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Almost regular c-partite tournaments with $c \geq 8$ contain an n-cycle through a given arc for $4 \leq n \leq c$

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Abstract

If x is a vertex of a digraph D, then we denote by $d^+(x)$ and $d^-(x)$ the outdegree and the indegree of x, respectively. The global irregularity of a digraph D is defined by $i_g(D) = \max\{d^+(x), d^-(x)\} - \min\{d^+(y), d^-(y)\}$ over all vertices x and y of D (including $x = y$). If $i_g(D) = 0$, then D is regular and if $i_g(D) \leq 1$, then D is almost regular.

A c-partite tournament is an orientation of a complete c-partite graph. In a recent article, the authors proved that in an almost regular c-partite tournament with $c \geq 7$ and at least two vertices in each partite set, every arc of D is contained in a directed cycle of length n for each $n \in \{4, 5, \ldots, c\}$.

Now, the aim is to extend this result to those almost regular multipartite tournaments with only one vertex in the smallest partite set. In the case that $c = 7$, the above mentioned result does not rest valid, if there is only one vertex in the partite set of the smallest cardinality. But for $c \geq 8$ it does, as we will show in this paper.

1 Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and arc set of a digraph D is denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X,Y)$. If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner* *neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and *indegree* of x, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n-cycle*. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

There are several measures of how much a digraph differs from being regular. In [13], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)}\{d^+(x), d^-(x)\} - \min_{y \in V(D)}\{d^+(y), d^-(y)\}.$$

If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \le 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \dots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \le |V_2| \le \ldots \le |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D, and we define $\gamma(D) = |V_1|$.

It is very easy to see that every arc of a regular tournament belongs to a 3-cycle. The next example shows that this is not valid for regular multipartite tournaments in general.

Example 1.1 *Let C, C', and C'' be three induced cycles of length 4 such that $C \to C' \to C'' \to C$. The resulting 6-partite tournament D_1 is 5-regular, but no arc of the three cycles C, C', C'' is contained in a 3-cycle.*

Let H, H_1, and H_2 be three copies of D_1 such that $H \to H_1 \to H_2 \to H$. The resulting 18-partite tournament is 17-regular, but no arc of the cycles corresponding to the cycles C, C', and C'' is contained in a 3-cycle.

If we continue this process, we arrive at regular c-partite tournaments with arbitrary large c which contain arcs that do not belong to any 3-cycle.

In 1998, Guo [3] proved the following generalization of Alspach's classical result [1] that every regular tournament is arc pancyclic.

Theorem 1.2 (Guo [3]) *Let D be a regular c-partite tournament with $c \ge 3$. If every arc of D is contained in a 3-cycle, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \dots, c\}$.*

Now, the aim was to carry this result forward to almost regular multipartite tournaments, however without the strong hypothesis that every arc is contained in a 3-cycle. To reach this, Volkmann [8] started with the following theorem.

Theorem 1.3 (Volkmann [8]) *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| = |V_2| = \ldots = |V_c| = r \ge 2$. If $c \ge 6$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \dots, c\}$.*

However, not all almost regular c-partite tournaments were considered in this theorem. According to an article of Tewes, Volkmann and Yeo [7], the following lemma holds:

Lemma 1.4 (Tewes, Volkmann, Yeo [7]) *If $V_1, V_2, \ldots, V_c$ are the partite sets of an almost regular c-partite tournament, then $||V_i| - |V_j|| \leq 2$ for $1 \leq i, j \leq n$.*

Using this lemma, Volkmann and Winzen [11] extended Theorem 1.3 to the following result.

Theorem 1.5 (Volkmann, Winzen [11]) *Let D be an almost regular c-partite tournament with at least two vertices in every partite set. If $c \geq 7$, then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

Now, the aim is to carry this result over to those almost regular multipartite tournaments, which also contain partite sets consisting of only one vertex. If $V_1, V_2, \ldots, V_c$ are the partite sets of the almost regular multpartite tournament D such that $c \geq 8$ and $|V_1| = |V_2| = \ldots = |V_c| = 1$, then D is a tournament and a theorem of Jakobsen [5] yields the desired result.

Theorem 1.6 (Jacobson [5]) *Let D be an almost regular tournament with $c \geq 8$ vertices. Then every arc of D is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.*

A first result for all almost regular c-partite tournaments with $c \geq 8$, was presented by Volkmann [10].

Theorem 1.7 (Volkmann [10]) *Let D be an almost regular c-partite tournament.*
If $c \geq 8$, then every arc of D is contained in a 4-cycle.
If $c = 7$ and there are at least two vertices in every partite set, then every arc of D is contained in a 4-cycle.

In this paper, we show that in all almost regular c-partite tournaments with $c \geq 8$, every arc is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$. Because of the Theorems 1.5, 1.6, and Lemma 1.4, we have to investigate the case that $1 = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq 3$ and $|V_c| \geq 2$.

For more information on multipartite tournaments see [2, 3, 4, 6, 9, 12].

2 Preliminary results

The following results play an important role in our investigations:

Lemma 2.1 (Tewes, Volkmann, Yeo [7]) *If D is an almost regular multipartite tournament, then for every vertex x of D we have*

$$\frac{|V(D)| - \alpha(D) - 1}{2} \leq d^+(x), d^-(x) \leq \frac{|V(D)| - \gamma(D) + 1}{2}$$

If we know the cardinality of the partite set $V(x)$, then we can specialize the previous lemma:

Lemma 2.2 (Volkmann, Winzen [11]) *If D is an almost regular multipartite tournament and x a vertex of D with $|V(x)| = p$, then*

$$\frac{|V(D)| - p - 1}{2} \le d^+(x), d^-(x) \le \frac{|V(D)| - p + 1}{2}$$

In this article we treat the case of an almost regular multipartite tournament D with $\alpha(D) = 2$ or $\alpha(D) = 3$ and $\gamma(D) = 1$. Therefore, we note that the digraphs of this paper cannot be regular. Furthermore, we can remark the following.

Remark 2.3 *If $\alpha(D) = 3$, $\gamma(D) = 1$ and $i_g(D) \le 1$, then the value of $|V(D)| - 1$ has to be even. So the bounds in Lemma 2.2 can be improved by*

$$d^+(x), d^-(x) = \frac{|V(D)| - 3}{2} \quad \textit{if} \quad |V(x)| = 3$$

or

$$d^+(x), d^-(x) = \frac{|V(D)| - 1}{2} \quad \textit{if} \quad |V(x)| = 1.$$

Now let us summarize some results of Lemma 2.2 and Remark 2.3.

Corollary 2.4 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of an almost regular c-partite tournament D. If $1 = |V_1| \le |V_2| \le \ldots \le |V_c| \le 3$, then for every vertex x of D we have*

$$\frac{|V(D)| - 3}{2} \le d^+(x), d^-(x).$$

3 Main results

Theorem 3.1 *Let D be an almost regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $1 = |V_1| \le |V_2| \le \ldots \le |V_c| \le 3$ and $|V_c| \ge 2$. If $c \ge 8$, then every arc of D is contained in an n-cycle for each $n \in \{4, \ldots, c\}$.*

Proof. We prove the theorem by induction on n. For $n = 4$, the result follows from Theorem 1.7. Now let e be an arc of D and assume that e is contained in an n-cycle $C = a_n a_1 a_2 \ldots a_n$ with $e = a_n a_1$ and $4 \le n \le c - 1$. Suppose that $e = a_n a_1$ is not contained in any $(n+1)$-cycle.

Obviously, $|V(D)| = c + k$ with $1 \le k \le c - 1$, if $|V_c| = 2$ and $2 \le k \le 2c - 2$, if $|V_c| = 3$. Firstly, we observe that, if $n = 4$ and $|V_c| = 2$ or $n \le 5$ and $|V_c| = 3$, then $N^+(v) - V(C) \ne \emptyset$ for each $v \in V(C)$, because otherwise Corollary 2.4, the fact that $k \ge 1$ (respectively, $k \ge 2$) and $c \ge 8$ yield the contradiction

$$4 = |V(C)| \ge d^+(v) + 2 \ge \frac{c + k - 3}{2} + 2 \ge 5$$

or

$$5 \ge |V(C)| \ge d^+(v) + 2 \ge \frac{c + k - 3}{2} + 2 > 5.$$

Analogously, one can show that $N^-(v) - V(C) \neq \emptyset$ for each $v \in V(C)$, in these cases.

Next, let S be the set of vertices that belong to partite sets not represented on C and define

$$X = \{x \in S | C \to x\}, \qquad Y = \{y \in S | y \to C\}.$$

Assume that $X \neq \emptyset$ and let $x \in X$. It follows that $N^-(v) - V(C), N^+(v) - V(C) \neq \emptyset$ for each $v \in V(C)$, because otherwise, we have $d^-(v), d^+(v) \leq n - 2$ and $d^-(x) \geq n$, a contradiction to $i_g(D) \leq 1$. If there is a vertex $w \in N^-(a_n) - V(C)$ such that $x \to w$, then $a_n a_1 a_2 \dots a_{n-2} x w a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. If $(N^-(a_n) - V(C)) \to x$, then $|N^-(x)| \geq |N^-(a_n) - V(C)| + |V(C)| \geq |N^-(a_n)| + 2$, a contradiction to the hypothesis that $i_g(D) \leq 1$. If there exists a vertex $b \in (N^-(a_n) - V(C))$ such that $V(b) = V(x)$, then b is adjacent with all vertices of C. In the case that $N^-(b) \cap V(C) \neq \emptyset$, let $l = \max_{1 \leq i \leq n-1}\{i | a_i \to b\}$. Then $a_n a_1 \dots a_l b a_{l+1} \dots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. It remains to consider the case that $N^-(b) \cap V(C) = \emptyset$. If there is a vertex $u \in (N^-(b) - V(C)) = N^-(b)$ such that $x \to u$, then $a_n a_1 a_2 \dots a_{n-3} x u b a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Otherwise, $N^-(b) \to x$, and we arrive at the contradiction $d^-(x) \geq d^-(b) + |V(C)|$. Altogether, we have seen that $X \neq \emptyset$ is not possible, and analogously we find that $Y \neq \emptyset$ is impossible. Consequently, from now on we shall assume that $X = Y = \emptyset$.

By the definition of S, every vertex of $V(C)$ is adjacent to every vertex of S, and since $n \leq c - 1$, we deduce that $S \neq \emptyset$. Now we distinguish different cases.

Case 1. There exists a vertex $v \in S$ with $v \to a_n$. Since $Y = \emptyset$, there is a vertex $a_i \in V(C)$ such that $a_i \to v$. If $l = \max_{1 \leq i \leq n-1}\{i | a_i \to v\}$, then $a_n a_1 \dots a_l v a_{l+1} \dots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $a_n \to S$.

Case 2. There exists a vertex $v \in S$ with $a_1 \to v$. Since $X = \emptyset$, there is a vertex $a_i \in V(C)$ such that $v \to a_i$. If $l = \min_{2 \leq i \leq n-1}\{i | v \to a_i\}$, then $a_n a_1 \dots a_{l-1} v a_l \dots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. This implies $S \to a_1$.

Case 3. There exists a vertex $v \in S$ such that $v \to a_{n-1}$. If there is a vertex $a_i \in V(C)$ with $2 \leq i \leq n-2$ such that $a_i \to v$, then we obtain as above an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \dots, a_{n-1}\}$. Because of $S \to a_1$, we note that every vertex of $N^+(a_1)$ is adjacent to v. If there is a vertex $x \in (N^+(a_1) - V(C))$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \dots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Therefore we assume now that $v \to (N^+(a_1) - V(C))$. This leads to $d^+(v) \geq d^+(a_1) + 1$, and thus, because of $i_g(D) \leq 1$, it follows that $N^+(v) = N^+(a_1) \cup \{a_1\}$ and $a_1 \to \{a_2, a_3, \dots, a_{n-1}\}$.

If we define $H = N^+(a_1) - V(C)$ and $Q = N^-(v) - \{a_n\}$, then $H \cap Q = \emptyset$, $S \cap H = \emptyset$, and $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$.

If there is an arc $x a_2$ with $x \in H$, then $a_n a_1 x a_2 a_3 \dots a_n$ is an $(n+1)$-cycle through $a_n a_1$, a contradiction. Thus, we assume in the following that $a_2 \leadsto H$.

Subcase 3.1. Let $n = 4$. At first, let $|V_c| = 2$. If C consists of at most 3 partite sets, then it has to be $|S| \geq 5$ and thus, it follows that $d^+(a_4) \geq 6$. On the other hand, we see that $d^-(a_4) \leq |V(D)| - |S| - |V(a_4)| - |\{a_1\}| \leq 3$, a contradiction to $i_g(D) \leq 1$. Therefore, $D[V(C)]$ has to be a tournament.

Now, let $|V_c| = 3$. If $V(C)$ is 2-partite, then we observe that $d^+(a_4) \geq |S| + 1 \geq 7$ and $d^-(a_4) \leq |V(a_3) - \{a_1\}| \leq 2$, a contradiction to $i_g(D) \leq 1$. So, let C contain vertices of only 3 partite sets. If $|S| \geq 6$, then we see that $d^+(a_4) \geq 7$ and $d^-(a_4) \leq 5$, a contradiction. Consequently, it remains to investigate the case that $|S| = 5$, $c = 8$, $2 \leq k \leq 6$ and $10 \leq |V(D)| \leq 14$. Since $d^+(a_4) \geq 6$, it follows that $12 \leq |V(D)| \leq 14$. In view of Remark 2.3, it remains to treat the case that $|V(D)| = 13$. If $|V(a_4)| = 3$, then $d^+(a_4) = d^-(a_4) = 5$, a contradiction to $d^+(a_4) \geq 6$. If $|V(a_1)| = 3$, then $d^+(a_1) = d^-(a_1) = 5$, a contradiction to $d^-(a_1) \geq 6$. This implies $|V(a_1)|, |V(a_4)| \leq 2$ and thus $|V(D)| \leq 12$, a contradiction.

Consequently, if $n = 4$, then it is sufficient to investigate the case that $D[V(C)]$ is a tournament. We remind that we have shown above that $H \neq \emptyset$.

Subcase 3.1.1. Suppose that $|H| = 1$. This implies $d^+(v) = d^+(a_1) + 1 = 4$. On the other hand, we see that $d^+(a_4) \geq |S| + 1 \geq 5$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.1.2. Let $|H| \geq 2$.

Subcase 3.1.2.1. Assume that $|H| = 2$ and $E(D[H]) = \emptyset$, which means that $|V_c| = |V(h)| = 3$. Then, it follows that $d^+(v) = d^+(a_1) + 1 = 5$, which yields

$$4 = d^+(a_1) \leq d^-(v) = |Q| + 1 \leq d^+(v) = 5,$$

and hence $3 \leq |Q| \leq 4$. Because of $d^+(a_4) \geq |S| + 1 \geq 5$, it remains to consider the case that $|S| = 4$, $d^+(a_4) = 5$, $c = 8$ and $a_2 \to a_4$. Since $|S| = 4$ and $S = V(v) \cup (Q \cap S)$, we see that we have to investigate the case $|Q - S| \leq 1$. If $H \subseteq V(a_4)$, then $d^-(a_4) \leq |\{a_2, a_3\}| + |Q - S| \leq 3$, a contradiction to $i_g(D) \leq 1$. Consequently, it has to be $H \to a_4$ and therefore also $H \leadsto a_3$, since otherwise $a_4a_1a_2a_3ha_4$ is a 5-cycle, if $h \in H$, a contradiction. Since $|V(v)| = 1$, at least three vertices of Q have to belong to $N^+(a_3)$, because otherwise, we arrive at the contradiction $d^+(a_3) \leq 3$. If there are vertices $q \in N^+(a_3) \cap Q$ and $h \in H$ such that $q \to h$, then $a_4a_1a_3qha_4$ is a 5-cycle, a contradiction. It remains to consider the case that $H \to (N^+(a_3) \cap Q)$. If $q \in Q \cap N^+(a_3)$ such that $q \to a_2$, then $a_4a_1hqa_2a_4$ is a 5-cycle, a contradiction. Let $q_1 \in N^+(a_3) \cap Q \cap S \neq \emptyset$ be a vertex such that $|N^-(q_1) \cap Q \cap S| \geq 1$. Then we arrive at $d^-(q_1) \geq |H| + 1 + |\{a_2, a_3, a_4\}| = 6$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.1.2.2. Suppose now that $|H| \geq 2$ and $E(D[H]) \neq \emptyset$. Hence, there is an arc $p \to q$ in $E(D[H])$. If $q \to a_3$, then $a_4a_1pqa_3a_4$ is a 5-cycle, a contradiction. Hence, let $a_3 \leadsto q$. If $x \in N^+(q)$, then $a_4 \leadsto x$, because otherwise, $a_4a_1pqxa_4$ is a 5-cycle, a contradiction.

Firstly, let $a_4 \to a_2$. Then, we have

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(q) - (V(C) \cup (V(a_4) - \{a_4\}))) \cup (N^-(q) \cap S) \cup \{v, a_1, a_2\} \quad \text{and} \\ N^+(q) &\subseteq (N^+(q) - V(C)) \cup \{a_4\}. \end{aligned}$$

If there is a vertex $x \in Q \cap S$ such that $x \to q$, then $|N^-(q) \cap S| \geq 1$ and we deduce that

$$d^+(a_4) \geq \begin{cases} d^+(q) + 1, & \text{if} \quad |V(a_4)| = 3 \\ d^+(q) + 2, & \text{if} \quad |V(a_4)| \leq 2 \end{cases},$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$. Therefore, let $q \to Q \cap S$. If $a_4 \to q$, then similarly, we arrive at a contradiction, and if $q \in V(a_4)$, then we observe that $N^+(a_4) \supseteq (N^+(q) - (V(C) \cup (V(a_4) - \{a_4, q\}))) \cup \{v, a_1, a_2\}$ and we get the same contradiction as above. Hence, let $q \to a_4$. Furthermore, $p \in V(a_2)$, since otherwise, $a_4a_1a_2pqa_4$ is a 5-cycle, a contradiction. If there is a vertex $x \in Q \cap S$ such that $x \to a_3$, then $a_4a_1qxa_3a_4$ is a 5-cycle, a contradiction. Hence $Q \cap S \subseteq N^+(a_3)$. If there are vertices $x \in N^+(a_3)$ and $y \in N^-(a_4)$ such that $x \to y$, then $a_4a_1a_3xya_4$ is a 5-cycle, a contradiction. Consequently, we conclude that $N^-(a_4) \leadsto N^+(a_3)$. Let $v_1 \to v_2$ be an arc in $E(D[Q \cap S])$. Then, we observe that $d^+(v_2) \leq d^+(a_4) - 2 + |V(a_4) - \{a_4\}|$, and thus $|V(a_4)| \geq 2$. If $(V(a_4) - \{a_4\}) \to v_2$, then we see that $d^+(a_4) \geq d^+(v_2) + 2$, a contradiction. If $|V(a_4)| = 3$ and $|N^+(v_2) \cap (V(a_4) - \{a_4\})| = 1$, then it follows that $d^+(a_4) \geq d^+(v_2) + 1$, a contradiction to Remark 2.3. Hence, let $v_2 \to (V(a_4) - \{a_4\})$. Analogously, we conclude that there is no vertex $w \in Q \cap S$ such that $|N^-(w) \cap Q \cap S| \geq 2$. Let x_1, x_2, x_3 be three vertices of $Q \cap S$ belonging to three different partite sets, then they have to form a 3-cycle and $\{x_1, x_2, x_3\} \to (V(a_4) - \{a_4\})$. Furthermore, we see that $a_3 \to (V(a_4) - \{a_4\})$, because otherwise, if $d \in V(a_4) - \{a_4\}$ such that $d \to a_3$, then

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(a_3) - (V(C) \cup (V(a_4) - \{a_4, d\}))) \cup \{v, a_1, a_2\} \quad \text{and} \\ N^+(a_3) &\subseteq (N^+(a_3) - V(C)) \cup \{a_4\}. \end{aligned}$$

If $|V(a_4)| = 3$, then this implies $d^+(a_4) \geq d^+(a_3) + 1$, a contradiction to Remark 2.3. If $|V(a_4)| = 2$, then $d^+(a_4) \geq d^+(a_3) + 2$, also a contradiction. Let $f \in V(a_4) - \{a_4\}$. Since $N^-(a_4) \leadsto N^+(a_3)$ and $f \in N^+(a_3)$, f has outer neighbors only in $N^+(a_4) - \{x_1, x_2, x_3\}$, a contradiction to $i_g(D) \leq 1$.

Secondly, let $a_2 \to a_4$. As above, we observe that $a_4 \leadsto (N^+(q) - V(C))$. If especially $V(q) \neq V(a_3)$, then $a_4 \leadsto q$ and thus

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(q) \cup \{q\} - (V(C) \cup (V(a_4) - \{a_4\}))) \cup \{v, a_1\} \quad \text{and} \\ N^+(q) &= N^+(q) - V(C). \end{aligned}$$

This implies

$$d^+(a_4) \geq \begin{cases} d^+(q) + 1, & \text{if} \quad |V(a_4)| = 3 \\ d^+(q) + 2, & \text{if} \quad |V(a_4)| \leq 2 \end{cases}.$$

The first case is a contradiction to Remark 2.3, and the second case is a contradiction to $i_g(D) \leq 1$. Analogously, we arrive at a contradiction, if $V(q) = V(a_3)$ and $a_4 \to q$.

Let $A \subseteq H$ be the set of vertices having an inner neighbor in H. Then, it remains to treat the case that $V(q) = V(a_3)$ for all $q \in A$, $A \to a_4$ ($|A| \leq 2$) and $2 \leq |H| \leq 4$. If $B = H - A$, then we conclude that $B \subseteq V(a_2)$, because otherwise, if $p \in B - V(a_2)$ and $q \in A$, then $a_4a_1a_2pqa_4$ is a 5-cycle, a contradiction.

If $|H| = 2$, then $d^+(v) = d^+(a_1) + 1 = 5$. Since $a_4 \to (V(v) \cup (Q \cap S) \cup \{a_1\})$ and thus $d^+(a_4) \geq 5$, this implies that $d^+(a_4) = 5$, $|V(v)| = 1$, $|Q \cap S| = 3$ and $H \to a_4$. If there is a vertex $v_1 \in Q \cap S$ such that $v_1 \to a_3$, then, as for the vertex v, it follows that $v_1 \to H \cup \{a_2\}$. Hence, we deduce that $d^+(v_1) \geq |H| + |\{v, a_1, a_2, a_3\}| = 6$, a contradiction. Thus, let $a_3 \to Q \cap S$. If there is a vertex $v_1 \in Q \cap S$ such that $v_1 \to x$ with $x \in \{p, q\}$, then $a_4a_1a_3v_1xa_4$ is a 5-cycle through e, a contradiction. If there is a vertex $v_1 \in Q \cap S$ such that $v_1 \to a_2$, then $a_4a_1a_3v_1a_2a_4$ is a 5-cycle, also a contradiction. Let $v_1, v_2 \in Q \cap S$ such that $v_1 \to v_2$. Summarizing our results, we observe that $d^-(v_2) \geq |H| + |\{a_2, a_3, a_4, v_1\}| = 6$, a contradiction.

Let $|H| = 4$, $H = \{p_1, p_2, q_1, q_2\}$ such that $p_i \to q_j$ with $i, j \in \{1, 2\}$. Then $d^+(v) = d^+(a_1) + 1 = 7$, $|V(a_2)| = |V(a_3)| = 3$ and because of Remark 2.3 $d^+(a_2) = d^-(q_1) = 6$. Since $d^-(v) = |Q| + 1 \geq 6$, we arrive at $|Q| \geq 5$. Furthermore, we see that $N^-(q_1) \supseteq \{p_1, p_2, v, a_1, a_2\}$. This implies $|N^-(q_1) \cap Q| \leq 1$, which means that $|N^+(q_1) \cap Q| \geq |Q| - 1 \geq 4$. If there exists a vertex $w \in N^+(q_1) \cap Q$ such that $w \to a_2$, then $a_4a_1q_1wa_2a_4$ is a 5-cycle, a contradiction. Therefore, we have

$$d^+(a_2) \geq |N^+(q_1) \cap Q| + |\{a_3, a_4, q_1, q_2\}| \geq 8,$$

a contradiction.

Assume now that $|H| = 3$, $H = \{p_1, p_2, q\}$ such that $p_i \to q$ for $i = 1, 2$. Then $d^+(v) = d^+(a_1) + 1 = 6$, $|V(a_2)| = 3$ and $d^+(a_2) = 5$. Since $d^-(v) = |Q| + 1 \geq 5$, we arrive at $|Q| \geq 4$. Furthermore, we see that $N^-(q) \supseteq \{p_1, p_2, v, a_1, a_2\}$. Since $d^-(q) = 5$, if $|V(q)| = 3$, and $d^-(q) \leq 6$, if $|V(q)| = 2$, we conclude that $|N^+(q) \cap Q| \geq |Q| - 1 \geq 3$. As above, we see that $a_2 \to N^+(q) \cap Q$. Therefore, we have

$$d^+(a_2) \geq |N^+(q) \cap Q| + |\{a_3, a_4, q\}| \geq 6,$$

a contradiction.

Consequently, it remains to treat the case that $|H| = 3$ and $H = \{p, q_1, q_2\}$ such that $p \to q_i$ for $i = 1, 2$. Then $d^+(v) = d^+(a_1) + 1 = 6$ and because of Lemma 2.2 and Remark 2.3 we observe that $|V(v)| \leq 2$ and $|V(D)| = 13$. Suppose that there is a vertex $x \in \{q_1, q_2\}$ such that $a_4 \to x$. This implies that $N^-(x) \supseteq \{a_1, a_2, a_4, p, v\}$. Since $|V(x)| = 3$, Remark 2.3 yields that $d^-(x) = 5$ and $x \to Q$. If $|V(v)| \geq 2$, then we conclude that $|S| \geq 5$ and thus $d^+(a_4) \geq 7$, a contradiction. Hence, let $|V(v)| = 1$ and therefore $|Q| = |V(D)| - |V(C)| - |H| - |V(v)| = 5$. If there is a vertex $y \in Q$ such that $y \to a_2$, then $a_4a_1xya_2a_4$ is a 5-cycle containing the arc e, a contradiction. Summarizing our results, we observe that $a_2 \to (Q \cup \{a_3, a_4, q_1, q_2\})$ and thus $d^+(a_2) \geq 9$, a contradiction. Hence, let $\{q_1, q_2\} \to a_4$. If $a_4 \to p$, then we define the cycle $C' = b_4b_1b_2b_3b_4 := a_4a_1pq_1a_4$. We observe that $v \to (\{b_1, b_2, b_3\} \cup (N^+(b_1) - V(C)))$, $|N^+(b_1) - V(C)| = 3$, $b_1 \to b_3$ and $b_4 \to b_2$ and as above we find a 5-cycle containing the arc $b_4b_1 = a_4a_1$, a contradiction. Hence, let $p \to a_4$. Let us take three vertices of $Q \cap S$ belonging to three different partite sets. Then, since $a_4 \rightsquigarrow N^+(a_3) - V(C)$, at least two of them have to be outer neighbors of a_3, because otherwise, there are vertices $v_1, v_2 \in Q \cap S$ such that $a_4 \to \{v_1, v_2\} \to a_3$, and thus, it follows that

$$\begin{aligned} N^+(a_4) &\supseteq (N^+(a_3) - (V(C) \cup (V(a_4) - \{a_4\}))) \cup \{v, v_1, v_2, a_1\} \quad \text{and} \\ N^+(a_3) &= (N^+(a_3) - V(C)) \cup \{a_4\}. \end{aligned}$$

This implies that

$$d^+(a_4) \geq \begin{cases} d^+(a_3) + 1, & \text{if} \quad |V(a_4)| = 3 \\ d^+(a_3) + 2, & \text{if} \quad |V(a_4)| \leq 2 \end{cases},$$

in both cases a contradiction.

Consequently, let $N^+(a_3) \cap Q \cap S \supseteq \{x, y\}$ such that $x \to y$. If $y \to a_2$, then $a_4a_1a_3ya_2a_4$ is a 5-cycle, a contradiction. Hence, we have $a_2 \to y$. If $y \to u$ with $u \in \{p, q_1, q_2\}$, then $a_4a_1a_3yua_4$ is a 5-cycle, a contradiction. Hence, let $\{p, q_1, q_2\} \to y$. Altogether, we have that $N^-(y) \supseteq \{p, q_1, q_2, x, a_2, a_3, a_4\}$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2. Let $n \geq 5$. If there are vertices $x \in H$ and $y \in Q$ such that $x \to y$, then $a_na_1xyva_4 \dots a_n$ is an $(n+1)$-cycle, a contradiction. Hence, let $Q \leadsto H$.

Subcase 3.2.1. Assume that $|H| \geq 2$. At first, let there be an arc $p \to q$ in $E(D[H])$. If $q \to a_3$, then $a_na_1pqa_3 \dots a_n$ is an $(n+1)$-cycle through the arc a_na_1, a contradiction. Altogether, we observe that $d^-(q) \geq |\{p, v, a_1, a_2, a_3\}| + |Q| - |V(q) - \{q\}| \geq |Q| + 3 = d^-(v) + 2$, a contradiction to $i_g(D) \leq 1$.

Consequently it remains to consider the case that $E(D[H]) = \emptyset$, which means that $|H| = 2$ and thus $d^+(v) = d^+(a_1) + 1 = n + 1$. According to Lemma 2.2 and Remark 2.3, we have $|V(v)| \leq 2$. If $h \in H$, then we see that $d^+(h) \leq |V(v) - \{v\}| + |\{a_3, \dots, a_n\}| \leq n - 1$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2.2. Suppose that $|H| = 1$ and $h \in H$. In this case, we observe that $d^+(v) = d^+(a_1) + 1 = n$. According to Lemma 2.2 and Remark 2.3, we have $|V(v)| \leq 2$. Since $d^+(h) \leq |V(v) - \{v\}| + |\{a_3, \dots, a_n\}| \leq n - 1$, it follows that $d^+(h) = n - 1$, $h \in V(a_2)$ and $|V(v)| = 2$. Let $q \in Q - V(h) \neq \emptyset$. Because of $H \cap Q = \emptyset$, we conclude that $Q \leadsto a_1$. If $a_2 \to q$, then $a_na_1a_2qha_4a_5 \dots a_n$ is an $(n+1)$-cycle, a contradiction. If $a_i \to q$ with $3 \leq i \leq n - 1$, then $a_na_1a_3 \dots a_iqha_{i+1} \dots a_n$ is an $(n+1)$-cycle, also a contradiction. This implies that $Q \cap S \to \{v, h, a_1, a_2, \dots, a_{n-1}\}$, which means that $d^+(p) \geq n + 1$, if $p \in Q \cap S$, a contradiction. Hence, we have $Q \cap S = \emptyset$ and thus $S = V(v)$, $n = c - 1$ and $D[V(C)]$ is a tournament. Let x be a vertex with $V(x) = \{x\}$. Obviously, we have $x \in V(C)$. If $x = a_i$ with $i \in \{3, \dots, n-1\}$, then it follows that $d^-(a_i) \geq |Q - V(h)| + |\{a_{i-1}, a_1, v, h\}| = |Q| + 3 = d^-(v) + 2$, a contradiction to $i_g(D) \leq 1$. If $|V(a_1)| = 1$, then we conclude that $d^-(a_1) \geq |Q| + |V(v)| + |\{a_n\}| = d^-(v) + 2$, a contradiction. Because of $h \in V(a_2)$, we observe that $|V(a_n)| = 1$ and at least $n-1$ of the n vertices of $V(C)$ belong to partite sets with at least two vertices. If $|V_c| = 3$, then we have $|Q| \geq |V(a_1) \cup V(a_2) \cup \dots V(a_{n-1})| - |\{a_1, a_2, \dots, a_{n-1}\}| - |H| \geq n - 1$ and $d^-(v) \geq n$. Together with Remark 2.3, this implies the contradiction

$$2n + 1 = |V(D)| = d^+(v) + d^-(v) + 2 \geq 2n + 2.$$

Hence, let $|V_c| = 2$. But now, for every $q \in Q$ we have that $q \notin V(h)$. Let there be a vertex $q \in Q$ such that $d^+_{D[Q]}(q) \geq 1$, then we see that $d^+(q) \geq d^+_{D[Q]}(q) + |\{v, h, a_1, \dots, a_{n-1}\}| - |V(q) - \{q\}| \geq n + 1$, a contradiction to $i_g(D) \leq 1$.

Subcase 3.2.3. Assume that $|H| = 0$. This yields $d^+(v) = d^+(a_1) + 1 = n - 1$. Because of $i_g(D) \leq 1$, it follows that $n - 1 \geq d^-(v) = |Q| + 1 \geq n - 2$, which means that $n - 3 \leq |Q| \leq n - 2$. As above we see that $Q \leadsto a_1$. If there is a vertex $q \in Q$ such that $a_2 \to q$, then $a_na_1a_2qva_4 \dots a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If there are vertices $q \in Q$ and $a_i \in V(C)$ with $a_i \to q$ for $3 \leq i \leq n - 2$, then $a_na_1a_3 \dots a_iqva_{i+1} \dots a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results, we observe that $Q \leadsto \{a_1, a_2, \dots, a_{n-2}, v\}$. Let L_1 be the set of vertices of $Q \cap S$ having an outer neighbor in Q. If $L_1 \neq \emptyset$ and $q_1 \in L_1$, then it follows that $d^+(q_1) \geq n$, a contradiction to $i_g(D) \leq 1$. Hence, let $L_1 = \emptyset$. Let L_2 be the set of vertices of Q having an outer neighbor in Q. Since $|Q| \geq n-3$ and $|H| \neq 0$, if $|V_c| = 3$ and $n = 5$ (cf. the beginning of the proof of this theorem), we conclude that either $L_2 \neq \emptyset$ or $Q - S = \emptyset$ and $Q \cap S$ consists of vertices of only one partite set. At first let $Q - S = \emptyset$ and let $Q \cap S = S - V(v)$ be one partite set. If $q \in Q \cap S$, then we conclude that $d^+(q) \geq n-1$, and thus $d^+(q) = n-1$ and $|Q| = |V(q)| \leq 2$. Since S consists of only two partite sets, we see that $n = c-2 \geq 6$ and thus $|Q| \geq n-3 \geq 3$, a contradiction. Hence, let $L_2 \neq \emptyset$. If $q_2 \in Q_2$ and $q_2 \to q_1$ with $q_1 \in Q$, then we arrive at

$$d^+(q_2) \geq |\{a_1, a_2, \ldots, a_{n-2}, v, q_1\}| - |V(q_2) - \{q_2\}| \geq \begin{cases} n-2, & \text{if} \quad |V(q_2)| = 3 \\ n-1, & \text{if} \quad |V(q_2)| = 2 \end{cases} . \quad (1)$$

To get no contradiction to $i_g(D) \leq 1$ or to Remark 2.3, it follows that we have equality in (1), $d^+_{D[Q]}(q_2) = 1$ and $|V(q_2) \cap Q| = 1$ for all $q_2 \in L_2$, since otherwise, if there is a vertex $q_3 \in Q - \{q_1, q_2\}$ such that $q_2 \rightsquigarrow q_3$, then we observe that $N^+(q_2) \supseteq (\{a_1, a_2, \ldots, a_{n-2}, v, q_1, q_3\} - (V(q_2) - \{q_2\})$ and the right hand side of (1) enlarges by one, a contradiction. If S consists of vertices of at least three partite sets, then, because of $R = \emptyset$ and thus $S - V(v) \subseteq Q$, we conclude that $Q \cap S$ contains vertices of at least two partite sets, a contradiction to $L_1 = \emptyset$. Consequently, it remains to treat the case that S consists of vertices of at most two partite sets.

Firstly, let S consist of vertices of one partite set. This yields $n = c-1$, $Q \cap S = \emptyset$ and Q is a tournament with $|Q| \leq 3$. But now, we see that $n-3 \leq |Q| \leq 3$, which means that $n = c-1 \leq 6$, a contradiction to $c \geq 8$.

Secondly, let S consist of vertices of two partite sets. This implies that $n \geq c-2$. To get no contradiction in (1), we deduce that $|Q \cap S| = 1$ and $q_2 \to Q \cap S$ for all $q_2 \in L_2$. Since $|V(q_2) \cap Q| = 1$, it follows that $|Q| \leq 2$, and thus $n-3 \leq |Q| \leq 2$, which means that $c-2 \leq n \leq 5$, a contradiction to $c \geq 8$.

Summarizing the investigations of Case 3, we see that there remains to consider the case that $a_{n-1} \to S$.

Case 4. There exists a vertex $v \in S$ such that $a_2 \to v$. If we consider the converse of D, then, analogously to Case 3, it remains to treat the case that $S \to a_2$.

If $C = a_n a_1 a_2 \ldots a_n$ and $v \in S$, then the following three sets play an important role in our investigations

$$H = N^+(a_1) - V(C), \qquad F = N^-(a_n) - V(C), \qquad Q = N^-(v) - V(C).$$

Summarizing the investigations in the Cases 1 - 4, we can assume in the following, usually without saying so, that

$$\{a_{n-1}, a_n\} \to S \to \{a_1, a_2\} \rightsquigarrow H \quad (2)$$

Case 5. Let $n = 4$. Because of (2), we see that $\{a_3, a_4\} \to S \to \{a_1, a_2\}$. Hence, we conclude that $N^+(a_4) \supseteq S \cup \{a_1\}$. Analogously as in Subcase 3.1, we observe that $D[V(C)]$ is a tournament.

Subcase 5.1. Let $a_1 \to a_3$. If $a_2 \to a_4$ and $v \in S$, then $a_4a_1a_3va_2a_4$ is a 5-cycle, a contradiction. Consequently, let $a_4 \to a_2$. If there are vertices $v \in S$ and $x \in F$ such that $v \to x$, then $a_4a_1a_3vxa_4$ is a 5-cycle, a contradiction. Hence, let $F \to S$. If we take vertices $v, w \in S$ such that $v \to w$, then we have $N^-(a_4) = F \cup \{a_3\}$ and $N^-(w) \supseteq F \cup \{a_3, a_4, v\}$, a contradiction to $i_g(D) \le 1$.

Subcase 5.2. Let $a_3 \to a_1$ and assume that $a_2 \to a_4$. If there are vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_4a_1xva_2a_4$ is a 5-cycle, a contradiction. Otherwise, we have $S \to H$. If we take two vertices $v, w \in S$ such that $v \to w$, then we observe that $N^+(a_1) = H \cup \{a_2\}$ and $N^+(v) \supseteq \{a_1, a_2, w\} \cup H$, a contradiction to $i_g(D) \le 1$.

Finally, let $a_4 \to a_2$. Because of Corollary 2.4, it follows that

$$\begin{aligned} c + k = |V(D)| &\ge |H| + |F| + |S| + |V(C)| - |H \cap F| \\ &\ge \frac{c+k-3}{2} - 1 + \frac{c+k-3}{2} - 1 + 4 + 4 - |H \cap F| \\ &= c + k + 3 - |H \cap F|, \end{aligned}$$

which leads to $|H \cap F| \ge 3$. Thus, $H \cap F$ contains vertices of at least two partite sets. Now, we take two vertices $u_2, u_3 \in H \cap F$ such that $u_2 \to u_3$. Then, $C' = a_4a_1u_2u_3a_4$ is a cycle through a_4a_1 such that $a_1 \to u_3$ and $u_2 \to a_4$. Analogously to Subcase 5.1 with $a_2 \to a_4$, this yields a contradiction.

Therefore, we have seen that every arc of D is contained in a 5-cycle. From now on, let us suppose that $n \ge 5$.

Case 6. Let $n \ge 5$ and assume that there exists a vertex $v \in S$ such that $v \to a_{n-2}$. If there is a vertex $a_i \in V(C)$ with $3 \le i \le n-3$ such that $a_i \to v$, then we obtain, as in Case 1, an $(n+1)$-cycle through a_na_1, a contradiction. Thus, we investigate now the case that $v \to \{a_1, a_2, \ldots, a_{n-2}\}$. If there is a vertex $h \in H$ such that $h \to v$, then $a_na_1hva_3a_4 \ldots a_n$ is an $(n+1)$-cycle through a_na_1, a contradiction. Therefore, we assume now that $v \to H$. This leads to $d^+(v) \ge d^+(a_1)$, and thus, because of $i_g(D) \le 1$, it follows that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$ or $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\}$ for some $j \in \{3, 4, \ldots, n-1\}$ and $a_j \to a_1$ or $V(a_1) = V(a_j)$.

Subcase 6.1. Assume that $a_1 \to \{a_2, a_3, \ldots, a_{n-1}\}$. If there is a vertex $h \in H$ such that $h \to a_n$, then $a_na_1a_3a_4 \ldots a_{n-1}vha_n$ is an $(n+1)$-cycle, a contradiction. Therefore, we may assume now that $a_n \to (H - V(a_n))$. If $a_{i-1} \to a_n$ for $3 \le i \le n-1$, then $a_na_1a_ia_{i+1} \ldots a_{n-1}va_2a_3 \ldots a_{i-1}a_n$ is an $(n+1)$-cycle, a contradiction. Hence, it remains to treat the case that $a_n \to a_{i-1}$ or $a_{i-1} \in V(a_n)$ for $2 \le i \le n-1$. If there is a vertex $x \in H \cap F$, then $a_na_1a_3 \ldots a_{n-1}vxa_n$ is an $(n+1)$-cycle, a contradiction. Let $R = V(D) - (H \cup F \cup S \cup V(C))$. Since $a_n \leadsto \{a_1, \ldots, a_{n-2}\}$, Corollary 2.4 leads to

$$|R| \le c + k - \left\{ \frac{c+k-3}{2} - (n-2) + \frac{c+k-3}{2} - 1 + 1 + n \right\} = 1,$$

if $|S| = 1$, $|R| \le 0$, if $|S| = 2$ and the contradiction $|R| \le -1$, if $|S| \ge 3$. Hence, it follows that $|S| \le 2$, and thus $n \ge 6$. If there are vertices $h \in H$ and $y \in F$ such that $h \to y$, then $a_na_1a_4 \ldots a_{n-1}vhya_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Consequently, let $F \leadsto H$.

Subcase 6.1.1. Suppose that $|H| \geq 2$. This implies that there are vertices $h_1, h_2 \in H$ such that $h_1 \rightsquigarrow h_2$. On the one hand, we have $d^+(v) \geq n-2+|H|$ and on the other hand, since $|S|+|R| \leq 2$, we conclude that $d^+(h_2) \leq |H-\{h_1,h_2\}|+|\{a_3,\dots,a_{n-1}\}|+|S-\{v\}|+|R| \leq |H|-2+n-3+1 = |H|+n-4$. Combining these results we arrive at $d^+(v)-d^+(h_2) \geq 2$, a contradiction to $i_g(D) \leq 1$.

Subcase 6.1.2. Let $|H| = 1$ and $h \in H$. In this case, we have

$$d^-(h) \geq |F| + |\{v, a_n, a_1, a_2\}| - |V(h) - \{h\}| \geq \begin{cases} |F|+2, & \text{if} \quad |V(h)| = 3 \\ |F|+3, & \text{if} \quad |V(h)| = 2 \end{cases},$$

whereas $d^-(a_n) \leq |F| + |\{a_{n-1}\}| = |F|+1$, which means that $d^-(h) - d^-(a_n) \geq 1$, if $|V(h)| = 3$ and $d^-(h) - d^-(a_n) \geq 2$, if $|V(h)| = 2$, in both cases a contradiction.

Subcase 6.1.3. Assume that $H = \emptyset$. This implies that $d^+(a_1) = n-2$ and $d^+(v) \geq n-2$. If there are vertices $w \in S$ and $f \in F$ such that $w \to f$, then $a_n a_1 a_3 \dots a_{n-1} w f a_n$ is an $(n+1)$-cycle, a contradiction. Hence, we have $F \to S$. Since $n-3 \leq d^-(a_n) \leq |F|+1$, we conclude that $|F| \geq n-4 \geq 2$, and thus $F \neq \emptyset$. Furthermore, we observe that

$$n-1 \geq d^-(v) \geq |F|+2 \quad \Rightarrow \quad |F| \leq n-3. \tag{3}$$

Since $H = \emptyset$, we see that $F \rightsquigarrow a_1$. If there is a vertex $f \in F$ such that $a_{n-1} \to f$, then $a_n a_1 \dots a_{n-1} f a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If there is a vertex $f \in F$ such that $a_i \to f$ with $3 \leq i \leq n-3$, then $a_n a_1 a_3 \dots a_i f v a_{i+1} \dots a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results we observe that $F \rightsquigarrow (S \cup \{a_1, a_3, a_4, \dots, a_{n-3}, a_{n-1}, a_n\})$. Let $f \in F$ with $d^-_{D[F]}(f) \leq \frac{|F|-1}{2}$. This yields

$$d^-(f) \leq d^-_{D[F]}(f) + |\{a_2, a_{n-2}\}| + |R| \leq \frac{|F|-1}{2} + 2 + |R|. \tag{4}$$

Subcase 6.1.3.1. Suppose that $d^-(f) = n-3$. In this case, the bound in (3) can be improved by $|F|+2 \leq d^-(v) \leq n-2$, which means that $|F| \leq n-4$ and thus $|F| = n-4$. Combining this with (4) we arrive at $n-3 \leq \frac{n-5}{2}+2+|R| \leq \frac{n+1}{2} \Rightarrow n \leq 7$.

Firstly let $n = 6$. Because of $|S| \leq 2$, it follows that $n \geq c-2$, and thus $c = 8$, $|S| = 2$ and $|R| = 0$. But now, with (4) yields $n-3 \leq \frac{n-5}{2}+2 = \frac{n-1}{2} \Rightarrow n \leq 5$, a contradiction.

Secondly let $n = 7$. If $|R| = 0$, then we arrive at a contradiction as above. Hence, let $|R| = 1$. Since $d^-(f) = n-3$ we conclude that $d^+(v) = n-2$ and $d^-(v) \geq |F|+2 = n-2$ and thus $d^-(v) = n-2$. If $x \in R$, then x is adjacent to v, a contradiction to $d^-(v) = d^+(v) = n-2$.

Subcase 6.1.3.2. Assume that $d^-(f) \geq n-2$. Combining (3) and (4) we see that

$$n-2 \leq \frac{n-4}{2} + 2 + |R| \leq \frac{n+2}{2} \Rightarrow n \leq 6.$$

This implies that $n = 6$ and the inequalities in the last inequality-chain have to be equalities, which especially means that $|R| = 1$ and thus $|S| = 1$. This yields the contradiction $6 = n = c-1 \geq 7$.

Subcase 6.2. Assume that $n = 5$ and there is exactly one $j \in \{3, 4\}$ such that $a_1 \to (\{a_2, a_3, a_4\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_j) = V(a_1)$. In this case, we observe that $d^+(v) \geq d^+(a_1) + 1$.

Subcase 6.2.1. Let $a_1 \to \{a_2, a_3\}$ and $a_4 \to a_1$ or $V(a_4) = V(a_1)$. If there is a vertex $h \in H$ such that $h \to a_5$, then $a_5a_1a_3a_4vha_5$ is a 6-cycle, a contradiction. Therefore, we may assume that $a_5 \to (H - V(a_5))$. If $a_2 \to a_5$, then $a_5a_1a_3a_4va_2a_5$ is a 6-cycle, a contradiction. Hence, it remains to consider the case that $a_5 \to a_2$ or $V(a_5) = V(a_2)$. Let $\{a_1, a_2\} = A \cup B$ such that $a_5 \to A$ and $B \subseteq V(a_5)$. Then $N^+(a_1) = H \cup \{a_2, a_3\}$ and $N^+(a_5) \supseteq A \cup S \cup (H - (V(a_5) - (B \cup \{a_5\})))$. This leads to

$$d^+(a_5) \geq |A| + |S| + |H| - (3 - (|B| + 1)) = d^+(a_1) + |S| - 2.$$

This implies $|S| \leq 3$ and thus $c = 8$ and $|S| = 3$. Then we see that $d^+(a_5) \geq d^+(a_1) + 1$ such that we have equality in the last inequality chain. Especially, we observe that $|V(a_5)| = 3$, a contradiction to Lemma 2.2 and Remark 2.3.

Subcase 6.2.2. Let $a_1 \to \{a_2, a_4\}$ and $a_3 \to a_1$ or $V(a_3) = V(a_1)$. Since $N^+(v) = H \cup \{a_1, a_2, a_3\}$, we observe that $R = V(D) - (H \cup Q \cup V(v) \cup V(C)) = \emptyset$. If $a_3 \to a_5$, then $a_5a_1a_4va_2a_3a_5$ is a 6-cycle, a contradiction. If there exists a vertex $h \in H$ such that $h \to a_5$ and if $q \in Q \cap S \neq \emptyset$, then $a_5a_1a_4qvha_5$ is a 6-cycle, a contradiction. Let $A \cup B = \{a_1, a_3\}$ such that $a_5 \to A$ and $B \subseteq V(a_5)$, then it follows that $N^+(a_1) = H \cup \{a_2, a_4\}$ and $N^+(a_5) \supseteq S \cup A \cup (H - (V(a_5) - (B \cup \{a_5\})))$, and thus, we have

$$d^+(a_5) \geq |A| + |H| + |S| - (3 - (|B| + 1)) = d^+(a_1) + |S| - 2.$$

This implies $|S| \leq 3$ and thus $c = 8$ and $|S| = 3$. Then we see that $d^+(a_5) \geq d^+(a_1) + 1$ such that we have equality in the last inequality chain. Especially, we observe that $|V(a_5)| = 3$, because of Lemma 2.2 and Remark 2.3 a contradiction.

Subcase 6.3. Suppose that $n \geq 6$ and there exists exactly one $j \in \{3, \ldots, n-1\}$ such that $a_1 \to (\{a_2, a_3, \ldots, a_{n-1}\} - \{a_j\})$ and $a_j \to a_1$ or $V(a_1) = V(a_j)$. In this case, we observe that $d^+(v) \geq d^+(a_1) + 1$ and thus $d^+(v) = d^+(a_1) + 1$. Since $Q \to v \to H$, it follows that $Q \cap H = \emptyset$. If $R = V(D) - (H \cup Q \cup V(v) \cup V(C))$, then obviously $R = \emptyset$. If there are vertices $x \in H$ and $y \in Q$ such that $x \to y$, then $a_na_1xyva_4 \ldots a_n$ is an $(n+1)$-cycle through e, a contradiction. Summarizing our results, we see that

$$(Q \cup \{a_1, a_2, v\}) \leadsto H.$$

Subcase 6.3.1. Let $|H| \geq 2$. If there are vertices $h_1, h_2 \in H$ such that $h_1 \to h_2$, then it follows that $a_3 \leadsto h_2$, since otherwise $a_na_1h_1h_2a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Hence we have

$$\begin{aligned} d^-(h_2) &\geq |Q| + |\{v, h_1, a_1, a_2, a_3\}| - |V(h_2) - \{h_2\}| \\ &\geq \begin{cases} |Q| + 3 = d^-(v) + 1, & \text{if} \quad |V(h_2)| = 3 \\ |Q| + 4 = d^-(v) + 2, & \text{if} \quad |V(h_2)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction, either to $i_g(D) \leq 1$ or to Remark 2.3.

Consequently it remains to consider the case that $E(D[H]) = \emptyset$, which means that $H = \{h_1, h_2\}$ such that $h_1 \in V(h_2)$. If there are vertices $a_i \in V(C)$ with $i \in \{3, 4, \ldots, n\}$ and $h \in H$ such that $a_i \rightsquigarrow h$, then analogously as above we arrive at a contradiction. Hence let $H \rightarrow \{a_3, a_4, \ldots, a_n\}$. This yields that $a_n a_1 h_1 a_4 \ldots a_{n-1} v h_2 a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction.

Subcase 6.3.2. Assume that $|H| = 1$ and $h \in H$. If there is a vertex $a_i \in N^+(h)$ with $3 \leq i \leq n$, then we conclude that $(Q - V(h)) \rightsquigarrow a_{i-2}$, since otherwise, if $q \in Q - V(h)$ such that $a_{i-2} \rightarrow q$, then $a_n a_1 \ldots a_{i-2} q h a_i \ldots a_n$ is an $(n+1)$-cycle, a contradiction. If $N^+(h) \cap V(C) = \{a_{i_1}, a_{i_2}, \ldots, a_{i_g}\}$, then we define $M = \{a_{i_1-2}, a_{i_2-2}, \ldots, a_{i_g-2}\}$. Furthermore we observe that $d^+(v) = n - 1 = d^+(a_1) + 1$. According to Remark 2.3, we have $|V(v)| \leq 2$. Because of $|Q| = d^-(v) - 2 \geq n - 4 \geq 2$, we see that there are vertices $q_1, q_2 \in Q$ such that $q_1 \rightsquigarrow q_2$.

Firstly, let $q_1 \notin V(h)$. This implies that

$$|N^+(h)| \leq |M| + |V(v) - \{v\}| \leq |M| + 1 \tag{5}$$

and

$$\begin{aligned} |N^+(q_1)| &\geq |M| + |\{q_2, v, h\}| - |V(q_1) - \{q_1\}| \\ &\geq \begin{cases} d^+(h), & \text{if } \ |V(q_1)| = 3 \\ d^+(h) + 1, & \text{if } \ |V(q_1)| = 2 \end{cases}. \end{aligned} \tag{6}$$

To get no contradiction, all inequalities in the inequality-chain of (5) and (6) have to be equalities, which especially means that $|V(v)| = 2$. If $a_3 \notin N^+(h)$, then, noticing that $q_1 \rightsquigarrow a_1$, we conclude that $a_1 \notin M$ and thus $N^+(q_1) \supseteq ((M \cup \{q_2, v, h, a_1\}) - (V(q_1) - \{q_1\}))$. Then similarly to (6), we arrive at a contradiction. Therefore, let $h \rightarrow a_3$. If $V(h) \neq V(a_2)$, then $a_n a_1 a_2 h a_3 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $V(h) = V(a_2)$. Let $v' \in V(v) - \{v\}$. Because of (5) and (6), it follows that $h \rightarrow v'$ and thus $a_3 \rightarrow v'$ since otherwise $a_n a_1 h v' a_3 \ldots a_n$ is an $(n+1)$-cycle through e, a contradiction. This implies $\{a_3, \ldots, a_n, h\} \rightarrow v'$ and thus $d^-(v') \geq n - 1$. Since $i_g(D) \leq 1$ we conclude that $d^-(v') = n - 1$ and $v' \rightarrow Q$. If $n \geq 7$, then $a_n a_1 h v' q v a_5 \ldots a_n$ is an $(n+1)$-cycle for any $q \in Q$, a contradiction. Hence, let $n = 6$, and thus $|S| \geq 3$ and $Q \cap S \neq \emptyset$. If there are vertices $s_1 \in Q \cap S$ and $\hat{q}_2 \in Q$ such that $s_1 \rightarrow \hat{q}_2$, then similarly as in (6), we arrive at the contradiction $d^+(s_1) \geq d^+(h) + 2$. This implies $Q \cap S$ consists of vertices of only one partite set, and thus we conclude that $c = 8$ and $D[V(C)]$ is a tournament. If there is a vertex a_i with $2 \leq i \leq 4$ such that $a_i \rightarrow a_6$, then $a_6 a_1 h v' q v a_i a_6$ is a 7-cycle for every $q \in Q$, a contradiction. This yields $d^+(a_6) \geq |\{a_1, a_2, a_3, a_4\}| + |S| \geq 7 = d^+(a_1) + 3$, a contradiction to $i_g(D) \leq 1$.

Secondly, let $q_1 \in V(h)$. If $|Q| \geq 3$, then there are vertices $q_1', q_2' \in Q$ such that $q_1' \rightsquigarrow q_2'$ and $q_1' \notin V(h)$ and as above this leads to a contradiction. Hence, let $|Q| = 2$ and thus, because of $|Q| \geq n - 4 \geq 2$, let $n = 6$. Since $c \geq 8$, we conclude that $Q \cap S \neq \emptyset$, $\{q_2\} = Q \cap S$ which implies that $c = 8$ and $D[V(C)]$ is a tournament. Furthermore we observe that

$$d^+(q_2) \geq |N^+(h) \cap V(C)| + |\{v, h\}| \geq d^+(h) + 1.$$

To get no contradiction to $i_g(D) \leq 1$, the equalities in the last inequality-chain and in (5) have to be equalities, which means that $|V(v)| = 2$, $h \to (V(v) - \{v\})$, and because of $q_2 \to \{a_1, a_2\}$, similarly as above it follows that $h \to \{a_3, a_4\}$, and thus $V(h) = \{h, a_2, q_1\}$. Let $v' \in V(v) - \{v\}$. If $v' \to a_3$, then $a_6a_1hv'a_3a_4a_5a_6$ is a 7-cycle, a contradiction. Consequently, we have $a_3 \to v'$ and analogously as in Case 2, we arrive at $\{a_3, a_4, a_5, a_6, h\} \to v'$. Since $d^+(a_1) = 4$, this implies that $d^-(v') = 5$ and $v' \to Q$. If $h \to a_6$, then either $a_6a_1a_3a_4a_5q_2ha_6$ or $a_6a_1a_4a_5q_2vha_6$ is a 7-cycle, a contradiction. It follows that $a_6 \to \{h, a_1, v, v', q_2\}$ and thus $a_3 \to a_6$. But now $a_6a_1hv'q_2a_2a_3a_6$ is a 7-cycle, a contradiction.

Subcase 6.3.3. Suppose that $|H| = 0$. If $a_1 \to a_i$ for some $i \in \{3, \dots, n-1\}$ and $a_{i-1} \to a_n$, then $a_na_1a_ia_{i+1} \dots a_{n-1}va_2a_3 \dots a_{i-1}a_n$ is an $(n+1)$-cycle, a contradiction. Let $N^+(a_1) = \{a_{i_1}, \dots, a_{i_{n-3}}\}$ and $A \cup B = \{a_{i_1-1}, \dots, a_{i_{n-3}-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. Then $|B| \leq 2$, $|S| \geq |B| + 1$, $N^+(a_n) \supseteq A \cup S$ and thus

$$d^+(a_n) \geq |A| + |S| = d^+(a_1) - |B| + |S| \geq d^+(a_1) + 1, \tag{7}$$

which means that $|S| = 1$, if $|B| = 0$, $|S| = 2$, if $|B| = 1$, and $|S| = 3$, if $|B| = 2$. According to Remark 2.3, the combination $|B| = 2$ and $d^+(a_n) \geq d^+(a_1) + 1$ is impossible. Hence let $|B| \leq 1$ and $|S| = |B| + 1 \leq 2$.

Since $|H| = 0$, we conclude that $d^+(a_1) = n - 3$, $d^+(v) = n - 2$ and $1 \leq n - 5 \leq |Q| = d^-(v) - 2 \leq n - 4$.

Firstly, let $|Q| = 1$. In this case we have $d^+(v) = n - 2 \geq 4$ and $d^-(v) = 3$ which implies that $n = 6 \leq c - 2$. Hence, we see that $|S| \geq 2$ and (7) yields $|S| = 2$, $Q = S - \{v\}$ and $D[V(C)]$ is a tournament, which means that $|B| = 0$, a contradiction to $|S| = 2$.

Secondly, let $|Q| = 2$ and $|V_c| = 3$. Then $d^+(v) = n - 2$ and $d^-(v) = 4$ and thus $n = 6$ or $n = 7$. If $n = 6 \leq c - 2$, then we conclude that $|S| \geq 2$ and (7) yields that $|S| = 2$ and $D[V(C)]$ is a tournament, which means that $|B| = 0$, a contradiction to $|S| = 2$. Consequently, let $n = 7$. In this case we have $d^+(v) = 5$ and $d^-(v) = 4$ and Remark 2.3 yields that $|V(v)| = 2$. Since $|S| \leq 2$ and $c \geq 8$ we obtain that $|S| = 2$, $c = 8 = n + 1$ and $D[V(C)]$ is a tournament and thus $|B| = 0$, also a contradiction to $|S| = 2$.

Thirdly, let $|Q| \geq 3$ or $|Q| = 2$ and $|V_c| = 2$. This implies that there are vertices $q_1, q_2 \in Q$ such that $q_1 \to q_2$. Because of (7), we have $N^+(a_n) \cap (Q - S) = \emptyset$. Let $q \in Q$ be arbitrary. Since $H = \emptyset$ we conclude that $q \leadsto a_1$. If $a_2 \to q$, then $a_na_1a_2qva_4 \dots a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction.

Assume that $a_1 \to a_3$. If $a_i \to q$ with $3 \leq i \leq n-3$, then $a_na_1a_3 \dots a_iqva_{i+1} \dots a_n$ is an $(n+1)$-cycle, a contradiction. Altogether, we see that $q_1 \leadsto \{v, a_1, \dots, a_{n-3}, a_n, q_2\}$, if $q_1 \in Q - S$ and $q_1 \to \{v, a_1, \dots, a_{n-3}, q_2\}$, if $q_1 \in Q \cap S$. It follows that

$$d^+(q_1) \geq \begin{cases} n - 1 = d^+(a_1) + 2, & \text{if} \quad |V(q_1)| = 2 \\ n - 2 = d^+(a_1) + 1, & \text{if} \quad |V(q_1)| = 3 \end{cases},$$

if $q_1 \in Q - S$ and $d^+(q_1) \geq n - 1$, if $q_1 \in Q \cap S$, in all cases a contradiction either to $i_g(D) \leq 1$ or to Remark 2.3.

Consequently, it remains to consider the case that $a_3 \to a_1$ or $V(a_3) = V(a_1)$ and $a_1 \to \{a_2, a_4, \dots, a_{n-1}\}$. If $n = 6$, then we deduce that $|S| = 2$ and $|B| = 0$, a contradiction to (7). Consequently, let $n \geq 7$. If $a_i \to q_1$ for $i \in \{4, \dots, n-3\}$, then $a_n a_1 a_4 \dots a_i q_1 q_2 v a_{i+1} \dots a_n$ is an $(n+1)$-cycle containing the arc $a_n a_1$, a contradiction. At first let $q_1 \in Q \cap S$. This implies that $q_1 \to \{v, a_1, a_2, a_3, \dots, a_{n-3}, q_2\}$ and thus $d^+(q_1) \geq n-1 = d^+(a_1)+2$, a contradiction to $i_g(D) \leq 1$. Hence, we have $q_1 \in Q - S$ and $q_1 \rightsquigarrow \{v, a_1, a_2, a_4, \dots, a_{n-3}, a_n, q_2\}$, which means that

$$d^+(q_1) \geq \begin{cases} n-2, & \text{if} \quad |V(q_1)| = 2 \\ n-3, & \text{if} \quad |V(q_1)| = 3 \end{cases}.$$

To get no contradiction to $i_g(D) \leq 1$, it has to be equality. This implies that $V(q_1) \neq V(a_{n-2})$ and $V(q_1) \neq V(a_{n-1})$ and $V(q_1) \neq V(a_3)$ and thus $\{a_3, a_{n-2}, a_{n-1}\} \to q_1$. The inequality-chain (7) yields that $|V(a_n)| \leq 2$. If $V(q_1) \neq V(a_n)$, then $a_n a_1 \dots a_{n-1} q_1 a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $V(q_1) = V(a_n)$ and thus $a_4 \notin V(q_1)$. This implies that $a_n a_1 a_2 a_3 q_1 a_4 \dots a_n$ is an $(n+1)$-cycle through e, a contradiction.

Summarizing the investigations of Case 6, we see that there remains to treat the case that $a_{n-2} \to S$.

Case 7. Let $n = 5$. If we consider the cycle $C^{-1} = a_1 a_5 a_4 a_3 a_2 a_1 = b_5 b_1 b_2 b_3 b_4 b_5$ in the converse D^{-1} of D, then $\{b_4, b_5\} \to S \to \{b_1, b_2, b_3\}$. Since this is exactly the situation of Case 6, there exists in D^{-1} a 6-cycle, containing the arc $b_5 b_1 = a_1 a_5$, and hence there exists in D a 6-cycle through $a_5 a_1$.

Case 8. Let $n \geq 6$. Assume that there exists a vertex $v \in S$ such that $a_3 \to v$. If we consider the converse of D, then in view of Case 6, it remains to consider the case that $S \to a_3$.

Case 9. Let $c > n \geq 6$. If there are vertices $v \in S$ and $x \in H$ such that $x \to v$, then $a_n a_1 x v a_3 a_4 \dots a_n$ is an $(n+1)$-cycle through e, a contradiction. Consequently, we have $S \to H$. If there is a vertex $x \in H$ such that $x \to a_n$, then $a_n a_1 a_2 \dots a_{n-2} v x a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results, we see that $(S \cup \{a_1, a_2, a_n\}) \rightsquigarrow H$. If $a_1 \to a_i$ with $3 \leq i \leq n-1$ and $a_{i-1} \to a_n$, then $a_n a_1 a_i \dots a_{n-1} v a_2 \dots a_{i-1} a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Let $N = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$ be exactly the subset of $V(C) - \{a_2\}$ such that $a_1 \to N$. Then we define $A \cup B = \{a_{i_1-1}, a_{i_2-1}, \dots, a_{i_k-1}\}$ such that $a_n \to A$ and $B \subseteq V(a_n)$. Obviously $|B| \leq 2$. Since $a_n \to (H - V(a_n))$, we deduce that $N^+(a_1) = \{a_2\} \cup N \cup H$ and $N^+(a_n) \supseteq \{a_1\} \cup A \cup S \cup (H - (V(a_n) - (B \cup \{a_n\})))$, and thus

$$d^+(a_n) \geq \begin{cases} |A| + |S| + 1 + |H| - (3 - (|B|+1)) = d^+(a_1) + |S| - 2, & \text{if} \quad |V(a_n)| = 3 \\ |A| + |S| + 1 + |H| - (2 - (|B|+1)) = d^+(a_1) + |S| - 1, & \text{if} \quad |V(a_n)| \leq 2 \end{cases}. \tag{8}$$

This implies that $|S| = 1$ or $|S| = 2$ and thus $|B| \leq 1$. Let $R_2 = V(D) - (H \cup F \cup S \cup V(C))$. Since $F \to a_n \rightsquigarrow H$, it follows that $H \cap F = \emptyset$. If there are vertices $\tilde{v} \in S$ and $f \in F$ such that $\tilde{v} \to f$, then $a_n a_1 \dots a_{n-2} \tilde{v} f a_n$ is an $(n+1)$-cycle, a contradiction.

Hence, let $F \to S$. Because of $F \cap H = \emptyset$, we observe that $F \leadsto a_1$. If there is a vertex $f \in F$ such that $a_{n-1} \to f$, then $a_n a_1 \ldots a_{n-1} f a_n$ is an $(n+1)$-cycle through e, a contradiction. Let $f \in F$ be arbitrary. If there is an index $i \in \{3, 4, \ldots, n-2\}$ such that $a_1 \to a_i$ and $a_{i-1} \to f$, then $a_n a_1 a_i \ldots a_{n-2} v a_2 \ldots a_{i-1} f a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If $a_1 \to a_{n-1}$ and $a_{n-2} \to f$, then $a_n a_1 a_{n-1} v a_3 \ldots a_{n-2} f a_n$ is an $(n+1)$-cycle, also a contradiction. Summarizing our results, we observe that

$$F \leadsto (S \cup A \cup B \cup \{a_1, a_n, a_{n-1}\}). \tag{9}$$

Subcase 9.1. Assume that there is a vertex $v \in S$ such that $v \to a_{n-3}$. As in Case 1, we see that $v \to \{a_1, a_2, \ldots, a_{n-3}\}$.

Subcase 9.1.1. Let $H = \emptyset$. If there is a vertex $f \in F$, then (9) implies

$$\begin{aligned} d^+(f) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| - |V(f) - \{f\}| \\ &\geq \begin{cases} |N| + 1 + |S| = d^+(a_1) + |S| \geq d^+(a_1) + 1, & \text{if } |V(f)| = 3 \\ |N| + 2 + |S| = d^+(a_1) + 1 + |S| \geq d^+(a_1) + 2, & \text{if } |V(f)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$. Hence, it remains to consider the case that $F = \emptyset$. According to (8), we have

$$d^+(a_n) \geq |A| + |S| + 1 \geq |A| + |S| + |B| = d^+(a_1) - 1 + |S|,$$

which means that there remain to treat the two following cases:

i) $|S| = 2$, $d^+(a_n) = d^+(a_1) + 1$, $|B| = 1$, $n = c - 1$, $|V(v)| = 1$ and $|V(a_n)| \leq 2$. If $|V_c| = 3$, then we have $|V(a_1)| \geq 2$.

ii) $|S| = 1$ and thus $|B| = 0$, $n = c - 1$, $D[V(C)]$ is a tournament, $d^+(a_n) = d^+(a_1) + 1$ and $|V(a_n)| \leq 2$. If $|V_c| = 3$, then we have $|V(a_1)| \geq 2$.

Let $a_1' \in V(a_1) - \{a_1\}$. If $a_1' \in V(C)$, then, because of $n = c - 1$, we conclude that $|S| \geq 2$ and $|B| = 0$ or $|B| \geq 1$ and $|S| \geq 3$, in both cases a contradiction to i) and ii). Since $F = \emptyset$, it follows that $a_n \to a_1'$, and similarly as in i) and ii) we deduce that $d^+(a_n) \geq d^+(a_1) + 2$, a contradiction to $i_g(D) \leq 1$.

Hence, let $V(a_1) = \{a_1\}$ and thus $|V_c| = 2$. We observe that

$$\begin{aligned} |V(D)| &= d^+(a_n) + d^-(a_n) + |V(a_n)| = d^+(a_1) + 1 + d^-(a_n) + |V(a_n)| \\ &\geq d^+(a_1) + d^-(a_1) + |V(a_n)| = d^+(a_1) + d^-(a_1) + 1 + |V(a_n)| - 1 \\ &= |V(D)| + |V(a_n)| - 1. \end{aligned}$$

It follows that $|V(a_n)| = 1$ and thus $|B| = 0$, which means that it remains to treat the Case ii). If $R_2 \neq \emptyset$ and $x \in R_2$, then, because of $|V(a_n)| = 1$ we have $x \notin V(a_n)$. If $x \to a_n$, then $x \in F$, a contradiction to $F = \emptyset$. If $a_n \to x$, then as in ii) we conclude that $d^+(a_n) \geq d^+(a_1) + 2$, a contradiction to $i_g(D) \leq 1$. Consequently, it remains to investigate the case that $R_2 = \emptyset$. Since the Case ii) yields that $D[V(C)]$ is a tournament and $|S| = 1$, we conclude that $k = 0$, a contradiction to the hypothesis of this theorem.

Subcase 9.1.2. Suppose that H consists of vertices of only one partite set, which means that $|H| \leq 2$.

Subcase 9.1.2.1. Let $H \subseteq V(a_n) - \{a_n\}$.

Firstly, let $|B| = 0$. This yields that $\{a_2, a_3, \dots, a_{n-1}\} \to H$, since otherwise for $l = \min\{2 \leq i \leq n-1 | \exists h \in H \text{ with } h \to a_i\}$, we have the $(n+1)$-cycle $a_n a_1 \dots a_{l-1} h a_l \dots a_n$, a contradiction. If there are vertices $h \in H$ and $f \in F$ such that $h \to f$, then $a_n a_1 a_2 \dots a_{n-2} h f a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Summarizing our results, we observe that $(S \cup F \cup \{a_1, a_2, \dots a_{n-1}\}) \rightsquigarrow H$. If $h \in H$, then, because of $|N| \geq 1$, we have $d^-(a_n) \leq |F| + n - 3$ and thus

$$d^-(h) \geq \begin{cases} |S| + d^-(a_n) \geq d^-(a_n) + 1, & \text{if} \quad |V(h)| = 3 \\ |S| + d^-(a_n) + 1 \geq d^-(a_n) + 2, & \text{if} \quad |V(h)| = 2 \end{cases},$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$.

Secondly, let $|B| = 1$ and thus $|H| = 1$, $|V(a_n)| = 3$, $|S| = 2$ and $n = c-1$. To get no contradiction using (8), we have $(Q - S) \to a_n$. If $n = 6 \leq c - 2$, then it follows that $|S| = 2$ and $D[V(C)]$ is a tournament, a contradiction to $|B| = 1$. Hence let $n \geq 7$. If there are vertices $q \in Q$ and $h \in H$ such that $h \to q$, then $a_n a_1 h q v a_4 \dots a_n$ is an $(n+1)$-cycle, a contradiction. This yields $Q \to H$. If $B \neq \{a_2\}$, then it follows that $d^-(h) \geq |Q| + |S| + |\{a_1, a_2\}| = |Q| + 4 \geq d^-(v) + 1$, a contradiction to Remark 2.3, since $|V(h)| = 3$. Consequently, it remains to consider the case that $B = \{a_2\}$, which means that $V(h) = \{a_n, a_2, h\}$, if $h \in H$, and $a_1 \to a_3$. Analogously we see that $h \to \{a_3, a_4, \dots, a_{n-1}\}$. But now $a_n a_1 a_3 \dots a_{n-2} v h a_{n-1} a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction.

Subcase 9.1.2.2. Assume that $H \cap V(a_n) = \emptyset$. It follows that

$$d^+(a_n) \geq |A| + |S| + 1 + |H| \geq |A| + |B| + |S| + |H| = d^+(a_1) - 1 + |S|,$$

and there remain to treat the same two Cases i) and ii) as in Subcase 9.1.1.

Firstly, let $F = \emptyset$. If $|V_c| = 3$, then we arrive at a contradiction following the same lines as in Subcase 9.1.1. Hence let $|V_c| = 2$. Similarly as in Subcase 9.1.1 we conclude that it is sufficient to treat the Case ii) with $|V(a_n)| = 1$, $|B| = 0$ and $|R_2| = 0$ and thus $N^-(v) = \{a_{n-2}, a_{n-1}, a_n\}$. The fact that $4 = d^-(v) + 1 \geq d^+(v) \geq |\{a_1, a_2, \dots, a_{n-3}, h\}| = n - 2$ yields $n \leq 6 \leq c - 2$ and thus $|S| \geq 2$, a contradiction to the Case ii).

Secondly, let $F \neq \emptyset$. If there is a vertex $f \in F$ such that $d^-_{D[F]}(f) \geq 3$, then there is a vertex $\tilde{f} \in F$ with $d^+_{D[F]}(\tilde{f}) \geq 2$ and (9) implies that

$$\begin{aligned} d^+(\tilde{f}) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + 2 + |S| - |V(\tilde{f}) - \{\tilde{f}\}| \\ &\geq \begin{cases} |N| + 4 \geq d^+(a_1) + 1, & \text{if} \quad |V(\tilde{f})| = 3 \\ |N| + 5 \geq d^+(a_1) + 2, & \text{if} \quad |V(\tilde{f})| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$. Hence, let $d^-_{D[F]}(f) \leq 2$ for all $f \in F$.

Suppose that there is a vertex $a_1' \in V(a_1) - \{a_1\}$. If $a_1' \in V(C)$, then the fact that $n \leq c-1$ leads to $|S| \geq 2$ and $|B| = 0$ or $|S| \geq 3$ and $|B| \geq 1$, in both cases a contradiction to the Cases i) and ii). If $a_n \to a_1'$, then as in i) and ii) we see that $d^+(a_n) \geq d^+(a_1) + 2$, a contradiction. Hence, let $a_1' \to a_n$ and thus $a_1' \in F$. It follows that $a_1' \to \{a_2, a_3, \ldots, a_n\}$ and since $F \to S$, we observe that $d^+(a_1') \geq n - 1 + |S|$. If there is a vertex $x \in R_2 - V(a_n)$, then $x \notin (F \cup V(C) \cup H)$ and thus $a_n \to x \to a_1$ and we arrive at the contradiction $d^+(a_n) \geq d^+(a_1) + 2$. Consequently, let $R_2 \subseteq V(a_n) - \{a_n\}$, and $|V(a_n)| \leq 2$ implies that $|R_2| \leq 1$. Altogether, it follows that

$$6 \geq |H| + |R_2| + d^-_{D[F]}(a_1') + 1 \geq d^-(a_1') + 1 \geq d^+(a_1') \geq n - 1 + |S|,$$

which means that either $|S| = 1$ and $n \leq 6$ or $|S| = 2$ and $n \leq 5$, in both cases a contradiction.

Consequently, it remains to consider the case that $V(a_1) = \{a_1\}$ and thus, because of i) and ii), $|V_c| = 2$. Let $f \in F$ be an arbitrary vertex. If $|S| = 2$ (Case i)), then (9) implies that

$$d^+(f) \geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| - |V(f) - \{f\}| \geq |N| + 4 = d^-(a_1) + 2,$$

a contradiction to $i_g(D) \leq 1$. Hence, let $|S| = 1$ (Case ii)). To get no contradiction as in the case $|S| = 2$, we deduce that $|F| = 1$ and $d^+(f) = d^+(a_1) + 1$. This leads to

$$|V(D)| \geq d^+(f) + d^-(f) + 2 = d^+(a_1) + d^-(f) + 3 \geq d^+(a_1) + d^-(a_1) + 2 = |V(D)| + 1,$$

a contradiction.

Subcase 9.1.3. Assume that H contains vertices of at least two partite sets, which means that there exist two vertices $p, q \in H$ such that $p \to q$. If $q \to a_3$, then $a_n a_1 p q a_3 \ldots a_n$ is an $(n+1)$-cycle containing the arc $a_n a_1$, a contradiction. Hence, let $a_3 \rightsquigarrow q$.

Subcase 9.1.3.1. Suppose that $n \geq 7$. If there are vertices $x \in Q$ and $h \in H$ such that $h \to x$, then $a_n a_1 h x v a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. Consequently, let $Q \rightsquigarrow H$. Let $q \in H$ with $d^-_{D[H]}(q) \geq \max\{1, \lceil \frac{|H|-2}{2} \rceil\}$. It follows that

$$\begin{aligned} d^-(q) &\geq |Q| + |S| + d^-_{D[H]}(q) + |\{a_1, a_2, a_3\}| - |V(q) - \{q\}| \\ &\geq \begin{cases} |Q| + |S| + 1 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 3 \\ |Q| + |S| + 2 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 2 \end{cases} \end{aligned}$$

and $d^-(v) \leq |Q| + 3$. Summarizing these results, we arrive at

$$d^-(q) - d^-(v) \geq \begin{cases} |S| - 2 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 3 \\ |S| - 1 + d^-_{D[H]}(q), & \text{if} \quad |V(q)| = 2 \end{cases}. \tag{10}$$

If $|H| \geq 5$, then (10) yields

$$d^-(q) - d^-(v) \geq \begin{cases} 1, & \text{if} \quad |V(q)| = 3 \\ 2, & \text{if} \quad |V(q)| = 2 \end{cases},$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$. Hence, let $|H| \leq 4$.

Firstly, let $|H| = 4$. If H consists of vertices of 3 or 4 partite sets, then there is a vertex $\tilde{q} \in H$ such that $d^-_{D[H]}(\tilde{q}) \geq 2$ and (10) yields a contradiction, if we replace q by $\tilde{q}$. If H consists of vertices of only two partite sets, then it follows that $D[H]$ is a 4-cycle $h_1h_2h_3h_4h_1$ without any chord since otherwise (10) leads to a contradiction. This implies that

$$d^-(h_1) \geq |Q| + |S| + 1 + |\{a_1, a_2, a_3\}| - 1 = |Q| + |S| + 3 \quad \text{and} \quad |V(q)| = 3$$

and $d^-(v) \leq |Q| + 3$. Combining these results we arrive at $d^-(h_1) - d^-(v) \geq |S| \geq 1$ and $|V(h)| = 3$, a contradiction to Remark 2.3.

Secondly, let $|H| = 3$. If H contains vertices of 3 partite sets, then, to get no contradiction with (10), we deduce that $D[H]$ is a 3-cycle $h_1h_2h_3h_1$. If without loss of generality $h_1 \notin V(a_4)$, then we observe that $a_4 \to h_1$, since otherwise $a_na_1h_2h_3h_1a_4 \ldots a_n$ is an $(n+1)$-cycle, a contradiction. But together with (10), this leads to a contradiction to $i_g(D) \leq 1$ or to Remark 2.3. If H contains vertices of only 2 partite sets, then either there is a vertex $q \in H$ with $d^-_{D[H]}(q) \geq 2$ or there are two vertices $h_1, h_2 \in H$ such that $h_1 \in V(h_2)$ and $d^-_{D[H]}(h_1) \geq 1$. Using (10), we arrive at a contradiction in both cases.

Finally, let $|H| = 2$ with the vertices $p, q \in H$ such that $p \to q$. This implies

$$d^-(q) \geq \begin{cases} |Q| + |S| + 2, & \text{if} \quad |V(q)| = 3 \\ |Q| + |S| + 3, & \text{if} \quad |V(q)| = 2 \end{cases},$$

and thus

$$d^-(q) - d^-(v) \geq \begin{cases} |S| - 1, & \text{if} \quad |V(q)| = 3 \\ |S|, & \text{if} \quad |V(q)| = 2 \end{cases}.$$

This leads to $|S| = 1$, $n = c - 1$, $D[V(C)]$ is a tournament, $|B| = 0$ and $Q \cap S = \emptyset$. If $q \notin V(a_3)$, then it follows that $a_3 \to q$, and thus $a_4 \leadsto q$, and as above this yields a contradiction either to $i_g(D) \leq 1$ or to Remark 2.3. Hence, let $q \in V(a_3)$ and $q \to a_4$. If $p \notin V(a_2)$, then $a_na_1a_2pqa_4 \ldots a_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Consequently, we deduce that $p \in V(a_2)$ and $V(a_n) \cap H = \emptyset$. Analogously as in (8), reminding that $|B| = 0$, we arrive at

$$d^+(a_n) \geq |A| + |S| + 1 + |H| + |B| = d^+(a_1) + 1, \tag{11}$$

which implies that $d^+(a_n) = d^+(a_1) + 1$ and $|V(a_n)| \leq 2$. Since $F \to S$, it follows that $F \subseteq Q$ and thus $F \leadsto H$. If $f \in F$, then with (9), we conclude that

$$\begin{aligned} d^+(f) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| + |H| - |V(f) - \{f\}| \\ &\geq \begin{cases} |N| + 2 + |H| = d^+(a_1) + 1, & \text{if} \quad |V(f)| = 3 \\ |N| + 3 + |H| = d^+(a_1) + 2, & \text{if} \quad |V(f)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction. Consequently, it remains to consider the case that $F = \emptyset$. Since $|S| = 1$, this implies that $a_n \leadsto Q \leadsto a_1$ and, because of (11), we have

$Q \subseteq V(a_n) - \{a_n\}$, which means that $|Q| \leq 1$, and thus $d^-(v) \leq 4$. Summarizing our results, we arrive at

$$5 \geq d^+(v) \geq |\{p, q, a_1, \ldots, a_{n-3}\}| = n - 1 \;\Rightarrow\; n \leq 6,$$

a contradiction to the assumption of this subcase.

Subcase 9.1.3.2. Suppose that $n = 6 \leq c - 2$. In this case, we observe that $|S| \geq 2$. To get no contradiction to (8), it follows that $|S| = 2$, $|V(v)| = 1$, $|B| = 0$, $D[V(C)]$ is a tournament and $V(a_n) - \{a_n\} \subseteq H$. Since $F \to a_6 \rightsquigarrow H$, it follows that $H \cap F = \emptyset$. Since $|B| = 0$ and $a_6 \to a_{i-1}$, if $a_1 \to a_i$ with $2 \leq i \leq n-1$ we conclude that $|N^+(a_1) \cap V(C)| + |N^-(a_6) \cap V(C)| \leq l + 5 - l = 5$, if $|N^+(a_1) \cap V(C)| = l$, and thus

$$|R_2| \leq c + k - \left\{\frac{c+k-3}{2} + \frac{c+k-3}{2} - 5 + |S| + n\right\} = 0.$$

Summarizing the results of the Cases 1-8, we observe that $\{a_4, a_5, a_6\} \to S \to \{a_1, a_2, a_3\}$. Without loss of generality let $S = \{v, w\}$ such that $v \to w$. Since $v \to (H \cup \{w, a_1, a_2, a_3\})$ and $a_1 \to (H \cup (N^+(a_1) \cap V(C)))$, the fact that $i_g(D) \leq 1$ implies that $|N^+(a_1) \cap V(C)| \geq 3$ and thus $|N^-(a_6) \cap V(C)| \leq 2$ and $a_1 \to a_3$ or $a_1 \to a_4$. If there are vertices $h \in H$ and $f \in F$ such that $h \to f$, then $a_6a_1a_3a_4vhfa_6$ or $a_6a_1a_4a_5vhfa_6$ is a 7-cycle, a contradiction. Hence, let $F \rightsquigarrow H$. Let $p, q \in H$ such that $p \to q$. Then we see that

$$d^-(q) \geq |F| + |S| + |\{p, a_1, a_2, a_3\}| - |V(q) - \{q\}| \geq |F| + |S| + 2 = |F| + 4,$$

whereas $d^-(a_6) \leq |F| + 2$. This implies that $d^-(q) - d^-(a_6) \geq 2$, a contradiction to $i_g(D) \leq 1$.

Subcase 9.2. Assume that $a_{n-3} \to S$. Since $S \to a_3$, we conclude that $n \geq 7$. Let $v \in S$. If there is a vertex $w \in H \cap F$, then $a_na_1a_2\ldots a_{n-2}vwa_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. Hence, let $H \cap F = \emptyset$. If there are vertices $x \in H$ and $y \in F$ such that $x \to y$, then $a_na_1a_2\ldots a_{n-3}vxya_n$ is an $(n+1)$-cycle through e, a contradiction. Consequently, let $F \rightsquigarrow H$. If $f \in F$, then together with (9), we arrive at

$$\begin{aligned} d^+(f) &\geq |N| + |\{a_1, a_n, a_{n-1}\}| + |S| + |H| - |V(f) - \{f\}| \\ &\geq \begin{cases} |N| + |H| + 2 = d^+(a_1) + 1, & \text{if} \quad |V(f)| = 3 \\ |N| + |H| + 3 = d^+(a_1) + 2, & \text{if} \quad |V(f)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$. Consequently it remains to treat the case that $F = \emptyset$. If there is a vertex $x \in H$ such that $x \to a_{n-1}$, then $a_na_1a_2\ldots a_{n-3}vxa_{n-1}a_n$ is an $(n+1)$-cycle, a contradiction. Hence, let $a_{n-1} \rightsquigarrow H$. Let $h \in H$. If $a_i \to a_n$ and $h \to a_{i+1}$ for some $i \in \{3, 4, \ldots, n-2\}$, then $a_na_1ha_{i+1}\ldots a_{n-1}va_3\ldots a_ia_n$ is an $(n+1)$-cycle containing the arc e, a contradiction. If $a_2 \to a_n$ and $h \to a_3$, then $a_na_1ha_3\ldots a_{n-2}va_2a_n$ is an $(n+1)$-cycle, also a contradiction. Let $N^-(a_n) \cap V(C) = N^-(a_4) = \{a_{j_1}, a_{j_2}, \ldots, a_{j_l}\}$ and

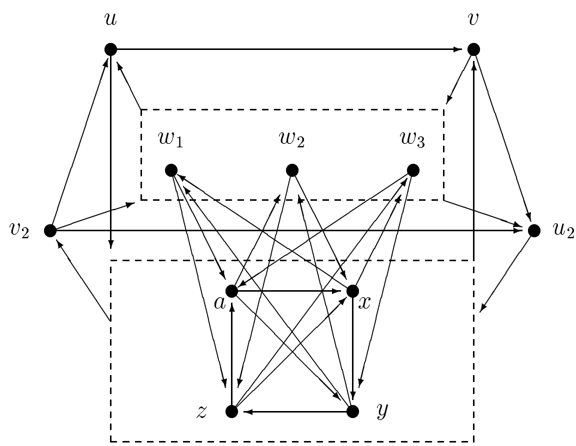

Figure 1: An almost regular 7-partite tournament with the property that the arc uv is not contained in a 4-cycle

$\tilde{N} = \{a_{j_1+1}, a_{j_2+1}, \ldots, a_{j_l+1}\}$. Summarzing our results, we observe that $(S \cup \{a_1, a_2\} \cup \tilde{N}) \rightsquigarrow H$ and thus

$$\begin{aligned} d^-(h) &\geq |\tilde{N}| + |S| + 2 - |V(h) - \{h\}| \\ &\geq \begin{cases} |\tilde{N}| + |S| \geq d^-(a_n) + 1, & \text{if} \quad |V(h)| = 3 \\ |\tilde{N}| + |S| + 1 \geq d^-(a_n) + 2, & \text{if} \quad |V(h)| = 2 \end{cases}, \end{aligned}$$

in both cases a contradiction either to Remark 2.3 or to $i_g(D) \leq 1$. Hence, let $H = \emptyset$. This leads to a contradiction analogously as in Subcase 9.1.1.

This completes the proof of this theorem. □

Combining this result with the Theorems 1.5 and 1.6 we arrive at the following corollary.

Corollary 3.2 *If D is an almost regular c-partite tournament and $e \in E(D)$ is an arbitrary arc of D, then the following holds.*

a) If $c \geq 8$, then e is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.

b) If $c = 7$ and there are at least two vertices in every partite set, then e is contained in an n-cycle for each $n \in \{4, 5, \ldots, c\}$.

The bound $c \geq 8$ in Theorem 3.1 and Corollary 3.2 a) is best possible as the following example (cf. [10]) demonstrates.

Example 3.3 *Let* $V_1 = \{u, u_2\}$, $V_2 = \{v, v_2\}$, $V_3 = \{w_1, w_2, w_3\}$, $V_4 = \{x\}$, $V_5 = \{y\}$, $V_6 = \{z\}$, *and* $V_7 = \{a\}$ *be the partite sets of a 7-partite tournament such that* $u \to v \to u_2 \to \{a, x, y, z\} \to v_2 \to u \to \{a, x, y, z\} \to v \to V_3 \to u$, $v_2 \to u_2$, $v_2 \to V_3 \to u_2$, $w_1 \to a \to x \to y \to z \to a \to y \to w_1 \to z \to x \to w_1$, $w_2 \to z \to w_3 \to a \to w_2 \to x \to w_3 \to y \to w_2$ *(see Figure 1). The resulting 7-partite tournament is almost regular, however, the arc* uv *is not contained in a 4-cycle. Consequently, the condition* $c \geq 8$ *in Theorem 3.1 and Corollary 3.2 a) is best possible.*

A further example by Volkmann [10] shows that $c = 7$ in Corollary 3.2 b) is also best possible.

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