AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **31** (2005), Pages 107–117

The minimal polynomial of $2\cos\frac{\pi}{p}$ and its application to regular maps of large planar width

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Abstract

For any prime p and any r we present an improved upper bound on the size of the smallest regular map of valence 3 and face length p whose planar width is larger than r.

1 Introduction

Regular maps on orientable surfaces are graph embeddings with the largest possible "degree" of orientation-preserving symmetry. In this paper we focus on regular maps with "large planar neighborhood" of each vertex. In what follows we first explain the basic concepts.

A (clockwise) oriented 2-manifold without boundary will be simply called a *surface*. Any cellular decomposition of a surface $\mathcal{S}$ will be refered to as a map; the 0-cells, 1-cells and 2-cells of the manifold are the *vertices*, *edges*, and *faces* of the map. The union of all 0-cells and 1-cells forms the *underlying graph of* M. We will restrict ourselves to maps whose underlying graphs have no loops, multiple edges, and semi-edges. A *dart* is an edge endowed with a direction; $D(M)$ will denote the set of all darts of a map M.

A permutation of $D(M)$ which extends to an orientation preserving self-homeomorphism of the supporting surface, respecting incidence between vertices, darts and faces, is an automorphism of M. The collection of all automorphisms of M forms the *automorphism group* $Aut(M)$; it acts freely on $D(M)$. A map M is *regular* if the automorphism group acts regularly on $D(M)$. In a regular map, all faces are bounded by closed walks of the same length (say m) and all vertices have the same valence (say n); we then speak about a map of type $\{m,n\}$. In order to avoid trivial cases we will assume throughout that $m,n\geq 3$.

Literature on regular maps is abundant, see e.g. [6, 8] and references therein. Besides central topics such as classification of regular maps with a given graph, or a given automorphism group, or on a given surface, researchers have also studied various special classes and properties of regular maps. Our aim is to discuss regular maps M with *planar width larger than* r, that is, when every non-contractible simple closed curve on the supporting surface of M intersects the underlying graph of M in more than r points.

Given a triple m, n, r such that $1/m + 1/n < 1/2$, the existence of (an infinite number of) regular maps of type $\{m, n\}$ with planar width larger than r follows, for example, from [2, 7]. However, the proofs do not give reasonable upper bounds on the number of darts of the corresponding regular maps. Recently, it was shown in [9] that the smallest number of darts $\delta(m, n, r)$ of a regular map of type $\{m, n\}$ and of planar width larger than r satisfies

$$\delta(m, n, r) < 2^{72r(m+n)(mn)^3} . \tag{1}$$

In the last section of [9] the author stated that (1) is clearly far from the best possible bound and suggested several ways of possible improvements. (In fact, the author's intention was to show that an easily computable upper bound does exist in the first place.) The aim of this paper is to improve this bound for regular maps of type $\{p, 3\}$ for prime numbers $p \geq 11$.

Theorem 1 *For any prime number $p \geq 11$ and any $r \geq 1$ we have*

$$\delta(p, 3, r) < 2^{0.94rp^4} .$$

To achieve this goal it turns out to be of advantage to represent automorphism groups of regular maps as quotients of linear representations of triangle groups. We will briefly outline the basic facts.

It is well known that the automorphism group of each regular map M of type $\{m, n\}$ can be presented in the form

$$Aut(M) = \langle \rho, \sigma |\ \rho^n = \sigma^m = (\sigma\rho)^2 = \ldots = 1 \rangle , \tag{2}$$

where the exponents are true orders of the corresponding elements. The generators ρ and σ may be chosen by fixing a corner at a vertex v and letting ρ be the automorphism of M that clockwise cyclically permute the successive darts emanating from v and letting σ be the automorphism that clockwise cyclically permute the edges that are successive sides of the face corresponding to the fixed corner.

First, note that the regular action of $Aut(M)$ on $D(M)$ in a regular map M allows us to identify $D(M)$ with $Aut(M)$ in an obvious way. Therefore, any abstract group

$$G = \langle x, y |\ x^n = y^m = (yx)^2 = \ldots = 1 \rangle \tag{3}$$

can be viewed as the automorphism group of a regular map $M = M(G; x, y)$ whose darts are elements of G and whose edges, vertices, and faces are (left) cosets of the subgroups $\langle yx \rangle$, $\langle x \rangle$ and $\langle y \rangle$; their incidence is given by non-empty intersection. We will refer to the map $M(G;x,y)$ as to the *generic regular map* associated with the presentation (3).

When the presentation (3) contains no additional independent relations then we have a *triangle group*

$$T(2,m,n) = \langle x,y|\ x^n = y^m = (yx)^2 = 1\rangle\,.$$

The generic regular map $M(T(2,m,n);x,y)$ is the *universal tessellation* $U(m,n)$ of a simply connected surface – the sphere, the Euclidean plane, or the hyperbolic plane, depending on whether $1/2 - 1/m - 1/n$ is negative, zero, or positive.

It is rather difficult to work with triangle groups just on the basis of their presentations. If $1/m + 1/n < 1/2$ then the triangle group $T(2,m,n)$ is a group of hyperbolic isometries leaving the universal tessellation $U(m,n)$ invariant, and so there is a geometric representation of $T(2,m,n)$. It can be shown [5] that, with respect to a suitable coordinate system in the hyperboloidal model of a hyperbolic plane, the two rotations x,y can be represented by the following two matrices in $SL_3(\mathcal{Z}[\xi,\eta])$, where $\eta = 2\cos(\pi/m)$ and $\xi = 2\cos(\pi/n)$:

$$X = \begin{pmatrix} -1 & -\xi & 0 \\ \xi & \xi^2-1 & 0 \\ \eta & \eta\xi & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} \eta^2-1 & 0 & \eta \\ \xi & 1 & 0 \\ -\eta & 0 & -1 \end{pmatrix}. \tag{4}$$

The assignment $x \mapsto X,\ y \mapsto Y$ induces a faithful representation of $T(2,m,n)$ in $SL_3(\mathcal{Z}[\xi,\eta])$.

The proof of the bound (1) in [9] is based on reducing the above representation to a homomorphism $T(2,m,n) \to SL_3(\mathcal{Z}_k[\xi,\eta])$ for a suitable positive integer k. In order to obtain an improvement over (1) for maps of type $\{p,3\}$ we need to study in some detail the minimal polynomials of $\xi = 2\cos(\pi/n)$; this is done in Section 3 and 4. The results obtained there are then applied in Section 5 to establish an improved bound on $\delta(p,3,r)$ for prime numbers $p \geq 11$.

2 The minimal polynomial of $2\cos\frac{\pi}{p}$

For any integer $k \geq 3$ we denote by k^+ the set of all primitive roots of unity of the form $\cos(2\pi d/k) + i\sin(2\pi d/k)$ where $1 \leq d < k/2$ and $(d,k) = 1$, and let

$$\Phi_k(x) = \prod_{\omega\in k^+} (x - (\omega + \omega^{-1}))\,.$$

The polynomial $\Phi_k(x)$ is monic and irreducible of degree $\varphi(k)/2$, with integer coefficients, whose roots have the form $2\cos(2\pi d/k)$, cf. [4]. The function $\varphi(k)$ is the *Euler function* giving the number of positive integers smaller than k and coprime with k. If we take $d = 1$ and $k = 2n$ we obtain $\Phi_{2n}(x)$, the minimal polynomial of $2\cos(\pi/n)$. We will compute this minimal polynomial using *modified Chebyshev polynomials*

$$P_n(x) = 2\cos(n \arccos \frac{x}{2}) \ ,$$

for positive integers n and real $x \in \langle -2, 2 \rangle$. The degree of $P_n(x)$ is n and the leading coefficient is 1. In particular, for $n \geq 1$ (using $P_1(x) = x$) we have

$$P_{n+1}(x) + P_{n-1}(x) = xP_n(x) \ . \tag{5}$$

It can be shown that for all $k > 0$ we have

$$P_k(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} b_j x^{k-2j} \quad \text{where} \quad b_j = (-1)^j \frac{k}{k-j} \binom{k-j}{j} . \tag{6}$$

The following Theorem shows the relation between polynomials $\Phi_k(x)$ and $P_k(x)$, cf. [3].

Theorem 2 *Let $\Phi_n(x)$ be the minimal polynomial of $2\cos(2\pi/n)$ and let $P_s(x)$ denote the s-th modified Chebyshev polynomial.*
a) If $n = 2s + 1$ is odd, then

$$P_{s+1}(x) - P_s(x) = \prod_{d|n} \Phi_d(x) \, , \tag{7}$$

and
b) if $n = 2s$ is even, then

$$P_{s+1}(x) - P_{s-1}(x) = \prod_{d|n} \Phi_d(x) \, . \tag{8}$$

For n odd, the polynomials $\Phi_n(x)$ and $\Phi_{2n}(x)$ are very similar. They differ only in the sign of every other coefficient. This is explained in the next Theorem and in its proof.

Theorem 3 *Let n be odd and let $n \geq 3$ (so that $\varphi(n) = \varphi(2n)$) and let $\Phi_n(x) = \sum_{r=0}^{\varphi(n)/2} c_r x^r$, $\Phi_{2n}(x) = \sum_{r=0}^{\varphi(n)/2} c'_r x^r$. Then*

$$c'_r = (-1)^{\frac{\varphi(n)}{2} - r} c_r \, . \tag{9}$$

Proof. If $n = 2s+1$ is odd, the $\varphi(n)/2$ roots of the polynomial $\Phi_n(x)$ are $2\cos(2\pi k/n)$ for $1 \leq k \leq s$ and $(k, n) = 1$. The roots of the polynomial $\Phi_{2n}(x)$ are $2\cos(2\pi k'/2n)$ $= 2\cos(\pi k'/n)$ for $1 \leq k' \leq n$ and $(k', 2n) = 1$; their number is $\varphi(2n)/2 = \varphi(n)/2$. It is easy to prove that for every k, $1 \leq k \leq s$, we have $2\cos(\pi k'/n) = -2\cos(2\pi k/n)$, where $k' = n - 2k$ and $1 \leq k' \leq n$. We have to show that for every k the following holds: if $(n, k) = 1$ then $(2n, k') = 1$. Clearly, $(2n, k') = (2n, n - 2k) = (4k, n - 2k)$. If n is odd then $n - 2k$ is also odd, so $(n - 2k, 4) = 1$ and $(n, k) = (n - 2k, k) = 1$. Therefore $(4k, n - 2k) = 1$ and $(2n, k') = 1$. In other words, the roots of $\Phi_n(x)$ and

$\Phi_{2n}(x)$ have opposite signs and $c'_r = (-1)^{\frac{\varphi(n)}{2}-r} c_r$. □

An explicit formula for the minimal polynomial $\Phi_p(x)$ for prime numbers $p > 2$ was presented in [1] without proof. We reproduce the formula here with a simple proof.

Theorem 4 *The minimal polynomial $\Phi_p(x)$ of $2\cos(2\pi/p)$ where $p = 2s+1$ is an odd prime has the form*

$$\Phi_p(x) = \sum_{r=0}^{s} (-1)^{\lfloor \frac{s-r}{2} \rfloor} \binom{\lfloor \frac{s+r}{2} \rfloor}{r} x^r .$$

Proof. For the prime number $p = 2s+1$ we obtain from the equation (7) that

$$\Phi_p(x) = \frac{P_{s+1}(x) - P_s(x)}{\Phi_1(x)} . \tag{10}$$

With help of (5) it is a matter of routine to show that

$$P_{s+1}(x) - P_s(x) = (x-2)(1 + \sum_{i=1}^{s} P_i(x)) . \tag{11}$$

If we realize that $\Phi_1(x) = x-2$ and we substitute (11) onto (10) then we can observe that $\Phi_p(x) = 1 + \sum_{i=1}^{s} P_i(x)$. Every modified Chebyshev polynomial can be written in the form (cf. (6))

$$\Phi_p(x) = 1 + \sum_{i=1}^{s} \sum_{j=0}^{\lfloor i/2 \rfloor} (-1)^j \frac{i}{i-j} \binom{i-j}{j} x^{i-2j} . \tag{12}$$

We have to find the coefficient at x^r, so we put $r = i - 2j$ and replace j with $\frac{i-r}{2}$. This enables us to rewrite (12) in the form

$$\Phi_p(x) = 1 + \sum_{r} \sum_{i} (-1)^{\frac{i-r}{2}} \frac{2i}{i+r} \binom{\frac{i+r}{2}}{\frac{i-r}{2}} x^r ,$$

where $0 \le r \le s$ and the second sum ranges over all i such that $r \le i \le s$, $i \equiv r \bmod 2$; moreover, if $r = 0$ then $i \ne 0$. Now we have to evaluate the coefficient $c_r = \sum_i (-1)^{\frac{i-r}{2}} \frac{2i}{i+r} \binom{\frac{i+r}{2}}{\frac{i-r}{2}}$ at x^r. To do this we use the substitution $u = \frac{i-r}{2}$, so that $0 \le u \le \lfloor \frac{s-r}{2} \rfloor$ and if $r = 0$ we have $u \ne 0$:

$$c_r = \sum_u (-1)^u \frac{2u+r}{u+r} \binom{u+r}{u} = \sum_u (-1)^u \binom{u+r}{u} + (-1)^u \binom{u+r-1}{u-1} .$$

For $r \ge 1$ there is a telescopic effect in the sum and we easily obtain

$$c_r = (-1)^{\lfloor \frac{s-r}{2} \rfloor} \binom{\lfloor \frac{s+r}{2} \rfloor}{r} .$$

For $r = 0$ we have to include the $+1$ term, and therefore

$$c_0 = 1 + \sum_{u=1}^{\lfloor \frac{s}{2} \rfloor} (-1)^u 2 = (-1)^{\lfloor \frac{s}{2} \rfloor} = (-1)^{\lfloor \frac{s-0}{2} \rfloor} \binom{\lfloor \frac{s+0}{2} \rfloor}{0} .$$

It follows that the minimal polynomial of $2\cos(2\pi/p)$ is

$$\Phi_p(x) = \sum_{r=0}^{s} (-1)^{\lfloor \frac{s-r}{2} \rfloor} \binom{\lfloor \frac{s+r}{2} \rfloor}{r} x^r .$$

□

Corollary 1 *The minimal polynomial $\Phi_{2p}(x)$ of $2\cos(\pi/p)$ where $p = 2s+1$ is an odd prime has the form*

$$\Phi_{2p}(x) = \sum_{r=0}^{s} (-1)^{\lfloor \frac{s-r+1}{2} \rfloor} \binom{\lfloor \frac{s+r}{2} \rfloor}{r} x^r . \tag{13}$$

Proof. Since $\frac{\varphi(p)}{2} = s$ and we know the coefficients of the polynomial $\Phi_p(x)$ (see Theorem 4) we can evaluate the coefficients of the polynomial $\Phi_{2p}(x)$:

$$c'_r = (-1)^{s-r} (-1)^{\lfloor \frac{s-r}{2} \rfloor} \binom{\lfloor \frac{s+r}{2} \rfloor}{r} .$$

We have $\lfloor \frac{s-r}{2} \rfloor \equiv \lfloor \frac{r-s+1}{2} \rfloor \bmod 2$ and $\lfloor \frac{s-r}{2} \rfloor + (s-r) \equiv \lfloor \frac{r-s+1}{2} \rfloor + (s-r) \bmod 2$, where $\lfloor \frac{r-s+1}{2} \rfloor + (s-r) = \lfloor \frac{s-r+1}{2} \rfloor$. So the coefficients of the polynomial $\Phi_{2p}(x)$ have the form

$$c'_r = (-1)^{\lfloor \frac{s-r+1}{2} \rfloor} \binom{\lfloor \frac{s+r}{2} \rfloor}{r} .$$

□

While $\Phi_{2p}(x)$ is irreducible over the integers, it may be of interest to note that over Z_p it reduces to a power of a linear polynomial.

Proposition 1 *p In $\mathcal{Z}_p[x]$, $p = 2s+1$ is prime, we have $\Phi_{2p}(x) = (x+2)^s$.*

Proof. In [1] is shown that $\Phi_p(x) = (x-2)^s = \sum_{r=0}^{s} (-1)^{s-r} \binom{s}{r} 2^{s-r} x^r$. The relation between coefficients of $\Phi_p(x)$ and $\Phi_{2p}(x)$ established in Theorem 3 projects onto Z_p. Therefore,

$$\Phi_{2p}(x) = \sum_{r=0}^{s} (-1)^{s-r} (-1)^{s-r} \binom{s}{r} 2^{s-r} x^r = \sum_{r=0}^{s} \binom{s}{r} 2^{s-r} x^r = (x+2)^s .$$

□

3 The norm of the polynomial $\Phi_{2p}(x)$

Let $\mathcal{Z}(x)$ denote the ring of polynomials in x with integer coefficients. We define the *norm* $||g(x)||$ of a polynomial $g(x) \in \mathcal{Z}(x)$ as the largest absolute value of all its coefficients.

In what follows we estimate the norm of $\Phi_{2p}(x)$ for prime $p \geq 11$.

Theorem 5 *Let p be a prime number, $p \geq 11$, and let $\Phi_{2p}(x)$ be a minimal polynomial of $2\cos(\pi/p)$. The norm of $\Phi_{2p}(x)$ is bounded by*

$$||\Phi_{2p}(x)|| < 0.316 \cdot 1.274^p . \tag{14}$$

Proof. By (13), we have to find r for which the coefficient $c_r = \binom{\lfloor\frac{s+r}{2}\rfloor}{r}$ is the largest. We first find the maximum r for which $c_r \leq c_{r+1}$, or, equivalently,

$$\binom{\lfloor\frac{s+r}{2}\rfloor}{r} \leq \binom{\lfloor\frac{s+r+1}{2}\rfloor}{r+1} . \tag{15}$$

If $s+r = 2t$ we obtain from (15) the inequality $(3r - s + 2)/((s-r)(r+1)) \leq 0$. The solution is $0 \leq r \leq (s-2)/3$, which means that for even $s+r$ the inequality $c_r \leq c_{r+1}$ holds up to $r \leq (s-2)/3$.

For $s + r = 2t + 1$ we obtain from (15) that $c_r \leq c_{r+1}$ for every odd $s + r$. It follows that we have to determine the largest r (such that $s+r$ is even) for which $c_r \leq c_{r+2}$, that is,

$$\binom{\lfloor\frac{s+r}{2}\rfloor}{r} \leq \binom{\lfloor\frac{s+r+2}{2}\rfloor}{r+2} .$$

This is equivalent with the inequality $(5r^2+14r-s^2-2s+8)/((s-r)(r+2)(r+1)) \leq 0$, and the largest r satisfying it has value $r_0 = \lfloor\frac{-7+\sqrt{5s^2+10s+9}}{5}\rfloor$. The maximum coefficient is therefore c_{r_0+2} obtained for

$$r_0 + 2 = \lfloor\frac{3+\sqrt{5(s+1)^2+4}}{5}\rfloor , \tag{16}$$

with value

$$c_{r_0+2} = \binom{\lfloor\frac{s+r_0+2}{2}\rfloor}{r_0+2} .$$

To facilitate further analysis we set

$$n = \lfloor\frac{s+r_0+2}{2}\rfloor \quad \text{and} \quad cn = r_0 + 2 ; \qquad \text{then} \quad c_{r_0+2} = \binom{n}{cn} . \tag{17}$$

With help of the formula for r_0 and using $s = (p-1)/2$ we find that $n < 0.362p+2.086$. For approximation of $\binom{n}{cn}$ we use Stirling's formula

$$\sqrt{2\pi n}\, \left(\frac{n}{e}\right)^n < n! < \sqrt{2\pi n}\, \left(\frac{n}{e}\right)^n (1 + \frac{1}{12n-1}) ,$$

which yields

$$\binom{n}{cn} < \frac{1+\frac{1}{12n-1}}{\sqrt{2\pi nc(1-c)}\cdot c^{cn}\cdot(1-c)^{(1-c)n}}\,.$$

From (16) and (17) we have

$$c = \frac{r_0+2}{\lfloor\frac{s+r_0+2}{2}\rfloor} = \frac{\lfloor\frac{3+\sqrt{5(s+1)^2+4}}{5}\rfloor}{\lfloor\frac{\lfloor\frac{5s+3+\sqrt{5(s+1)^2+4}}{5}\rfloor}{2}\rfloor}\,.$$

We need to work with reasonable bounds on c in order to determine an upper bound for $\binom{n}{cn}$. It is easy to check that

$$\frac{2}{\sqrt{5}+1+\frac{8}{\sqrt{5s}}} < c < \frac{2}{\sqrt{5}+1-\frac{15}{s+4}}\,.$$

For $s\to\infty$ both bounds approach $2/(\sqrt{5}+1)\approx 0.62$. To keep the value of c approximately equal to 0.62 within the accuracy to two decimal places, we will find the smallest s for which the number c fulfills the conditions $0.615\le c\le 0.624$ and $0.376\le 1-c\le 0.385$. A routine verification shows that this is the case provided that $s\ge 481$. Thus for every prime $p=2s+1>963$

$$c_{r_0+2} = \binom{n}{cn} < \frac{1+\frac{1}{12n-1}}{\sqrt{2\pi n\cdot 0.615\cdot 0.376}\cdot 0.615^{0.615n}\cdot 0.376^{0.376n}}\,,$$

so that

$$c_{r_0+2} \le \frac{1+\frac{1}{12n-1}}{\sqrt{0,462n}}\cdot 1,948^n \quad \text{if} \quad n = 0.362p+2.086\,.$$

Now by substituting for n and taking $p=967$ (the nearest larger prime to 963) to the fraction at left we obtain the announced bound for the norm of the polynomial $\Phi_{2p}(x)$

$$||\Phi_{2p}(x)|| < 0.316\cdot 1.274^p\,, \tag{18}$$

for every prime $p\ge 967$.

Finally, comparing (for example, with the help of the system *Mathematica*) the norms of the polynomial $\Phi_{2p}(x)$ given by (13) with the bound (18) for primes $p\le 967$ we conclude that the bound (18) is valid for each prime $p\ge 11$. □

4 Regular maps with large planar width

Recall that a map M on a compact surface $\mathcal{S}$ of positive genus has *planar width larger than* r if every non-contractible simple closed curve on $\mathcal{S}$ intersects the underlying graph of M in more than r points. Let $1/m+1/n\le 1/2$ and let r be a positive integer. Denote by S_r the set of all non-identity elements $u\in T(2,m,n)=\langle x,y|\, x^n =$

$y^m = (yx)^2 = 1\rangle$ with the property that u can be expressed as a word of length at most r in the symbols x, y. A representation $\vartheta : T(2,m,n) \to H$ in a finite group H will be said to be r-*locally faithful* if $\vartheta(u) \neq 1_H$ for each $u \in S_r$.

The following two results were proved in [9]. The two concepts introduced above are related by Proposition 2 of [9]. In the special case of maps of type $\{p, 3\}$ this result can be reformulated as follows.

Proposition 2 *Let $r \geq 1$ and let $\vartheta : T(2,p,3) \to H$ be an epimorphism of an infinite triangle group onto a finite group H. If ϑ is an $r(p-1)$-locally faithful representation then the generic map $M(H; \vartheta(x), \vartheta(y))$ is of type $\{p, 3\}$ and has planar width larger than r.*

For any $f(x) \in \mathcal{Z}(x)$ let $w(f)$, the *width* of $f(x)$, be the number of non-zero coefficients of $f(x)$.

If $A \in SL_q(\mathcal{Z}[\xi])$ is a matrix with entries $A_{ij} = A_{ij}(\xi) \in \mathcal{Z}[\xi]$, then the width $w(A)$ and the norm $||A||$ of A are defined as the maximum of $w(A_{ij}(\xi))$ and $||A_{ij}(\xi)||$, respectively, taken over all indices $1 \leq i, j \leq q$.

Theorem 6 *Let $\mathcal{Z}[\xi] = \mathcal{Z}(x)/(h(x))$ where $h(x) \in \mathcal{Z}(x)$ is a polynomial of degree $d > 0$ with leading coefficient 1; let $s = 1 + ||h(x)||$. Assume that we have a faithful (that is injective) representation $\vartheta : T(2,m,n) \to SL_q(\mathcal{Z}[\xi])$ of the triangle group $T(2,m,n) = \langle x, y |\ x^n = y^m = (yx)^2 = 1\rangle$, where $1/m + 1/n < 1/2$. Further, let both matrices $\vartheta(x)$ and $\vartheta(y)$ have norm $\leq t$ and width $\leq w$. Then, for any positive integer r there exists an r-locally faithful representation of $T(2,m,n)$ in a finite group H such that*

$$|H| < (qwts^{d-1})^{q^2dr} \ . \tag{19}$$

As stated in Section 1, using the faithful representation (4) of $T(2,m,n)$ in $SL_3(\mathcal{Z}[\xi,\eta])$ for $1/m + 1/n < 1/2$, in [9] the author derived an upper bound on the number of darts $\delta(m,n,r)$ of the *smallest* regular map of type $\{m,n\}$ and of planar width larger than r:

$$\delta(m,n,r) < 2^{72r(m+n)(mn)^3} \ .$$

We now improve this bound for maps of type $\{p, 3\}$ for prime $p \geq 11$.

Let $m = p \geq 7$ be a prime and let $n = 3$; then $\eta = 2\cos(\pi/p)$ and $\xi = 2\cos(\pi/3) = 1$. Substituting $\xi = 1$ in (4) we obtain a faithful representation $\vartheta : T(2,p,3) \to SL_3(\mathcal{Z}[\eta])$ of the triangle group $T(2,p,3) = \langle x, y |\ x^3 = y^p = (yx)^2 = 1\rangle$, where the ring $\mathcal{Z}[\eta]$ is obtained by adjoining to $\mathcal{Z}$ the root η of the minimal polynomial $\Phi_{2p}(x)$:

$$\vartheta(x) = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ \eta & \eta & 1 \end{pmatrix} \quad \text{and} \quad \vartheta(y) = \begin{pmatrix} \eta^2 - 1 & 0 & \eta \\ 1 & 1 & 0 \\ -\eta & 0 & -1 \end{pmatrix}.$$

In order to apply Theorem 6 we need upper bounds w and t on the width and the norm of the matrices $\vartheta(x)$ and $\vartheta(y)$. The polynomials $A_{ij}(\eta) \in \vartheta(x)$ and $A'_{ij}(\eta) \in$

$\vartheta(y)$ are linear (except for $A'_{11}(\eta)$ which is quadratic), so the norm and the width of matrices $\vartheta(x)$ and $\vartheta(y)$ are $t = 1$ and $w = 2$. The polynomial $\Phi_{2p}(x)$ is of degree $d = (p-1)/2$. From (14) we have, for $p \geq 11$, $||\Phi_{2p}(x)|| < 0.316 \cdot 1.274^p$. Therefore $s = 1 + ||\Phi_{2p}(x)|| < 1 + 0.316 \cdot 1.274^p < (1/3)(4/3)^p$. Combining these facts with Proposition 2 we obtain the upper bound of number of darts of the smallest regular map of type $\{p, 3\}$, p a prime, and of planar width larger than r.

Theorem 7 *For any prime number $p \geq 11$ and any $r \geq 1$ we have*

$$\delta(p,3,r) < \left(\left(\frac{1}{3}\right)^{p-5} \cdot \left(\frac{4}{3}\right)^{p(p-3)}\right)^{9r(p-1)^2/4} . \tag{20}$$

Let us compare the bound on the number of darts $\delta(p,3,r)$ derived from (1) with our bound (20). If we substitute $m = p$ and $n = 3$ into (1) we obtain, for $p \geq 3$,

$$\delta(p,3,r) < 2^{1944r(p+3)p^3} . \tag{21}$$

On the other hand, the bound (20) can be rewritten in the form

$$\delta(p,3,r) < 2^{\frac{9}{4}\log_2\frac{1}{3}\ r(p-5)(p-1)^2 + \frac{9}{4}\log_2\frac{4}{3}\ rp(p-3)(p-1)^2} .$$

The left-hand side of the exponent is negative and the right-hand side of the exponent can be bounded by $9/4 \log_2(4/3)\ rp^4$. Therefore, from (20) we have

$$\delta(p,3,r) < 2^{0.94rp^4} .$$

We see that this result (which is Theorem 1 from the Introduction) is much better than (21) for prime $p \geq 11$.

Acknowledgement

This research was supported by the VEGA research grant No. 1/9176/02 and by the APVT grant No. 20-023302.

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(Received 5 Apr 2003)