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On equality in Berge's classical bound for the domination number

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Abstract

Let $\gamma(G)$ denote the cardinality of a minimum dominating set of a graph G. A well-known upper bound for $\gamma(G)$, due to Berge (1962), states that for any graph G of order n and maximum degree Δ, $\gamma(G) \leq n - \Delta$. Similarly, Hedetniemi and Laskar (1984) proved $\gamma_c(G) \leq n - \Delta$, where $\gamma_c(G)$ denotes the cardinality of a minimum connected dominating set of G. In this paper, we characterize the regular graphs with $\gamma(G) = n - \Delta$, the regular graphs with $\gamma_c(G) = n - \Delta$ and the triangle-free graphs with $\gamma_c(G) = n - \Delta$. Moreover, we prove that both the problem of deciding whether $\gamma(G) = n - \Delta$ and the problem of deciding whether $\gamma_c(G) = n - \Delta$ are $co - NP$-complete.

1 Introduction

We consider finite, simple graphs $G = (V(G), E(G))$ with maximum degree $\Delta(G)$ and $n(G)$. Given any subset $U \subseteq V(G)$ the induced subgraph on U is denoted by $G[U]$. For any vertex $v \in V(G)$, the *open neighbourhood* $N(v)$ of v is defined by $N(v) = \{u \in V(G) \,|\, uv \in E(G)\}$ and the *closed neighbourhood* $N[v]$ of v is defined by $N[v] = N(v) \cup \{v\}$. For any set $U \subseteq V(G)$, let $N(U) = \cup_{u \in U} N(u)$ and $N[U] = N(U) \cup U$. For sets $U, W \subseteq V(G)$, we say that U *dominates* W if $W \subseteq N[U]$. If $U \subseteq V(G)$ dominates $V(G)$, then U is called a *dominating set* of G. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. A *connected dominating set* of G is a dominating set D of G with the additional property that the induced graph $G[D]$ is connected. The *connected domination number* $\gamma_c(G)$ is the cardinality of a minimum connected dominating set of G. For any undefined concept the reader may refer to [2] and [6]. When no confusion is possible, we may denote any parameter $f(G)$ of G by f.

A classical result by Berge [1] states that for any graph G, $\gamma(G) \leq n(G) - \Delta(G)$. It seems that Domke et al. [3] were the first to consider the problem of characterizing the graphs with $\gamma = n - \Delta$. They obtained a characterization of the connected bipartite graphs with $\gamma = n - \Delta$. Favaron and Mynhardt [4] continued the study of the problem, and gave the following characterization of graphs with $\gamma = n - \Delta$.

Theorem 1.1 (Favaron and Mynhardt [4])
Let G be a graph and x a vertex of G with maximum degree Δ. Let $B = N(x)$, $C = V(G) - N[x]$ and $R = B - N(C)$. For each $c \in C$, let $B_c = N(c) \cap B$. Then $\gamma = n - \Delta$ if and only if

(i) C *is independent,*

(ii) *every vertex of* B *is adjacent to at most one vertex in* C *and*

(iii) *for every non-empty subset* C' *of* C, *the subset* $B' = (\cup_{u \in C'} B_u) \cup R$ *of* B *is either empty or not dominated by a set consisting of exactly one vertex of each* B_u, $u \in C'$.

This characterization does not lead to a polynomial algorithm for determining whether $\gamma = n - \Delta$. One of the main results of this paper shows that the general problem of determining whether $\gamma = n - \Delta$ is $co - NP$-complete. However, for some classes of graphs characterizations leading to polynomial algorithms can be found.

If G is a disconnected graph with components $H_1, \ldots, H_k$, $\Delta(G) \geq 1$ and $\gamma(G) = n(G) - \Delta(G)$, then all but one component of G are K_1-components. This shows that it is sufficient to consider the connected graphs with $\gamma = n - \Delta$.

Favaron and Mynhardt [4] gave the following characterization of connected triangle-free graphs with $\gamma = n - \Delta$.

Theorem 1.2 (Favaron and Mynhardt [4])
Let G denote a connected triangle-free graph, and let v denote any vertex of G with maximum degree. Then $\gamma(G) = n(G) - \Delta(G)$ if and only if

(i) G *is bipartite with partition sets* $N(v)$ *and* $V(G) - N(v)$,

(ii) $|V(G) - N(v)| \leq |N(v)|$,

(iii) $\deg(u) \leq 2$ *for every* $u \in N(v)$, *and*

(iv) *If* $\deg(u) = 2$ *for every* $u \in N(v)$, *then* $\deg(u) \geq 2$ *for every* $u \in V(G) - N(v)$.

This characterization gives a polynomial algorithm for recognizing triangle-free graphs with $\gamma = n - \Delta$.

In Section 2 we characterize the regular graphs with $\gamma = n - \Delta$.

We also consider the problem of characterizing the graphs with $\gamma_c = n - \Delta$. Hedetniemi and Laskar [7] characterized the trees with $\gamma_c = n - \Delta$.

Proposition 1.3 (Hedetniemi and Laskar [7])
Let T denote a tree of order $n \geq 2$ and let $l(T)$ denote the number of leaves of T. Then

$$\gamma_c(T) = n - l(T) \leq n - \Delta(T)$$

Furthermore, $\gamma_c(T) = n - \Delta$ if and only if T has at most *one vertex of degree greater than two.*

It follows that $\gamma_c(G) \leq n(G) - \Delta(G)$ for any connected graph G. This seems to be the only work done on the problem of characterizing the graphs with $\gamma_c = n - \Delta$.

In Section 4 we characterize the class of triangle-free graphs with $\gamma_c = n - \Delta$, and in Section 5 we characterize the regular graphs with $\gamma_c = n - \Delta$. In Section 6, we show that, in general, the problem of deciding whether $\gamma_c = n - \Delta$ is $co - NP$-complete.

2 Regular Graphs with $\gamma = n - \Delta$

Theorem 2.1
Let G denote a connected regular graph. Then $\gamma = n - \Delta$ if and only if G is a complete graph, or n is even and G is a complete graph with a perfect matching removed, i.e. $G = K_n - M$, where M is a perfect matching of K_n.

Proof. First, suppose $\gamma(G) = n - \Delta$. Let v be any vertex of G. Since G is regular, the vertex v has maximum degree. If $G - N[v]$ does not contain any vertices, then $\Delta = n - 1$ and so G is complete. Hence assume that $G - N[v]$ contains at least one vertex. By Theorem 1.1, the graph $G - N[v]$ consists of $n - \Delta - 1$ isolated vertices. Hence every vertex of $V(G) - N[v]$ has all its neighbours in $N(v)$, and since G is Δ-regular, each vertex of $V(G) - N[v]$ is adjacent to every vertex of $N(v)$. Suppose that there is more than one vertex in $G - N[v]$. Then there are at least two vertices in $V(G) - N[v]$, say a and b, with a common neighbour, say x, in $N(v)$ and so $(V(G) - (N(v) \cup \{a, b\})) \cup \{x\}$ is a dominating set of G of cardinality $n - \Delta - 1$, a contradiction. Hence $G - N[v]$ contains exactly one vertex. This implies $\Delta = n - 2$, i.e. G is the connected $(n - 2)$-regular graph. It is easy to see that this graph is isomorphic to $K_n - M$, where M is a perfect matching of K_n.

Conversely, if G is complete, then $\gamma(G) = 1 = n - \Delta$, and if $G = K_n - M$, then $\gamma(G) = 2 = n - \Delta$. ■

3 Preliminary Results on Graphs with $\gamma_c = n - \Delta$

For extreme values of Δ the situation is simple.

Observation 3.1
Let G denote a connected graph with $\gamma_c(G) = n - \Delta(G)$.

- *If $\Delta(G) = 1$, then $G = K_2$.*

- *If $\Delta(G) = 2$, then $G \in \{C_n, P_n\}$.*
- *If $\Delta(G) = n-2$, then G can be any connected graph with $\Delta(G) = n-2$.*
- *If $\Delta(G) = n-1$, then G can be any connected graph with $\Delta(G) = n-1$.*

Proposition 3.2
Let G be a connected graph with $\gamma_c(G) = n(G) - \Delta(G)$. Then the following conditions (i-iii) are satisfied for every vertex v of degree $\Delta(G)$.

(i) All components of $G - N[v]$ are paths.

(ii) For every path $P : u_1, u_2, \ldots, u_r$ in $G - N[v]$ and $i \in \{2, \ldots, r-1\}$,

$$\deg_G(u_i) = 2.$$

(iii) Each vertex of $N(v)$ is adjacent to at most one vertex of $V(G) - N[v]$.

Proof. Assume $\gamma_c(G) = n - \Delta$. Let v denote a vertex of G for which $\deg_G(v) = \Delta$. If $\Delta = n-1$, then (i-iii) is satisfied. Suppose $\Delta \leq n-2$ and define $G' = G - N_G[v]$.

Let $H_1, H_2, \ldots, H_t$ $(t \geq 1)$ denote the components of G'. Note that in G at least one vertex x_i of H_i is adjacent to at least one vertex y_i in $N(v)$. Amongst all spanning trees of H_i, let T_i be one such that $l(T_i)$ is maximum and note that $l(T_i) \geq \Delta(H_i)$. Some of the trees T_i might be K_1's or K_2's. If there are any such trees, then let the trees be indexed such that $T_1, \ldots, T_s$ are K_1's and K_2's while $T_{s+1}, \ldots, T_t$ all have more than two vertices. If $s = t$, then every component of G' is a K_1 or a K_2 and (i) is satisfied. Hence we shall assume $s < t$.

By Proposition 1.3 we have $\gamma_c(T_i) = n(T_i) - l(T_i)$ for all $i = s+1, \ldots, t$ (this is not true for K_2). Furthermore, $l(T_i) \geq 2$ for all $i = s+1, \ldots, t$. We shall construct a connected dominating set D of G. Let v be in D. For each $T_i = K_1$ assign the vertex y_i to D. For each $T_i = K_2$ assign the vertices x_i and y_i to D. For each T_i, $(i > s)$ add y_i, x_i and the vertices of a $\gamma_c(T_i)$-set to D. Now we obtain

$$\begin{aligned}
\gamma_c(G) &\leq 1 + \sum_{i=1}^{s} n(T_i) + \sum_{j=s+1}^{t} (\gamma_c(T_j) + 2) \\
&= 1 + \sum_{i=1}^{s} n(T_i) + \sum_{j=s+1}^{t} (n(T_j) - l(T_j) + 2) \\
&\leq 1 + \sum_{i=1}^{s} n(T_i) + \sum_{j=s+1}^{t} n(T_j) \\
&\leq 1 + (n - \Delta - 1). \qquad (1)
\end{aligned}$$

Observe that if $l(T_i) > 2$ for some $i = s+1, \ldots, t$, then the argument of (1) implies $\gamma_c(G) < n - \Delta$, a contradiction. Hence $l(T_i) = 2$ for all $i = s+1, \ldots, t$ and therefore $\Delta(H_i) \leq 2$ for every $i = 1, \ldots, t$. This implies that H_i is either a path or a cycle.

If some vertex $u \in V(H_i)$ with $\deg_{H_i}(u) = 2$ is adjacent to some vertex of $N(v)$, then there exists a spanning tree T of G, where $\deg_T(v) = \Delta(G)$ and $\deg_T(u) \geq 3$. Now Proposition 1.3 implies $\gamma_c(G) \leq \gamma_c(T) < n - \Delta$, a contradiction. Hence only vertices $u \in V(H_i)$ with $\deg_{H_i}(u) = 1$ can be adjacent to vertices of $N(v)$. Since G is connected, it follows that some vertex $u \in V(H_i)$ with $\deg_{H_i}(u) = 1$ is adjacent to some vertex in $N(v)$. This implies that H_i is a path and so (i-ii) is satisfied.

If some vertex of $N(v)$ have more than one neighbour in $V(G) - N[v]$, then G has a spanning tree T with at least two vertices of degree greater than two, and so Proposition 1.3 implies $\gamma_c(G) < n - \Delta$, a contradiction. Thus each vertex of $N(v)$ has at most one neighbour in $V(G) - N[v]$. This establishes (iii). ■

The induced graph $G[N(v)]$ seem to elude characterization. One way to overcome this problem is to require the graph to be triangle-free, since in a triangle-free graph G the induced subgraph $G[N(v)]$ contains no edge.

4 Triangle-free Graphs with $\gamma_c = n - \Delta$

Theorem 4.1
Let G denote a connected triangle-free graph. Then $\gamma_c(G) = n - \Delta$ if and only if the following conditions (i-iii) are satisfied for every vertex v of degree $\Delta(G)$.

(i) All components of $G - N[v]$ are paths.

(ii) For every path $P : u_1, u_2, \ldots, u_r$ in $G - N[v]$ and $i \in \{2, \ldots, r-1\}$,

$$\deg_G(u_i) = 2.$$

(iii) Each vertex of $N(v)$ is adjacent to at most one vertex of $V(G) - N[v]$.

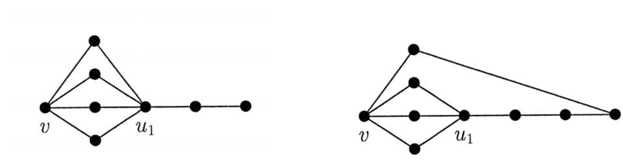
Proof. If $\gamma_c(G) = n - \Delta$, then it follows from Proposition 3.2 that (i-iii) is satisfied for every vertex v of G with $\deg_G(v) = \Delta$.

Now, suppose (i-iii) is satisfied for some vertex v of G with $\deg_G(v) = \Delta$. If $\Delta \leq 1$, then $G \in \{K_1, K_2\}$, and $\gamma_c(G) = n - \Delta$. If $\Delta = 2$, then G is either a path or a cycle. In either case, $\gamma_c(G) = n - \Delta$. Hence we may assume $\Delta \geq 3$. Let D denote a $\gamma_c(G)$-set. Suppose that $G - N[v]$ contains precisely one component, say $P^1 : u_1, \ldots, u_r$. Then D contains at least $n(P^1)$ vertices of $V(P^1) \cup N(u_1) \cup N(u_r)$. We shall consider three cases.

(i) P^1 is a singleton.

(ii) $n(P^1) \geq 2$ and only one end-vertex of P^1 is adjacent to a vertex of $N(v)$ in G.

(iii) $n(P^1) \geq 2$ and both end-vertices of P^1 are adjacent to vertices of $N(v)$ in G.



(a) Since v is not a stem, u_1 must be adjacent to every vertex of $N(v)$.

(b) If D does not contain v, then D contains every vertex of P^1.

Figure 1:

(i) In this case we have $\Delta = n - 2$, and so $\gamma_c(G) > 1$. On the other hand, $\gamma_c(G) \leq n - \Delta = 2$ and so $\gamma_c(G) = n - \Delta$.

(ii) Assume that $n(P^1) \geq 2$ and that, in G, only one end-vertex of P^1 is adjacent to a vertex of $N(v)$. Let u_1 denote the vertex of P^1 which is adjacent to one or more vertices of $N(v)$. Now D contains at least $n(P^1) = n - \Delta - 1$ vertices of $N[P^1]$ in order to dominate P^1. Suppose that these vertices dominate G. Then there are no leaves adjacent to v, and, since $G - N[v]$ contains only one component, $N(v) = N(v) \cap N(u_1)$ (See Figure 1a). But then u_1 has degree $\Delta + 1$ in G, a contradiction.

(iii) Assume that $n(P^1) \geq 2$ and that, in G, both end-vertices of P^1 are adjacent to vertices of $N(v)$. Again, D contains at least $n(P^1) = n - \Delta - 1$ vertices of $N[P^1]$ in order to dominate P^1. If $v \in D$, then $|D| \geq n - \Delta$. Hence we may assume that $v \notin D$. Now in order for $G[D]$ to be connected, the set D contains every vertex of P^1, and for D to dominate v, D contains at least one vertex of $(N(u_1) \cup N(u_r)) \cap N(v)$ (See Figure 1b). This shows that $|D| \geq n - \Delta$.

Suppose that $G - N[v]$ contains more than one component. Then D contains v, since otherwise $G[D]$ would be disconnected. Let $P^1, \ldots, P^r$ denote the components of $G - N[v]$. In order to be a connected dominating set, D contains at least $n(P^i)$ vertices of $N[P^i]$ for the domination of P^i and , since $N[P^1], \ldots, N[P^r]$ are all disjoint, we obtain $|D| \geq 1 + \sum_{i=1}^{r} n(P^i) = 1 + (n - \Delta - 1)$. Hence $\gamma_c(G) = n - \Delta$. ∎

Using Theorem 4.1 it is easy to design a polynomial algorithm for recognition of triangle-free graphs with $\gamma = n - \Delta$.

5 Regular Graphs with $\gamma_c = n - \Delta$

Theorem 5.1
Let G denote a connected regular graph. Then $\gamma_c(G) = n - \Delta$ if and only if G is one of the following: C_n, K_n, or $K_n - M$, where M is a perfect matching in K_n.

Proof. First, suppose that G is a Δ-regular connected graph. Let v denote any vertex of G. If $V(G) - N[v] = \varnothing$, then $\Delta = n - 1$ and G is a complete graph. Suppose $V(G) - N[v] \neq \varnothing$. If $\Delta = 2$, then $G = C_n$, so we may assume $\Delta \geq 3$. Let u_1 denote a vertex of $V(G) - N[v]$ which is adjacent to a vertex of $N(v)$. Proposition 3.2(ii) and the regularity of G implies that $V(G) - N[V]$ contains at most two vertices.

Suppose that u_1 is adjacent to a vertex u_2 in $V(G) - N[v]$. Now the regularity of G implies that both of u_1 and u_2 are adjacent to precisely $\Delta - 1$ vertices of $N(v)$. Since u_1 and u_2 do not have a common neighbour in $N(v)$, we obtain $(\Delta - 1) + (\Delta - 1) \leq |N(v)| = \Delta$, which implies $\Delta \leq 2$, a contradiction. Hence u_1 has all its neighbours in $N(v)$, i.e. u_1 is adjacent to every vertex of $N(v)$. Then it follows from Proposition 3.2 that $V(G) - N[v]$ only contains this one vertex u_1. Hence $\Delta = n - 2$, and, since G is regular, $G \simeq K_n - M$, where M is a perfect matching in K_n. ■

6 Complexity Results

In this section, we prove one of the main results of this paper, namely that the problem of deciding whether $\gamma = n - \Delta$ is $co - NP$-complete.

Decision Problem 6.1 (MDS $(n - \Delta)$)
MINIMUM DOMINATING SET OF CARDINALITY $n - \Delta$
INSTANCE: *A graph G.*
QUESTION: *Does G have a minimum dominating set of cardinality $n(G) - \Delta(G)$?*

In order to prove the $co - NP$-completeness of the above problem, we prove that the **3-SAT** problem can be reduced to the problem of deciding whether $\gamma \leq n - \Delta - 1$.

Decision Problem 6.2 (DS $(n - \Delta - 1)$)
DOMINATING SET OF CARDINALITY $\leq n - \Delta - 1$
INSTANCE: *A graph G.*
QUESTION: *Does G have a dominating set of cardinality $\leq n(G) - \Delta(G) - 1$?*

For any boolean variable u, let $\overline{u}$ denote the negation of u. Given a set of independent boolean variables $U = \{u_1, u_2, \ldots, u_p\}$ (independent in the sense that truth values can be assigned completely arbitrarily to the variables of U), we define a clause C of U to be a 3-element set $\{x_1, x_2, x_3\}$, where either $x_i \in U$ or $\overline{x}_i \in U$ for each $i \in \{1, 2, 3\}$.

Decision Problem 6.3 (3-SAT)
3-SATISFIABILITY
INSTANCE: *A set $U = \{u_1, u_2, \ldots, u_p\}$ of variables, and a set $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ of clauses.*
QUESTION: *Does $\mathcal{C}$ have a satisfying truth assignment, i.e. an assignment of True and False to the variables in U such that at least one variable in each clause C_i of $\mathcal{C}$ is assigned the value True ?*

Theorem 6.4
The decision problem $\boldsymbol{DS}(n-\Delta-1)$ *is* NP*-complete.*

The proof uses a construction which, to the best of my knowledge, was first introduced by [5].

Proof. The decision problem $\mathbf{DS}(n-\Delta-1)$ is in NP, since if $S \subseteq V(G)$ is a dominating set of cardinality $\leq n(G)-\Delta(G)-1$, then it can be verified in polynomial time that S is a dominating set.

Next, we show that **3-SAT** is reducible to $\mathbf{DS}(n-\Delta-1)$. Given any nontrivial instance $\mathcal{C}$ of **3-SAT**, we construct an instance $G_{\mathcal{C}}$ of $\mathbf{DS}(n-\Delta-1)$ as follows. For each variable u_i, construct a triangle with vertices labelled $u_i, \overline{u}_i, v_i$. For each clause $C_j = \{u_i, u_k, u_l\}$ add a vertex C_j, and edges u_iC_j, u_kC_j, u_lC_j. Finally, add a vertex x, and join x to every vertex of $V(G_{\mathcal{C}}) - (\{x\} \cup \{v_1, v_2, \ldots, v_p\})$ (See Figure 2).

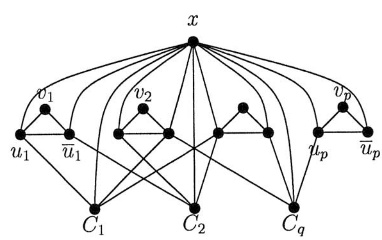


Figure 2: Construction of the graph $G_{\mathcal{C}}$ from an instance $\mathcal{C}$ of the **3SAT** problem.

Notice that the construction of $G_{\mathcal{C}}$ from $\mathcal{C}$ is done in polynomial time.

Claim 6.5
The vertex x *is a vertex of maximum degree in* $G_{\mathcal{C}}$ *and* $\Delta(G_{\mathcal{C}}) = n(G_{\mathcal{C}}) - p - 1$.

Argument. Since $\mathcal{C}$ is a nontrivial instance, we have $|U| = p \geq 2$. Every vertex C_i has degree four; every vertex v_i has degree two; every vertex u_i (and $\overline{u}_i$) has degree at most $q+3$. The vertex x has degree $n(G_{\mathcal{C}}) - p - 1 = 2p + q \geq 4 + q$. Hence x is a vertex of maximum degree, and $\Delta(G_{\mathcal{C}}) = n(G_{\mathcal{C}}) - p - 1$. ◇

Hence we have $p = n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.

Claim 6.6
The instance $\mathcal{C}$ *of* ***3-SAT*** *has a satisfying truth assignment if and only if the graph* $G_{\mathcal{C}}$ *has a dominating set of cardinality* $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.

Argument. First, suppose that $\mathcal{C}$ has a satisfying truth assignment. We construct a dominating set S as follows: If u_i is True, then assign u_i to S, else assign $\overline{u}_i$ to S. The set S is a dominating set, since (i) each triangle contains a vertex of S, (ii) each C_j is adjacent to a vertex in S, and (iii) certainly x is dominated by S. Since S contains exactly one vertex from each triangle $u_i\overline{u}_iv_i$, and no more vertices, we obtain $|S| = p = n(G_c) - \Delta(G_{\mathcal{C}}) - 1$. This shows that $G_{\mathcal{C}}$ has a dominating set of cardinality $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.

Conversely, suppose that $G_{\mathcal{C}}$ has a dominating set S of cardinality $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$. Then S contains at least one vertex from each triangle $u_i\overline{u}_iv_i$, and so $|S| \geq p = n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$. It follows that S contains exactly one vertex from each triangle $u_i\overline{u}_iv_i$, and no other vertices. We may assume without loss of generality that $S \subseteq \{u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p\}$. Now $y \in S$ if and only if $\overline{y} \notin S$, and so we obtain a correct assignment of truth values by letting $y \in \{u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p\}$ be assigned the value True if and only if $y \in S$. Every C_j is dominated by S in G. Suppose $w \in S$ is a vertex dominating C_j. Now w was assigned the value True and the construction of $G_{\mathcal{C}}$ implies w is a variable in the clause C_j. Hence the clause C_j is True and so $\mathcal{C}$ has a satisfying truth assignment.

◇

■

Corollary 6.7
The decision problem $\boldsymbol{MDS}(n - \Delta)$ *is* $co - NP$*-complete.*

Proof. Given any instance G, we find that the answer to the problem $\mathbf{MDS}(n - \Delta)$ is YES if and only if the answer to to $\mathbf{DS}(n - \Delta - 1)$ is NO. Hence $\mathbf{MDS}(n - \Delta)$ and $\mathbf{DS}(n - \Delta - 1)$ are complementary problems, and so $\mathbf{DS}(n - \Delta - 1) \in NPC$ implies $\mathbf{MDS}(n - \Delta) \in co - NPC$. ■

Decision Problem 6.8 (MCDS $(n - \Delta)$)
MINIMUM CONNECTED DOMINATING SET OF CARDINALITY $n - \Delta$
INSTANCE: *A graph* G.
QUESTION: *Does* G *have a minimum connected dominating set of cardinality* $n(G) - \Delta(G)$ *?*

Decision Problem 6.9 (CDS $(n - \Delta - 1)$)
CONNECTED DOMINATING SET OF CARDINALITY $\leq n - \Delta - 1$
INSTANCE: *A graph* G.
QUESTION: *Does* G *have a connected dominating set of cardinality* $\leq n(G) - \Delta(G) - 1$ *?*

Theorem 6.10
The decision problem $\boldsymbol{CDS}(n - \Delta - 1)$ *is* NP*-complete.*

The proof is similar to the proof of Theorem 6.4, and so we only present a sketch.

Sketch of proof. The decision problem $\mathbf{CDS}(n - \Delta - 1)$ is obviously in NP. The next step is to show that **3-SAT** is reducible to $\mathbf{CDS}(n - \Delta - 1)$. Given any instance $\mathcal{C}$ of **3-SAT**, let $G_{\mathcal{C}}$ denote the corresponding instance of $\mathbf{CDS}(n - \Delta - 1)$. If $\cap_{i=1}^{q} C_i \neq \varnothing$, then let $G_{\mathcal{C}} = K_3$. If $\cap_{i=1}^{q} C_i = \varnothing$, then construct $G_{\mathcal{C}}$ as follows. For each variable u_i, construct a triangle with vertices labelled $u_i, \overline{u}_i, v_i$. For each clause $C_j = \{u_i, u_k, u_l\}$ add a vertex C_j, and edges u_iC_j, u_kC_j, u_lC_j. Add edges such that the induced subgraph on $\{u_1, \overline{u}_1, \ldots, u_p, \overline{u}_p\}$ is a complete graph. Finally, add a vertex x, and join x to every vertex of $V(G_{\mathcal{C}}) - (\{x\} \cup \{v_1, v_2, \ldots, v_p\})$. Now the theorem follows by establishing the two following claims.

Claim 6.11
If $\bigcap_{i=1}^{q} C_i = \varnothing$, *then* x *is a vertex of maximum degree in* $G_{\mathcal{C}}$ *and* $\Delta(G_{\mathcal{C}}) = n(G_{\mathcal{C}}) - p - 1$.

Claim 6.12
The instance $\mathcal{C}$ *of* ***3-SAT*** *has a satisfying truth assignment if and only if the graph* $G_{\mathcal{C}}$ *has a connected dominating set of cardinality* $\leq n(G_{\mathcal{C}}) - \Delta(G_{\mathcal{C}}) - 1$.

The details are omitted. ■

Corollary 6.13
The decision problem ***MCDS***$(n - \Delta)$ *is* $co - NP$*-complete.*

Proof. **MCDS**$(n - \Delta)$ and **CDS**$(n - \Delta - 1)$ are complementary problems, and so, since **CDS**$(n - \Delta - 1) \in NPC$, we obtain **MCDS**$(n - \Delta) \in co - NPC$. ■

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