \\ &= R_1 R_{1+j-i}^T \end{aligned}$$

□

Corollary 1 *For a negacylic matrix of odd dimension n, there are only n inner products of interest, i.e. each row with the first. More precisely, for a negacylic matrix of odd dimension n, there are $\frac{n+1}{2}$ distinct inner products, p_1, p_2, p_3, ..., $p_{\frac{n+1}{2}}$, where $p_i = R_1 R_i^T$. The remaining $\frac{n-1}{2}$ inner products are related by the property that $-p_{\frac{n+1}{2}+q} = p_{\frac{n+3}{2}-q}$, for $1 \le q \le \frac{n-1}{2}$.*

Corollary 2 *For a negacyclic matrix of odd dimension n, $R_1 R_i^T = -R_1 R_{n-i+2}^T$ for $1 < i \le n$.*

Definition 1 *Let L, M be two negacyclic matrices of dimension n. We say L* and *M* are in the same inner product equivalence class *if for every $1 \le i, j \le n$, $R_i R_j^T$ of L equals $R_i R_j^T$ of M.*

Theorem 2 *Let M be a negacyclic matrix of dimension n, with first row*

$$m_1, m_2, \ldots, m_n.$$

Then the negacyclic matrices with first rows

$$-m_1, -m_2, \ldots, -m_n \quad \text{and} \quad m_n, m_{n-1}, \ldots, m_1$$

(denoted $-M$ and M^* respectively) are in the same inner product equivalence class as M.

Proof. By Theorem 1, we need only consider the inner products of rows R_1, R_{1+j} for $1 \leq j \leq n-1$. We observe that for the matrix M,

$$\begin{array}{rcccccccc} R_1 = & m_1 & m_2 & \cdots & m_j & m_{j+1} & m_{j+2} & \cdots & m_n \\ R_{1+j} = & -m_{n-j+1} & -m_{n-j+2} & \cdots & -m_n & m_1 & m_2 & \cdots & m_{n-j}. \end{array}$$

Likewise, for the matrix $-M$,

$$\begin{array}{rcccccccc} R_1 = & -m_1 & -m_2 & \cdots & -m_j & -m_{j+1} & -m_{j+2} & \cdots & -m_n \\ R_{1+j} = & m_{n-j+1} & m_{n-j+2} & \cdots & m_n & -m_1 & -m_2 & \cdots & -m_{n-j}. \end{array}$$

Similarly, for the matrix M^*,

$$\begin{array}{rcccccccc} R_1 = & m_n & m_{n-1} & \cdots & m_{n-j+1} & m_{n-j} & m_{n-j-1} & \cdots & m_1 \\ R_{1+j} = & -m_j & -m_{j-1} & \cdots & -m_1 & m_n & m_{n-1} & \cdots & m_{j+1}. \end{array}$$

The inner product of R_1 with R_{1+j} is the same in all three cases, namely

$$\begin{aligned} R_1 R_{1+j}^T &= -m_1 m_{n-j+1} - m_2 m_{n-j+2} - \cdots - m_j m_n \\ &\quad + m_1 m_{j+1} + m_2 m_{j+2} + \cdots + m_{n-j} m_n \\ &= -\sum_{i=1}^{j} m_i m_{n-j+i} + \sum_{i=1}^{n-j} m_i m_{i+j}. \end{aligned}$$

Hence M, $-M$ and M^* are in the same inner product equivalence class. □

Corollary 3 *Given a negacyclic matrix M, the negacyclic matrix $-M^*$ is in the same inner product equivalence class as M.*

Theorem 3 *Let M be a negacyclic matrix with first row $R_1 = (m_1, \ldots, m_n)$. Then every negacyclic matrix M' with a first row R_1' equal to a negashift of R_1 is in the same inner product equivalence class as M. (There are $2n-1$ negashifts of R_1.)*

Proof. The $2n-1$ negashifts of R_1 are

$$R_2, \ldots, R_n, -R_1, \ldots, -R_n$$

where R_i is the ith row of the matrix M.

By Theorem 2, the negacyclic matrix with first row equal to $-R_1$ is in the same inner product equivalence class as M. Likewise, the negacyclic matrices with first rows equal to $-R_2, \ldots, -R_n$ are in the same inner product equivalence class as the negacyclic matrices with first rows $R_2, \ldots, R_n$ respectively. Thus we need consider only the negacyclic matrices with first rows $R_1' = R_i$, where $2 \leq i \leq n$.

Let M' be a negacyclic matrix with first row $R_1' = R_i$ for some i in the range $2 \leq i \leq n$. By Corollary 1, M' has only $\frac{n+1}{2}$ distinct inner products, namely $R_1'(R_j')^T$, for $2 \leq j \leq \frac{n+1}{2}$.

Now, $R_1' = R_i$, and

$$R'_{\frac{n+1}{2}} = \begin{cases} R_{i+\frac{n-1}{2}} & \text{for } 2 \leq i \leq \frac{n+1}{2} \\ -R_{i+\frac{n-1}{2}-n} & \text{for } \frac{n+3}{2} \leq i \leq n. \end{cases}$$

We consider the two cases in turn.

Case I: When $2 \leq i \leq \frac{n+1}{2}$, the rows $R'_1, \ldots, R'_{\frac{n+1}{2}}$ are all from M. Then the inner product $R'_1(R'_j)^T$ for $2 \leq j \leq \frac{n+1}{2}$ is:

$$\begin{aligned} R'_1(R'_j)^T &= R_i R^T_{i+j-1} \\ &= R_1 R^T_j \qquad \text{(by Theorem 1).} \end{aligned}$$

Thus the negacyclic matrices M' with first row $R'_1 = R_i$ for $2 \leq i \leq \frac{n+1}{2}$ are in the same inner product equivalence class as M.

Case II: When $\frac{n+3}{2} \leq i \leq n$, the rows $R'_1, \ldots, R'_{n-i+1}$ are from M, and the rows $R'_{n-i+2}, \ldots, R'_{\frac{n+1}{2}}$ are from $-M$. For $2 \leq j \leq n-i+1$, $R'_1(R'_j)^T = R_1 R^T_j$ by the argument presented in Case I. For $n-i+2 \leq j \leq \frac{n+1}{2}$, we have

$$\begin{aligned} R'_1(R'_j)^T &= R_i R^T_{i+j-1-n} \\ &= -R_1 R_{n-j+2} \qquad \text{(observe } i+j-1-n < i \\ &\qquad\qquad\qquad\quad \text{and then apply Theorem 1)} \\ &= R_1 R_j \qquad\qquad \text{(by Corollary 2).} \end{aligned}$$

Thus the negacyclic matrices M' with first row $R_1 = R_i$ for $\frac{n+3}{2} \leq i \leq n$ are in the same inner product equivalence class as M.

Hence the $2n-1$ negacyclic matrices which have a first row equal to a negashift of the first row of M are all in the same inner product equivalence class as M. □

We note from Wallis and Whiteman [7] that circulant can be replaced by group-type or type 1 in abelian groups so that all results that follow for circulant also follow for group-type or type 1. Similarly we observe that group-type or type 1 negacyclic can be used instead of negacyclic and the corresponding results hold. So we have, modifying Goethals-Seidel construction [6]:

Theorem 4 *Suppose there exist four negacyclic* $(1, -1)$ *matrices* A, B, C, D *of order* n*. Further, suppose*

$$AA^T + BB^T + CC^T + DD^T = 4nI_n.$$

Then

$$SF = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & D^T R & -C^T R \\ -CR & -D^T R & A & B^T R \\ -DR & C^T R & -B^T R & A \end{bmatrix} \tag{1}$$

is an Hadamard matrix of order $4n$ *of* SF type*. (Here* R *is the back diagonal matrix.) If* A *is of skew-type, then* SF *is skew-Hadamard.*

3 Our results

Definition 2 *A set of four negacyclic matrices* A, B, C, D *is said to be* suitable *if*

$$AA^T + BB^T + CC^T + DD^T = fI$$

for some f*.*

Theorem 5 *If there exist four sequences with zero non-periodic autocorrelation function, then there exist four suitable negacyclic matrices.*

Proof. Let there be four sequences of length n, denoted

$$X = \{\{a_1, a_2, \ldots, a_n\}, \{b_1, b_2, \ldots, b_n\}, \{c_1, c_2, \ldots, c_n\}, \{d_1, d_2, \ldots, d_n\}\},$$

with zero non-periodic autocorrelation function. We now treat each of these sequences as the first row of a negacyclic matrix. The matrices generated in this way are denoted A, B, C, D respectively. We consider the sum

$$AA^T + BB^T + CC^T + DD^T.$$

For a negacyclic matrix M, the element (i, j) of the product MM^T is equal to $R_i R_j^T$, where R_i, R_j are the ith and jth rows of M respectively. By Theorem 1, it is sufficient to consider only the products $R_1 R_{1+j}^T$, for $1 \le j \le n-1$. We recall that

$$R_1 R_{1+j}^T = -\sum_{i=1}^{j} m_i m_{n-j+i} + \sum_{i=1}^{n-j} m_i m_{i+j}$$

(as shown in the proof of Theorem 2).

Thus the sum of the products $R_1 R_{1+j}^T$ of A, B, C and D, for $1 \le j \le n-1$, is equal to

$$\begin{aligned} & -\sum_{i=1}^{j}(a_i a_{n-j+i} + b_i b_{n-j+i} + c_i c_{n-j+i} + d_i d_{n-j+i}) \\ & +\sum_{i=1}^{n-j}(a_i a_{i+j} + b_i b_{i+j} + c_i c_{i+j} + d_i d_{i+j}) \\ = \; & -N_X(n-j) + N_X(j) \\ = \; & 0 \end{aligned}$$

By Theorem 1, the off-diagonal element (i, j), where $i \ne j$, of the sum $AA^T + BB^T + CC^T + DD^T$, is equal to element $(1, 1+j-i)$. As has just been shown, these elements are all equal to 0. Thus all off-diagonal elements of the sum are 0.

Lastly, we consider the diagonal elements. It is clear that element $(j, j) = \sum_{i=1}^{n}(a_i^2 + b_i^2 + c_i^2 + d_i^2)$ for $1 \le j \le n$. Write this sum as f. Thus we have

$$AA^T + BB^T + CC^T + DD^T = fI$$

and so the negacyclic matrices A, B, C, D are suitable. □

Corollary 4 *The four suitable negacyclic matrices A, B, C, D that were constructed in the last proof satisfy*

$$AA^T + BB^T + CC^T + DD^T = 4nI_n.$$

Hence, by Theorem 4, we can produce an Hadamard matrix of order $4n$ of SF-type.

We have found that suitable negacyclic matrices are not limited to those generated from sequences with a zero non-periodic autocorrelation function. The appendix contains a listing of negacyclic sequences of order $n = 3, 5, 7$ which are suitable, and so yield SF-type Hadamard matrices and orthogonal designs of order $4n$.

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A Negacyclic sequences with zero autocorrelation function

Order 12

```
Design          A          B          C          D
(1,1,1,1)   a  0  0    b  0  0    c  0  0    d  0  0
(1,1,1,4)   a  0  0    b  0  0    c  d  d    0  d -d
(1,1,1,9)   a  d  d    b  d  d    c  d  d    d -d  d
(1,1,2,2)   a  0  0    b  0  0    c -d  0    c  d  0
(1,1,2,8)   a  d  d    b  d  d   -c -d  d    c -d  d
(1,1,4,4)   a  d  d    b  c  c    0 -c  c    0 -d  d
(1,1,5,5)   a  d  d    b  c  c    d -c  c   -c -d  d
(1,2,2,4)   0  d -d    a  d  d    b  c  0    b -c  0
(1,2,3,6)   a  d -d    c  d  d    c  a -d    c -a  d
(2,2,2,2)   a  b  0    a -b  0    c  d  0    c -d  0
(2,2,4,4)   a  b  b    d  b -b    c  d  d   -b  d -d
(3,3,3,3)   a  b  c    b -c  d    c  d -a    d  a -b
```

Order 20

```
Design            A, C                B, D
(1,1,1,1)   a  0  0  0  0    b  0  0  0  0
            c  0  0  0  0    d  0  0  0  0
(1,1,1,4)   a  0  0  0  0    b  0  0  0  0
            c  0  d  d  0    0  0  d -d  0
(1,1,1,9)   a  0  d  d  0    b  0  d  d  0
            c  d  0  0  d    d -d  d  0  0
```

(1,1,2,2)	a	0	0	0	0	b	0	0	0	0
	c	-d	0	0	0	d	c	0	0	0
(1,1,2,8)	a	0	d	d	0	b	0	d	d	0
	c	0	d	-d	0	-c	0	d	-d	0
(1,1,4,4)	a	0	c	c	0	0	0	c	-c	0
	b	0	d	d	0	0	0	d	-d	0
(1,1,4,9)	a	0	d	d	0	b	0	d	d	0
	0	-d	c	-c	-d	d	-d	c	c	d
(1,1,5,5)	a	0	c	c	0	b	0	d	d	0
	c	0	d	-d	0	-d	0	c	-c	0
(1,1,8,8)	a	-c	d	d	-c	b	-c	-d	-d	-c
	0	-c	d	-d	c	0	-c	-d	d	c
(1,1,9,9)	a	-c	d	d	-c	b	-d	-c	-c	-d
	-c	-c	c	-c	c	-d	-d	d	-d	d
(1,2,2,4)	a	0	d	d	0	0	0	d	-d	0
	b	-c	0	0	0	c	b	0	0	0
(1,2,2,9)	a	-d	0	0	-d	d	-d	d	0	0
	b	-c	d	d	0	c	b	0	-d	-d
(1,2,3,6)	a	0	d	d	0	b	-c	0	-d	0
	b	c	0	d	0	c	0	-d	d	0
(1,2,4,8)	a	0	d	d	0	b	-c	d	-d	-c
	b	0	-d	d	0	d	d	c	0	c
(1,2,8,9)	a	-d	c	c	-d	d	-d	d	-c	-c
	c	-b	d	d	c	b	c	-c	-d	-d
(1,4,4,4)	a	0	b	b	0	0	0	b	-b	0
	0	-c	d	d	c	0	-c	d	-d	-c
(1,4,5,5)	a	-d	0	0	-d	-d	-c	0	0	c
	c	-d	b	b	d	0	-c	b	-b	-c
(1,5,5,9)	a	-b	c	c	-b	-c	-b	d	d	b
	b	-d	c	-c	-d	-d	-d	d	-d	d
(2,2,2,2)	a	-b	0	0	0	b	a	0	0	0
	c	-d	0	0	0	d	c	0	0	0
(2,2,2,8)	a	-d	b	d	0	a	d	-b	-d	0
	c	0	d	0	d	c	0	-d	0	-d
(2,2,4,4)	a	-b	0	0	0	b	a	0	0	0
	0	-c	d	d	c	0	-c	d	-d	-c
(2,2,4,9)	a	-b	d	d	0	a	b	d	d	0
	0	-d	c	-c	-d	d	-d	c	c	d
(2,2,8,8)	a	-d	c	-d	-c	a	d	-c	d	c
	b	-c	-d	-c	d	b	c	d	c	-d
(2,3,4,6)	b	-d	a	0	0	b	-d	-a	0	0
	-b	-d	c	c	d	0	-d	c	-c	-d
(2,3,6,9)	b	-d	c	-a	-d	a	-d	-d	b	-c
	-b	-d	-d	-c	c	d	-d	-c	-c	d
(2,4,4,8)	c	0	b	-d	d	c	0	b	d	-d
	b	-a	-c	-d	-d	b	a	-c	d	d
(2,5,5,8)	c	-b	d	d	b	a	-c	d	-d	-c
	-a	-b	d	-d	-b	-b	-c	d	d	c
(3,3,3,3)	a	-b	c	0	0	a	b	0	-d	0
	a	0	-c	d	0	b	c	d	0	0
(3,3,6,6)	0	-d	-a	-c	-b	0	-d	a	c	-b
	b	-c	d	-d	-c	a	-c	d	d	c
(4,4,4,4)	0	-a	b	b	a	0	-a	b	-b	-a
	0	-c	d	d	c	0	-c	d	-d	-c

(4,4,5,5)	d -c a a c	0 -d a -a -d
	0 -c b -b -c	-c -d b b d
(5,5,5,5)	a -b b -d -d	-b -a a -c -c
	d -c c a a	-c -d d b b
(1,1,13)	a -c -c -c -c	0 -c -c c -c
	b -c 0 0 -c	0 -c c -c 0
(1,2,17)	a -c -c -c -c	b -c -c c -c
	b c c -c c	c c -c c -c
(1,2,11)	a 0 c c 0	b 0 c c -c
	b 0 -c -c c	0 -c 0 -c c
(1,3,14)	a -c -c -c -c	-b -c c -c 0
	c -c c -b -c	c -c -c 0 b
(1,4,13)	a -c -c -c -c	0 -c -c c -c
	0 -c b -b -c	b -c c -c -b
(1,6,11)	a -c b b -c	c -c b -b -c
	c b c -c 0	c 0 c c b
(1,8,11)	a -c -b -b -c	-c -c b -b -c
	c b c -c b	c -b c c b
(2,5,7)	c -c -c 0 -a	a -c b b 0
	b -b 0 -c 0	c b 0 -c 0
(3,6,8)	0 -b c b -c	0 -b c -a c
	a -b -c 0 -c	a b c b -c
(7,10)	0 -a a b -b	b -a a a a
	0 -a a a b	b 0 b b a

Order 28

Design	A, C	B, D
(1,1,1,1)	a 0 0 0 0 0 0	b 0 0 0 0 0 0
	c 0 0 0 0 0 0	d 0 0 0 0 0 0
(1,1,1,4)	a 0 0 0 0 0 0	b 0 0 0 0 0 0
	d -c -d 0 0 0 0	d 0 d 0 0 0 0
(1,1,1,9)	d -a -d 0 0 0 0	d -b -d 0 0 0 0
	d 0 c 0 -d 0 0	d 0 d 0 d 0 0
(1,1,1,16)	a -d d d d d -d	b 0 d 0 0 d 0
	c 0 d 0 0 d 0	0 -d d d d -d d
(1,1,1,25)	a -d d d d d -d	b -d d d d d -d
	c -d d d d d -d	-d -d d -d d -d d
(1,1,2,2)	a 0 0 0 0 0 0	b 0 0 0 0 0 0
	c -d 0 0 0 0 0	c d 0 0 0 0 0
(1,1,2,8)	d -a -d 0 0 0 0	d -b -d 0 0 0 0
	d -c d 0 0 0 0	d c d 0 0 0 0
(1,1,2,18)	a -b b b b b -b	c -b b b b b -b
	d -b b 0 b 0 0	-d -b b 0 b 0 0
(1,1,4,4)	a 0 0 0 0 0 0	b 0 0 0 0 0 0
	c -c d d 0 0 0	d -d -c -c 0 0 0
(1,1,4,9)	a -d d d d d -d	0 -d d 0 d 0 0
	b 0 0 -c -c 0 0	0 0 0 -c c 0 0
(1,1,4,16)	b 0 b 0 b 0 b	b 0 b -a -b 0 -b
	b -c -b 0 -b -c b	b -c -b -d b c -b
(1,1,5,5)	a 0 0 0 0 0 0	b 0 0 0 0 0 0
	c -c -c -d 0 -d 0	d -d -d c 0 c 0
(1,1,8,8)	c -d a d -c 0 0	-c -d b d c 0 0
	c -d 0 -d c 0 0	c d 0 d c 0 0

(1,1,9,9)	a	-c	c	c	c	c	-c	0	-c	c	0	c	0	0
	b	-d	d	d	d	d	-d	0	-d	d	0	d	0	0
(1,1,10,10)	d	-a	a	a	a	a	-a	c	-b	b	b	b	b	-b
	-a	-b	b	0	b	0	0	b	-a	a	0	a	0	0
(1,1,13,13)	-a	-b	b	-a	b	-a	a	-b	a	-a	-b	-a	-b	b
	c	-a	a	a	a	a	-a	d	-b	b	b	b	b	-b
(1,2,2,4)	b	-a	-b	0	0	0	0	b	0	b	0	0	0	0
	c	-d	0	0	0	0	0	c	d	0	0	0	0	0
(1,2,2,9)	a	-d	d	d	d	d	-d	0	-d	d	0	d	0	0
	b	-c	0	0	0	0	0	b	c	0	0	0	0	0
(1,2,2,16)	a	-d	d	d	d	d	-d	b	0	d	-c	0	d	0
	b	0	d	c	0	d	0	0	-d	d	d	d	-d	d
(1,2,3,6)	a	-b	c	0	0	0	0	a	-b	-c	0	0	0	0
	b	a	b	0	0	0	0	b	-d	-b	0	0	0	0
(1,2,4,8)	c	-a	-c	0	0	0	0	c	0	c	0	0	0	0
	d	-b	-d	-d	0	-d	0	d	-b	-d	d	0	d	0
(1,2,8,9)	a	-d	d	d	d	d	-d	0	-d	d	0	d	0	0
	-b	-b	b	-c	b	0	0	-b	-b	b	c	b	0	0
(1,3,6,8)	-c	-c	c	0	c	-a	b	c	c	-c	0	-c	-a	b
	b	a	b	0	0	0	0	b	-d	-b	0	0	0	0
(1,4,4,4)	b	-a	-b	0	0	0	0	b	0	b	0	0	0	0
	c	-c	d	d	0	0	0	d	-d	-c	-c	0	0	0
(1,4,4,9)	a	-d	d	d	d	d	-d	0	-d	d	0	d	0	0
	c	-c	b	b	0	0	0	b	-b	-c	-c	0	0	0
(1,4,5,5)	b	-a	-b	0	0	0	0	b	0	b	0	0	0	0
	c	-c	-c	-d	0	-d	0	d	-d	-d	c	0	c	0
(1,5,5,9)	a	-d	d	d	d	d	-d	0	-d	d	0	d	0	0
	b	-b	-b	-c	0	-c	0	c	-c	-c	b	0	b	0
(2,2,2,2)	a	-b	0	0	0	0	0	a	b	0	0	0	0	0
	c	-d	0	0	0	0	0	c	d	0	0	0	0	0
(2,2,8,2)	a	-b	0	0	0	0	0	a	b	0	0	0	0	0
	c	-d	-c	-c	0	-c	0	c	-d	-c	c	0	c	0
(2,2,2,18)	a	-d	-d	0	b	-d	-d	a	-d	-d	0	-b	-d	-d
	c	-d	d	-d	d	d	0	-c	-d	d	-d	d	d	0
(2,2,4,4)	a	-b	0	0	0	0	0	a	b	0	0	0	0	0
	c	-c	d	d	0	0	0	d	-d	-c	-c	0	0	0
(2,2,4,16)	a	-b	-a	-a	c	-a	0	a	-b	-a	-a	-c	-a	0
	a	-b	-a	a	d	a	0	a	-b	-a	a	-d	a	0
(2,2,5,5)	a	-b	0	0	0	0	0	a	b	0	0	0	0	0
	c	-c	-c	-d	0	-d	0	d	-d	-d	c	0	c	0
(2,2,8,8)	a	-b	-a	-a	0	-a	0	a	-b	-a	a	0	a	0
	c	-d	-c	-c	0	-c	0	c	-d	-c	c	0	c	0
(2,2,9,9)	d	-b	-d	-c	a	c	0	d	-b	-d	c	-a	-c	0
	d	-c	0	-c	-d	-c	0	-c	-d	0	-d	c	-d	0
(2,2,10,10)	c	-c	-c	-d	a	-d	0	c	-c	-c	-d	-a	-d	0
	d	-d	-d	c	b	c	0	d	-d	-d	c	-b	c	0
(2,3,4,6)	a	-d	0	d	a	0	0	a	-d	c	-d	-a	0	0
	b	-c	-d	0	0	0	0	-b	-c	-d	0	0	0	0
(2,4,4,8)	a	-b	-a	-a	0	-a	0	a	-b	-a	a	0	a	0
	c	-c	d	d	0	0	0	d	-d	-c	-c	0	0	0
(2,4,6,12)	a	-b	c	-a	b	c	0	a	-b	c	a	-b	-c	0
	b	a	b	-b	d	b	0	b	a	b	b	-d	-b	0
(2,5,5,8)	a	-d	-a	-a	0	-a	0	a	-d	-a	a	0	a	0
	c	-c	-c	-b	0	-b	0	b	-b	-b	c	0	c	0

(2,8,8,8)	a -a b b c -d -c	a -a b b -c d c
	a a b -b c 0 c	a a b -b -c 0 -c
(3,3,3,3)	a -b -c 0 0 0 0	a b 0 -d 0 0 0
	a 0 c d 0 0 0	b -c d 0 0 0 0
(3,3,3,12)	-c -c c 0 c -a b	c c -c -d -c -a 0
	c c -c d -c 0 b	0 0 0 d 0 -a -b
(3,3,6,6)	a -d b d a 0 0	a -d c -d -a 0 0
	c d a 0 -b 0 0	c d -a 0 b 0 0
(3,4,6,8)	-c -c c 0 c -a b	c c -c 0 -c -a b
	d -b 0 b d 0 0	-d -b -a -b d 0 0
(4,4,4,4)	a -b c -d 0 0 0	a b c d 0 0 0
	a -b -c d 0 0 0	a b -c -d 0 0 0
(4,4,4,16)	c -d -c -c a -c b	c -d -c -c -a -c -b
	c -d -c c a c -b	c -d -c c -a c b
(4,4,5,5)	a -a b b 0 0 0	b -b -a -a 0 0 0
	c 0 c -d d d 0	d 0 d c -c -c 0
(4,4,8,8)	a -a b b c -d 0	a -a b b -c d 0
	a a b -b c d 0	a a b -b -c -d 0
(4,4,10,10)	b -c a -c d -d -d	b c a c -d d d
	b -d -a -d -c c c	b d -a d c -c -c
(5,5,5,5)	a -a -a -b 0 -b 0	-b b b -a 0 -a 0
	-c c c d 0 d 0	-d d d -c 0 -c 0
(6,6,6,6)	a -a b b c -d 0	-b b a a d c 0
	-c c -d -d a -b 0	-d d c c -b -a 0
(7,7,7,7)	a -a -a -b c -b d	-b b b -a d -a -c
	-c c c d a d b	-d d d -c -b -c a
(1,2,22)	a -b 0 -b b -b -b	a b 0 b -b b b
	0 b b b b -b b	c -b b b b b -b
(1,3,24)	a -c -c -c -c -c -c	b -c -c c -c c -c
	b c -c -c c -c c	b c c -c -c -c c
(1,4,20)	a -b b b b b -b	c c b -b b -b 0
	0 -b b b -b -b b	c 0 -b -b -b -b c
(1,6,12)	b b -b 0 -b -a a	-b -b b 0 b -a 0
	-b -b b 0 b 0 a	c 0 0 -a -a 0 0
(1,6,18)	c -a a -b -b a -a	-b b a a -a -a 0
	a -a a a a -b 0	a -a -a -a -b -a 0
(1,6,21)	c -b -b -b -b -b -b	a -b b -b b b -b
	a a b -b -a -b b	a -a b b -b -b b
(1,9,13)	a -b b b b b -b	0 -b b 0 b 0 0
	0 -c -c -c c c c	c -c c c -c -c c
(1,10,14)	c b -b -b -b -b b	0 b a -a a a a
	0 -b -a -b a -b a	0 -b b -a -b a b
(2,2,13)	a -b 0 0 0 0 0	a b 0 0 0 0 0
	0 -c -c -c c c c	c -c c c -c -c c
(2,7,19)	c -a a -a -a -b -a	-c -a b -a -a -a -a
	b b a -a -b -a a	b -b a a -a -a a
(2,8,13)	-b -b b -c b 0 0	-b -b b c b 0 0
	0 -a -a -a a a a	a -a a a -a -a a
(4,4,18)	a b -c b -b -b -b	a -b -c -b b b b
	a b -b -b b -c 0	a -b b b -b -c 0
(5,5,13)	a -a -a -b 0 -b 0	b -b -b a 0 a 0
	0 -c -c -c c c c	c -c c c -c -c c
(3,23)	-a 0 b -b b b b	a b b -b b b b
	a -b b b b b 0	b -b b -b b b -b

(4,19)	b -b 0 -a 0 a 0	b b a a a a a
	0 -a a 0 -a -a a	a -a a -a a a -a
(5,21)	b a a -b -b -a a	b -b a a -a -a -a
	a -a -a a -a -a 0	a -a a -a -a -a 0
(5,23)	a -a -a b b -b b	-a -b -a -b -b -b -b
	b -b b -b b b -b	b -b b b b -b -b
(6,17)	b b a -a a -a -a	b -b a -a -a a 0
	b -a -a -a a a a	b 0 -a 0 0 0 -a
(6,20)	b -a b -a -a 0 a	b -a -b -a 0 a -a
	b -a a a -a -a -a	b a -a a -a -a -a
(7,10)	a 0 0 -b 0 0 0	b a a 0 0 -a 0
	a 0 -b -b b 0 b	-b -b b 0 b a -a
(7,15)	b -a b -a -b 0 a	b a 0 -b 0 0 0
	b a a a 0 -a a	b a -a a a -a -a
(8,17)	a a a a -a -a 0	a -a b -b b b 0
	b a -a a b 0 a	b -a a a a -a -b
(9,16)	b -b b -a b a -b	b -a a -a -b -a 0
	b b 0 -a a a -a	a a a a -a -a 0
(9,17)	a a a a -a -a 0	a -a -a b -a a b
	a -a -b -b a -a 0	a -b -a -b b -b -b
(11,12)	a a a -a 0 0 0	-b -a a -a -a a a
	a -b a -b b b -b	b -b -b -b 0 -b 0
(11,15)	b -b -b -b 0 -b -a	b -b -b b 0 b -a
	b -a a a -a -a a	a -a a -a -a -a -a
(11,17)	b b b -b b -a -a	b b -b b b a -a
	b -a -a a -a a a	a -a a a a -a -a
(12,14)	b a b -b -b -a -a	b -a b -a b b a
	b -b -a a a 0 -a	a -a b a -b a 0

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