AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **33** (2005), Pages 169–181

Vertex magic total labeling of products of cycles

Dalibor Fronček*

Department of Mathematics and Statistics
University of Minnesota Duluth, Minnesota 55812
U.S.A.
dalibor@d.umn.edu

Petr Kovář Tereza Kovářová†

Department of Mathematics and Statistics
University of Minnesota Duluth, Minnesota 55812
U.S.A.
and
Department of Mathematics and Descriptive Geometry
Technical University Ostrava, 708 33
Czech Republic
{petr.kovar,tereza.kovarova}@vsb.cz

Abstract

A vertex magic total labeling of a graph $G(V, E)$ assigns to all vertices and edges of G labels from the set $\{1, 2, \ldots, |V| + |E|\}$ so that the sum (called the weight) of the vertex label and of labels of all adjacent edges does not depend on the vertex. A generalized (s, d)-vertex antimagic total labeling of G assigns positive integers to all vertices and edges of G so that the vertex weights form an arithmetic progression s, $s + d$, $\ldots$, $s + (|V| - 1)d$.

We present a construction of vertex magic total labelings of products of cycles $C_m \times C_n$ for $m, n \geq 3$, n odd. The construction is based on a generalized (s, d)-vertex antimagic labeling of cycles in which nonconsecutive integers are used.

* Research supported by the University of Minnesota Duluth Grant 177–1009.
† Research supported by the University of Minnesota Duluth Grant 177–5963.

1 Introduction

Graph labelings of many kinds have been studied extensively over the last forty years as many of them are very useful in various applications. In general, a *labeling* of a graph $G(V, E)$ is an injection from a set S, where typically $S = V, E$, or $V \cup E$, to the set of integers, $\{1, 2, \ldots, t\}$, where $t = |V|, |E|$, or $|V| + |E|$, respectively. One of the best known applications is the use of *graceful* and *ρ-labelings* in decompositions of complete graphs. For an excellent survey of labelings, see [3]. Another important family of labelings is the family of *magic type labelings.* In these labelings, the labels are assigned in such a way that the sum of labels, taken over each vertex or each edge, is constant. Magic type labelings can be useful for instance in network addressing. For more information, see [5].

In this paper we study *vertex magic total labelings* of products of cycles. A *vertex magic total labeling* (or VMT labeling for short) of a graph $G(V, E)$ is a bijection λ from the set $V \cup E$ to the set of integers, $\{1, 2, \ldots, |V| + |E|\}$, with the property that the sum of labels of a vertex x and of all edges incident to x is equal to a certain *magic constant*, h, for every vertex x of G. This sum is called the *weight* of the vertex x and is denoted $wt(x)$. More formally, if $N(x)$ is the set of all vertices adjacent to x, then

$$wt(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy) = h$$

for all vertices $x \in V$.

A graph that allows a VMT labeling will be often called a *VMT graph.* Vertex magic total labelings of several classes of graphs have been found so far: C_n, P_n, $K_{m,n}$ iff $|m - n| \leq 1$, K_n, Petersen $P(n, k)$, prisms $C_n \times P_2$, W_n iff $n \leq 11$, F_n iff $n \leq 10$, friendship graphs, $G + H$ if $G \cup H$ is a VMT graph and $|V(G)| = |V(H)|$, unions of stars, odd number of copies of an r-regular VMT graph G if r is even or any number of copies of an r-regular VMT graph G if r is odd.

We will show that a VMT labeling exists for each product $C_m \times C_n$ (also called the $m \times n$ grid), where n is odd and $n, m \geq 3$.

Our construction consists of two steps. In the first step we find a certain labeling, called a generalized (s, d)-vertex antimagic labeling, of the cycle C_m and a generalized VMT labeling of the cycle C_n. In the second step, we "glue" copies of these cycles together to obtain the desired VMT labeling of $C_m \times C_n$. A *generalized (s, d)-antimagic labeling* (or (s, d)-VAT labeling for short) of a graph $G(V, E)$ is an injection from the set $V \cup E$ to the set of integers, $\{1, 2, \ldots, t\}$. This time the weights of the vertices have the property that they form an arithmetic progression with the starting term s and difference d. The labels are not necessarily consecutive as t can be greater than $|V| + |E|$. For an explicit formula see Definition 2.3. The generalized VMT labeling also uses non-consecutive labels.

Our main results are summarized in the following theorems.

Theorem 5.1
For each $m, n \geq 3$ and n odd, there exists a VMT labeling of $C_m \times C_n$ with the magic constant

$$h = \frac{1}{2}m(15n+1)+2.$$

Although the above theorem is more general, we prove another theorem just for products of odd cycles with a different magic constant.

Theorem 5.2
For each $m, n \geq 3$ and m, n odd, there exists a VMT labeling of $C_m \times C_n$ with the magic constant

$$h = \frac{17}{2}mn + \frac{5}{2}.$$

2 Known results and methods

As already mentioned, we will use in our constructions both the VMT and (s,d)-VAT labelings of cycles and their generalizations. Therefore we start with their formal definitions.

Definition 2.1 *Let $G(V,E)$ be a graph with vertex set V and edge set E. A mapping $\lambda : V \cup E \to \{1, 2, \ldots, |V|+|E|\}$ is called a* vertex magic total labeling of G *if there exists a constant h such that*

$$\lambda(v) + \sum_{u \in N(v)} \lambda(vu) = h$$

for every vertex v of G.

We mention here a VMT labeling for odd cycles, since it proved to be useful in constructions of VMT labelings of products of cycles. A VMT labeling of a cycle is equivalent to an edge magic total (EMT) labeling of a cycle and results on EMT labelings of cycles of any length are due to Kotzig and Rosa [4].

Let C_n have vertices v_i and edges $v_i v_{i+1}$ for $i = 0, 1 \ldots, n-1$. Subscripts are taken modulo n. For n odd, labels of vertices and edges are assigned as follows:

$$\begin{aligned} \lambda(v_i) &= 2n - i, \qquad (1) \\ \lambda(v_i v_{i+1}) &= \begin{cases} \frac{i}{2}+1 & \text{for } i \text{ even}, \\ \frac{n+i}{2}+1 & \text{for } i \text{ odd}. \end{cases} \end{aligned}$$

The magic constant is

$$h = \frac{1}{2}(5n+3).$$

Figure 2.1 shows an example of this labeling for C_7. Notice that we use consecutive integers to label the vertices, and edges are labeled so that the sums of edge labels at a vertex also form an arithmetic progression.

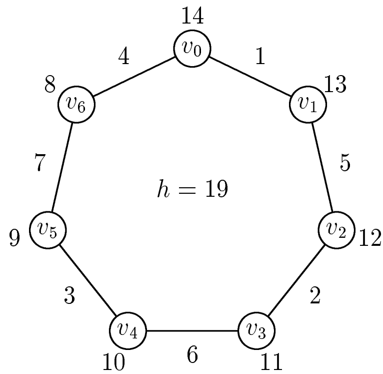

Figure 2.1: *VMT labeling of* C_7 *with the magic constant* 19.

Once a VMT labeling is known to exist, one can ask a question which values of the magic constant h can be obtained. In general for every graph $G(V, E)$ it is easy to verify the following inequality (see [5]), which gives bounds on the spectrum of the magic constant.

$$\binom{|E|+1}{2} + \binom{|V|+|E|+1}{2} \leq |V|h \leq 2\binom{|V|+|E|+1}{2} - \binom{|V|+1}{2} \qquad (2)$$

For $C_m \times C_n$, where $|V| = mn$ and $|E| = 2mn$, (2) gives

$$\frac{13}{2}mn + \frac{5}{2} \leq h \leq \frac{17}{2}mn + \frac{5}{2}.$$

Besides labelings in which all vertices have the same weight we will use labelings where the weights of the vertices form an arithmetic progression. Now we turn to (s, d)-vertex antimagic total labelings, which were introduced in [1].

Definition 2.2 *Let $G(V, E)$ be a graph with vertex set V and edge set E. Let $v = |V|$ and $e = |E|$. A mapping $\lambda : V \cup E \to \{1, 2, \ldots, v+e\}$ is called an (s, d)-*vertex antimagic total labeling of G *if the vertices $x_1, x_2, \ldots, x_v$ of G can be ordered in such a way that*

$$\lambda(x_i) + \sum_{y \in N(x_i)} \lambda(x_i y) = s + (i-1)d$$

for some constants s and d and for $i = 1, 2, \ldots, v$. It means that the weights form an arithmetic progression $\{s, s+d, \ldots, s+(v-1)d\}$.

Now we introduce a more general notion of antimagic labelings by allowing the set of labels to be distinct integers not necessarily $1, 2, \ldots, v+e$.

Definition 2.3 *Let $G(V, E)$ be a graph with vertex set V and edge set E. An injection $\lambda : V \cup E \to \mathbf{N}$ is called a* generalized (s, d)-vertex antimagic total labeling of G *if the set of sums*

$$\lambda(x) + \sum_{y \in N(x)} \lambda(xy) \quad \forall x \in V$$

forms an arithmetic progression $\{s, s+d, \ldots, s+(v-1)d\}$.

Generalized VMT labelings are sometimes called *vertex magic total injections* (VMTI) and similarly (s,d)-VAT labelings are called (s,d)-*vertex antimagic total injections.*

We can generalize the VMT labeling for regular graphs in a similar way to obtain a *generalized vertex magic total labeling.* In particular taking a VMT labeling λ with magic constant h of an r-regular graph G the labeling λ' defined as

$$\begin{aligned} \lambda'(x) &= a\lambda(x) + b \quad \forall x \in V, \\ \lambda'(xy) &= a\lambda(xy) + c \quad \forall xy \in E, \end{aligned}$$

for any $a, b, c \in \mathbf{N}$ is a generalized VMT labeling.

The magic constant is $ah + b + rc$, since the weight of any vertex $x \in V$ is

$$\begin{aligned} wt(x) &= a\lambda(x) + b + \sum_{y \in N(x)} a\lambda(xy) + c, \\ &= ah + b + rc. \end{aligned}$$

Given a vertex magic total labeling or an (s,d)-VAT labeling of any regular graph G we can construct another labeling called *dual.* The dual labeling is again a vertex magic total labeling or a generalized (s,d)-VAT labeling, respectively. Notice that a dual labeling can be found only for regular graphs.

The following theorem was proved in [5].

Theorem 2.4 *Let λ be a vertex magic total labeling of an r-regular graph $G(V,E)$ with v vertices and e edges and a magic constant h. The labeling λ' given by*

$$\begin{aligned} \lambda'(x) &= v + e + 1 - \lambda(x) \quad x \in V \\ \lambda'(xy) &= v + e + 1 - \lambda(xy) \quad xy \in E \end{aligned}$$

is also a vertex magic total labeling of G with the magic constant $(r+1)(e+v+1) - h$.

The following theorem was proved by Bača, Bertault, MacDougall, Miller, Simanjuntak, and Slamin (see [1]).

Theorem 2.5 *Let λ be an (s,d)-vertex antimagic total labeling of an r-regular graph $G(V,E)$ with v vertices and e edges. The labeling λ' given by*

$$\begin{aligned} \lambda'(x) &= v + e + 1 - \lambda(x) \quad x \in V \\ \lambda'(xy) &= v + e + 1 - \lambda(xy) \quad xy \in E \end{aligned}$$

is an $((r+1)(e+v+1) - s - (v-1)d, d)$-vertex antimagic total labeling of G.

3 (s,d)-vertex antimagic total labelings of cycles

There are three different types of $(s,2)$-VAT labelings known so far, see [1]. We will show two types of generalized $(s,2)$-VAT labelings of cycles C_n, which we use later in constructions of VMT labelings of products of cycles.

Theorem 3.1 (Type 1 generalized $(a+2b+2(n-1),2)$-VAT labeling) *Let a and b be positive integers and let $n \geq 3$ be an integer. Then there exists a generalized $(a+2b+2(n-1),2)$-vertex antimagic total labeling of C_n where $a, a+2, \ldots, a+2(n-1)$ are the vertex labels and $b, b+2, \ldots, b+2(n-1)$ are the edge labels.*

Proof. Let C_n be a cycle with vertices $v_0, v_1, \ldots, v_{n-1}$ and edges v_iv_{i+1} for $i = 0, 1, \ldots, n-1$, where the subscripts are taken mod n. Consider the following labeling

$$\begin{aligned} \lambda(v_i) &= \begin{cases} a & \text{for } i = 0 \\ a+2(n-i) & \text{for } i = 1, 2, \ldots, n-1 \end{cases} \\ \lambda(v_iv_{i+1}) &= b+2i \qquad \text{for } i = 0, 1, \ldots, n-1. \end{aligned}$$

This is a generalized $(a+2b+2(n-1),2)$-VAT labeling of C_n (see Figure 3.1). We refer to this labeling of cycles C_n as to *Type 1 generalized* $(a+2b+2(n-1),2)$-*VAT labeling*.

For $i = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} \lambda(v_i) + \lambda(v_{i-1}v_i) + \lambda(v_iv_{i+1}) &= a+2(n-i)+b+2(i-1)+b+2i \\ &= a+2b+2(n-1)+2i. \end{aligned}$$

For the vertex v_0 we have

$$\begin{aligned} \lambda(v_0) + \lambda(v_{n-1}v_0) + \lambda(v_0v_1) &= a+b+2(n-1)+b \\ &= a+2b+2(n-1)+0. \end{aligned}$$

Thus we have a VAT labeling with the sums

$$a+2b+2(n-1)+2i, \qquad \text{for } i = 0, 1, \ldots, n-1.$$

□

Taking $a = 1$ and $b = 2$ we get a $(2n+3,2)$-VAT labeling for every C_n with labels $1, 2, \ldots, 2n$. This special case was already known, see [1].

The dual labeling, also mentioned in [1], is a $(2n+2,2)$-VAT labeling of C_n. Taking $a = 2$ and $b = 1$ we get another $(2n+2,2)$-VAT labeling of C_n. Again the dual labeling is another $(2n+3,2)$-VAT labeling of C_n.

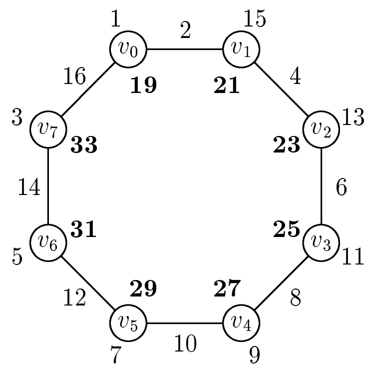

Figure 3.1: *Type 1 generalized* $(a+2b+2(n-1),2)$*-VAT labeling of* C_8*.*

Theorem 3.2 (Type 2 generalized $(a+2b+\frac{n-1}{2},2)$**-VAT labeling)**

Let a *and* b *be positive integers and let* $n \geq 3$ *be odd. Then there exists a generalized* $(a+2b+\frac{n-1}{2},2)$*-vertex antimagic total labeling of* C_n *where* $a, a+1, \ldots, a+n-1$ *are the vertex labels and* $b, b+1, \ldots, b+n-1$ *are the edge labels.*

Proof. Let C_n be a cycle with vertices $v_0, v_1, \ldots, v_{n-1}$, n odd, and edges v_iv_{i+1} for $i = 0, 1, \ldots, n-1$, where the subscripts are taken mod n. Consider the following labeling:

$$\begin{aligned}\lambda(v_i) &= a+i \text{ for } i = 0,1,\ldots,n-1\\ \lambda(v_iv_{i+1}) &= \begin{cases} b+\frac{i}{2} & \text{for } i \text{ even}\\ b+\frac{n+i}{2} & \text{for } i \text{ odd.}\end{cases}\end{aligned}$$

This is a generalized $(a+2b+\frac{n-1}{2},2)$-VAT labeling of C_n with difference 2 (see Figure 3.2). We refer to this labeling of odd cycles C_n as to *Type 2 generalized* $(a+2b+\frac{n-1}{2},2)$*-VAT labeling.*

For i even we have

$$\begin{aligned}\lambda(v_i)+\lambda(v_{i-1}v_i)+\lambda(v_iv_{i+1}) &= a+i+b+\frac{n+(i-1)}{2}+b+\frac{i}{2}\\ &= a+2b+\frac{n-1}{2}+2i\end{aligned}$$

and for i odd we have

$$\begin{aligned}\lambda(v_i)+\lambda(v_{i-1}v_i)+\lambda(v_iv_{i+1}) &= a+i+b+\frac{i-1}{2}+b+\frac{n+i}{2}\\ &= a+2b+\frac{n-1}{2}+2i.\end{aligned}$$

As a, b, n are constants, the sums form an arithmetic progression with difference 2.

□

Taking $a = 1$ and $b = n + 1$ we get a $(\frac{5n+5}{2}, 2)$-VAT labeling for every C_n with labels $1, 2, \ldots, 2n$. Taking $a = n+1$ and $b = 1$ we get the dual $(\frac{3n+5}{2}, 2)$-VAT labeling of C_n. This special case was already known, see [1].

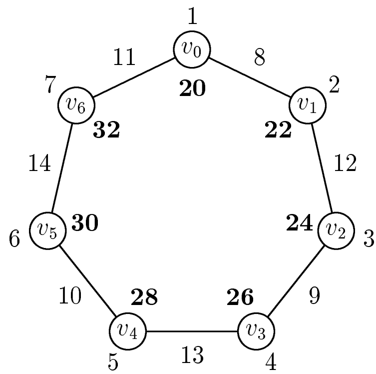

Figure 3.2: *Type 2 generalized* $(a + 2b + \frac{n-1}{2}, 2)$*-VAT labeling of* C_7.

4 Vertex magic total labelings of products of cycles

Now we are ready to prove our main results. We present two methods for constructing VMT labelings of products of cycles. The first method is more general, for products of cycles of any length with odd cycles. The second method can be used only for products of cycles of odd lengths. Using both methods we can construct several different labelings, which give distinct magic constants for the same graph. In the second method it is easier to follow the pattern as to how the labels are distributed, and both the lowest and highest bound of the spectrum for the magic constant are obtained.

Theorem 4.1 *For each $m, n \geq 3$ and n odd, there exists a VMT labeling of $C_m \times C_n$ with the magic constant*

$$h = \frac{1}{2}m(15n + 1) + 2.$$

Proof. We use Type 1 generalized $(a + 2b + 2(n - 1), 2)$-VAT labeling from Theorem 3.1 to assign the labels to the vertices and edges of the vertical cycles (C_m). The edge labels of the horizontal cycles (C_n) are assigned in accordance with the generalized form of the VMT labeling of odd cycles given by (1). Temporary sums on the vertices obtained only from the labels of the vertical cycles and the labels on the horizontal edges form the generalized VMT labeling (1) on the horizontal cycles.

Let $C_m \times C_n$ have vertices $v_{i,j}$, vertical edges $v_{i,j}v_{i+1,j}$ and horizontal edges $v_{i,j}v_{i,j+1}$ where $i = 0, 1, \ldots, m - 1$ and $j = 0, 1, \ldots, n - 1$ for $m, n \geq 3$ and n odd. Consider the following labeling, where the subscripts i and j are taken modulo m and n, respectively.

$$
\begin{aligned}
\lambda(v_{i,j}) &= 3mj + 2 && \text{for} \quad i = 0, \\
&= 3mj + 2m - 2\,(i-1) && \text{for} \quad i = 1, \dots, m-1, \\
\lambda(v_{i,j}v_{i+1,j}) &= 3m\,(n - (j+1)) + 2i + 1, \\
\lambda(v_{i,j}v_{i,j+1}) &= 3m\left(\tfrac{j}{2} + 1\right) - i && \text{for} \quad j \text{ even}, \\
&= 3m\left(\tfrac{n+j}{2} + 1\right) - i && \text{for} \quad j \text{ odd}.
\end{aligned}
$$

From the construction it is easy to observe that all $3mn$ numbers are assigned. The sequence of used labels $1, 2, \dots, 3mn$ is divided into $3n$ m-tuples. Let $a_t = \{tm+1, tm+2, \dots, tm+m\}$ be the t-th m-tuple of labels for $t = 0, 1, \dots, 3n-1$. We assign numbers from the m-tuples $a_0, a_1, a_3, a_4, \dots, a_{3n-3}, a_{3n-2}$ to the vertices and edges of vertical cycles. Vertices receive even labels and edges odd labels. Numbers from the remaining m-tuples $a_2, a_5, \dots, a_{3n-1}$ are assigned to the horizontal edges. See example of this labeling in Figure 4.1.

It is easy to verify that the sum of the labels at each vertex is the same.

(i) For $i = 0$ and j even we get

$$
\begin{aligned}
h &= \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) + \lambda(v_{i,j-1}v_{i,j}) + \lambda(v_{i,j}v_{i,j+1}) \\
&= 3mj + 2 + 3m\,(n-(j+1)) + 2(m-1) + 1 + 3m\,(n-(j+1)) + 1 + \\
&\quad 3m\left(\tfrac{n+j-1}{2} + 1\right) + 3m\left(\tfrac{j}{2} + 1\right) \\
&= 3mn + 3mn + \tfrac{3}{2}mn - 3m + 2m - 3m - \tfrac{3}{2}m + 3m + 3m + 2 \\
&= \tfrac{1}{2}m(15n+1) + 2.
\end{aligned}
$$

(ii) For $i = 1, \dots, m-1$ and j even the sum of the labels is

$$
\begin{aligned}
h &= \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) + \lambda(v_{i,j-1}v_{i,j}) + \lambda(v_{i,j}v_{i,j+1}) \\
&= 3mj + 2m - 2\,(i-1) + 3m\,(n-(j+1)) + 2(i-1) + 1 \\
&\quad + 3m\,(n-(j+1)) + 2i + 1 + 3m\left(\tfrac{n+j-1}{2} + 1\right) - i + 3m\left(\tfrac{j}{2} + 1\right) - i \\
&= 3mn + 3mn + \tfrac{3}{2}mn + 2m - 3m - 3m - \tfrac{3}{2}m + 3m + 3m + 2 \\
&= \tfrac{1}{2}m(15n+1) + 2
\end{aligned}
$$

(iii) and (iv) Similarly we get the same constant also for remaining two cases, $i = 0$ and j odd, or $i = 1, \dots, m-1$ and j odd.

□

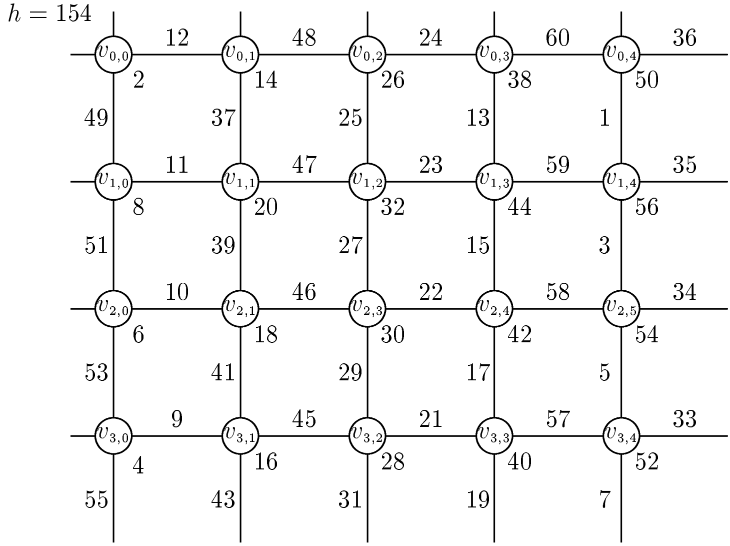

Figure 4.1: *VMT labeling of* $C_4 \times C_5$ *with the magic constant* $h = 154$.

Our labeling can be easily modified to obtain some other values for the magic constant h. In Table 4.1 we show four different ways to label the vertices, vertical edges and horizontal edges, so that the same pattern as in the given labeling is followed.

In Table 4.1 the index t runs from 0 to $n-1$.

1	$\lambda(v_{i,j})$	$3mt+2,\ 3mt+4,\ldots,3mt+2m$	$h =$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+1,\ 3mt+3,\ldots,3mt+2m-1$	$\frac{1}{2}m(15n+1)+2$
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+2m+1,\ 3mt+2m+2,\ldots,3mt+3m$	
2	$\lambda(v_{i,j})$	$3mt+1,\ 3mt+3,\ldots,3mt+2m-1$	$h =$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+2,\ 3mt+4,\ldots,3mt+2m$	$\frac{1}{2}m(15n+1)+3$
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+2m+1,\ 3mt+2m+2,\ldots,3mt+3m$	
3	$\lambda(v_{i,j})$	$3mt+m+2,\ 3mt+m+4,\ldots,3mt+3m$	$h =$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+m+1,\ 3mt+m+3,\ldots,3mt+3m-1$	$\frac{1}{2}m(15n-1)+2$
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+1,\ 3mt+2,\ldots,3mt+m$	
4	$\lambda(v_{i,j})$	$3mt+m+1,\ 3mt+m+3,\ldots,3mt+3m-1$	$h =$
	$\lambda(v_{i,j}v_{i+1,j})$	$3mt+m+2,\ 3mt+m+4,\ldots,3mt+3m$	$\frac{1}{2}m(15n-1)+3$
	$\lambda(v_{i,j}v_{i,j+1})$	$3mt+1,\ 3mt+2,\ldots,3mt+m$	

Table 4.1: *Intervals from which labels are taken for vertices, vertical and horizontal edges, and corresponding magic constants.*

By duality we do not obtain any new labeling. It is easy to show that the labeling which we get in case 1 (see Table 4.1) is dual to the labeling in case 4. Also the labelings which we obtain in cases 2 and 3 are dual to each other.

We cannot easily extend this method to VMT labelings of $C_m \times C_n$ with n even. The reason is that no nice VMT labeling of even cycles is known. By 'nice' labeling we mean that the labels of the vertices and edges are consecutive integers or follow some regular pattern.

Theorem 4.2 *For each $m, n \geq 3$ and m, n odd, there exists a VMT labeling of $C_m \times C_n$ with the magic constant*

$$h = \frac{17}{2}mn + \frac{5}{2}.$$

Proof. We construct the labeling as follows. We use a Type 2 generalized $(a + 2b + \frac{n-1}{2}, 2)$-VAT labeling from Theorem 3.2 to assign the labels to the vertices and edges of the vertical cycles C_m.

The labels of the edges of the horizontal cycles together with the sums on vertices obtained from the labels of the vertical cycles C_m again form a generalized VMT labeling. This construction is less general than in the proof of Theorem 4.1, since our Type 2 antimagic labeling (see Theorem 3.2) exists only for cycles of odd length.

Let $C_m \times C_n$ have vertices $v_{i,j}$, vertical edges $v_{i,j}v_{i+1,j}$ and horizontal edges $v_{i,j}v_{i,j+1}$ where $i = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, n-1$ and m and n are odd integers greater than 1. Consider the following labeling, where the subscripts i and j are taken modulo m and n, respectively.

$$\begin{aligned}
\lambda(v_{i,j}) &= jm + 1 + i, \\
\lambda(v_{i,j}v_{i+1,j}) &= (2n - (j+1))\, m + 1 + \tfrac{i}{2} \qquad i \text{ even}, \\
&= (2n - (j+1))\, m + 1 + \tfrac{m+i}{2} \qquad i \text{ odd}, \\
\lambda(v_{i,j}v_{i,j+1}) &= \left(2n + \tfrac{j}{2} + 1\right) m - i \qquad j \text{ even}, \\
&= \left(2n + \tfrac{n+j}{2} + 1\right) m - i \qquad j \text{ odd}.
\end{aligned}$$

The set of the labels we need for a VMT labeling of the product of $C_m \times C_n$ is $\{1, 2, \ldots, 3mn\}$. In our construction the vertices obtain labels $1, 2, \ldots, mn$. Labels assigned to the vertical edges are all integers $mn+1, mn+2, \ldots, 2mn$. We get the smallest vertical edge label for $i = 0$ and $j = n-1$ and the highest vertical edge label for $i = m-2$ and $j = 0$. Labels assigned to the horizontal edges are $2mn+1, 2mn+2, \ldots, 3mn$ with the smallest horizontal edge label for $i = m-1$, $j = 0$ and the highest horizontal edge label for $i = 0$, $j = n-2$.

Again we show that the sum of the labels at each vertex is the same.

(i) For both i and j even we get

$$\begin{aligned} h &= \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) + \lambda(v_{i,j-1}v_{i,j}) + \lambda(v_{i,j}v_{i,j+1}) \\ &= jm + 1 + i + (2n - (j+1))\,m + 1 + \tfrac{m+i-1}{2} + (2n - (j+1))\,m + 1 + \\ &\quad \tfrac{i}{2} + \left(2n + \tfrac{n+j-1}{2} + 1\right) m - i + \left(2n + \tfrac{j}{2} + 1\right) m - i \\ &= \tfrac{17}{2}mn + \tfrac{5}{2}. \end{aligned}$$

(ii) For i even and j odd the sum of labels is

$$\begin{aligned} h &= \lambda(v_{i,j}) + \lambda(v_{i-1,j}v_{i,j}) + \lambda(v_{i,j}v_{i+1,j}) + \lambda(v_{i,j-1}v_{i,j}) + \lambda(v_{i,j}v_{i,j+1}) \\ &= jm + 1 + i + (2n - (j+1))\,m + 1 + \tfrac{m+i-1}{2} + (2n - (j+1))\,m + 1 + \\ &\quad \tfrac{i}{2} + \left(2n + \tfrac{j-1}{2} + 1\right) m - i + \left(2n + \tfrac{n+j}{2} + 1\right) m - i \\ &= \tfrac{17}{2}mn + \tfrac{5}{2}. \end{aligned}$$

(iii) and (iv) The proof is similar for i odd, j even and both i, j odd.

□

In Figure 4.2 the labeling is shown for $C_3 \times C_5$.

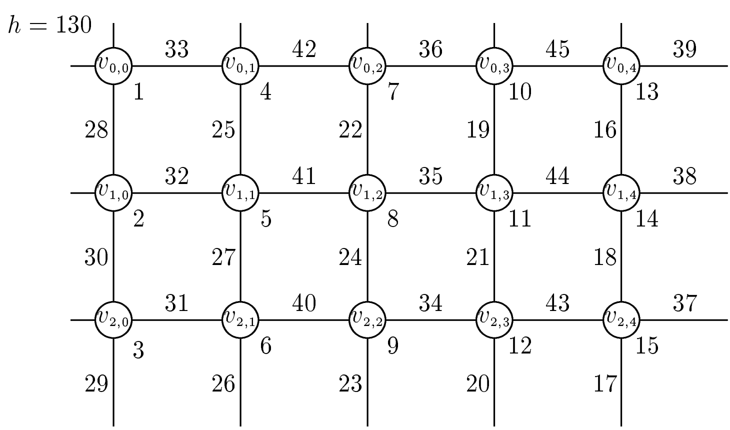

Figure 4.2: *VMT labeling of* $C_3 \times C_5$ *with magic constant* $h = 130$.

This labeling can be also slightly modified to obtain some other values of the magic constant h. In this method it is important that the m-tuples of the labels of the vertices or the vertical or horizontal edges are consecutive, so the sets of the labels between the vertices or the vertical or horizontal edges can be exchanged. We do not need to preserve the property that the first mn labels are assigned to vertices, the labels $mn+1, mn+2, \ldots, 2mn$ are assigned to vertical edges, and finally the labels

$2mn+1, 2mn+2, \ldots, 3mn$ are assigned to horizontal edges. Six possibilities of how to distribute the labels are summarized in Table 4.2 along with the corresponding magic constants.

From the spectrum of the magic constant h for the product of cycles the largest value is realized in cases 1 and 2 and the smallest value in cases 5 and 6.

By duality we do not obtain any new labeling. It is easy to observe from Table 4.2 that the labeling 1 is dual to the labeling 6, labeling 2 to 5, and 3 to 4.

	$\lambda(v_{i,j})$	$\lambda(v_{i,j}v_{i+1,j})$	$\lambda(v_{i,j}v_{i,j+1})$	h
1	$1, \ldots, mn$	$mn+1, \ldots, 2mn$	$2mn+1, \ldots, 3mn$	$\frac{17}{2}mn + \frac{5}{2}$
2	$1, \ldots, mn$	$2mn+1, \ldots, 3mn$	$mn+1, \ldots, 2mn$	$\frac{17}{2}mn + \frac{5}{2}$
3	$mn+1, \ldots, 2mn$	$1, \ldots, mn$	$2mn+1, \ldots, 3mn$	$\frac{15}{2}mn + \frac{5}{2}$
4	$mn+1, \ldots, 2mn$	$2mn+1, \ldots, 3mn$	$1, \ldots, mn$	$\frac{15}{2}mn + \frac{5}{2}$
5	$2mn+1, \ldots, 3mn$	$1, \ldots, mn$	$mn+1, \ldots, 2mn$	$\frac{13}{2}mn + \frac{5}{2}$
6	$2mn+1, \ldots, 3mn$	$mn+1, \ldots, 2mn$	$1, \ldots, mn$	$\frac{13}{2}mn + \frac{5}{2}$

Table 4.2: *Intervals from which labels are taken for vertices, vertical and horizontal edges, and the corresponding magic constants.*

References

[1] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R. Simanjuntak and Slamin, Vertex-antimagic total labelings of graphs, *Discussiones Mathematicae Graph Theory* **23** (2003), 67–83.

[2] J.A. MacDougall, M. Miller, Slamin and W.D. Wallis, Vertex-magic total labelings of graphs, *Utilitas Math.* **61** (1990), 3–21.

[3] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* Dynamic Survey 6, 43pp. (2002).

[4] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451–461.

[5] W.D. Wallis, Magic Graphs, *Birkhäuser* (2001).

(Received 6 Jan 2004)