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A tower of geometries related to the ternary Golay codes

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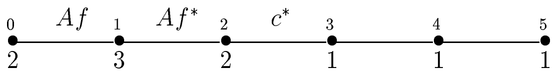
Abstract

The Steiner system $\Sigma = S(12, 6, 5)$ admits a unique lax projective embedding f in $PG(V)$, $V = V(6, 3)$. The embedding f induces a full projective embedding e of the dual Δ of Σ in the dual $PG(V^*)$ of $PG(V)$. The affine expansion $\mathrm{Af}_e(\Delta)$ of Δ to $AG(V^*)$ (also called linear representation of Δ in $AG(V^*)$) is a flag-transitive geometry with diagram and orders as follows:

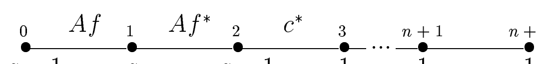


Its collinearity graph is the minimal distance graph of the 6-dimensional ternary Golay code. We shall prove that $\mathrm{Af}_e(\Delta)$ is the unique flag-transitive geometry with diagrams and orders as above. The $\{0, 1, 2, 3, 4\}$-residues of $\mathrm{Af}_e(\Delta)$ can also be obtained as affine expansions from the dual of $S(11, 5, 4)$ and are related to the 5-dimensional ternary Golay code. We shall characterize them too by their diagram and orders. Finally, the $\{0, 1, 2, 3\}$-residues of $\mathrm{Af}_e(\Delta)$ are isomorphic to the affine expansion of the dual of the classical inversive plane of order 3. A characterization will also be given for these expansions, in the same style as for $\mathrm{Af}_e(\Delta)$.

1 Introduction and main results

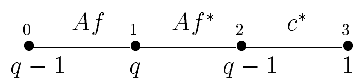
In this paper we consider geometries belonging to the following diagram of rank $n + 3 \geq 4$, where the integers $0, 1, ..., n + 2$ are the types, $q - 1, q, q - 1, 1, ..., 1$ are finite orders, the labels Af and Af^* stands for the class of affine planes and the class of dual affine planes and c^* denotes the class of dual circular spaces:

$(Af.Af^*.c^{*n})$

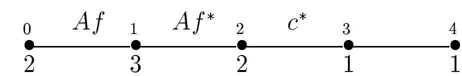


(We follow [18] for the definition of geometry; in particular, all geometries are residually connected, by definition.) If Γ is a geometry for the above diagram, then the residues of the 0-elements of Γ are dually isomorphic to n-point extensions of dual affine planes. We recall that 1-point extensions of affine planes are called inversive planes. It is well known that an affine plane of order $q > 3$ does not admit any n-point extension for $n > 2$ (see [18, Theorem 7.24]). $AG(2, 13)$ is the unique affine plane of order $q > 3$ that might possibly admit a 2-point extension [18, Theorem 7.24], but no such extension has been discovered so far. Anyhow, that extension, if it existed, would not be flag-transitive (Delandtsheer [10]; see also [11]). The affine plane $AG(2, 3)$ of order 3 admits no n-point extension for $n > 3$, but it admits a unique 3-point extension and a unique 2-point extension, namely the Steiner systems $S(12, 6, 5)$ and $S(11, 5, 4)$ for M_{12} and M_{11} respectively. Finally, $AG(2, 2)$ admits an n-point extension for any n, obtained as a truncation from the $(n + 2)$-dimensional symplex. Thus, the following are the only possibilities for $Af.Af^*.c^{*n}$ if flag-transitivity is assumed:

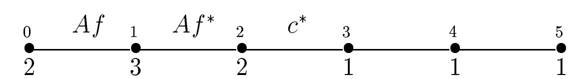
$(Af.Af^*.c^*)$



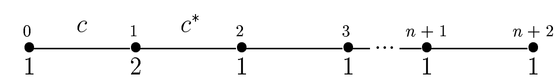
$(Af.Af^*.c^{*2})$

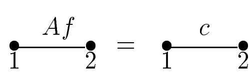


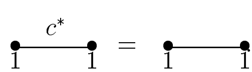
$(Af.Af^*.c^{*3})$



$(c.c^{*(n+1)})$



(Note that 

and 

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The geometries belonging to diagram $c.c^{*(n+1)}$ with orders $1, 2, 1, ..., 1$ have been classified by Ceccherini and Pasini [5, Theorem 3.5] (see also Huybrechts and Pasini [16]): all of them are homomorphic images of truncated Coxeter complexes. So, we will assume $q > 2$ in this paper.

Geometries for $Af.Af^*.c^*$ can be obtained as follows. Given an ovoid O of $PG(V)$, $V = V(4, q)$, let $\mathcal{I} = \mathcal{I}(O)$ be the inversive plane of points and secant planes of O, but regarded as a 3-dimensional matroid with the secant lines of O as lines. By applying a correlation of $PG(V)$ (a polarity, for instance), we obtain a (full) projective embedding $e : \mathcal{I}^* \to PG(V^*)$ of the dual $\mathcal{I}^*$ of $\mathcal{I}$ in the dual $PG(V^*)$ of $PG(V)$. The *affine expansion* $\mathrm{Af}_e(\mathcal{I}^*)$ of $\mathcal{I}^*$ by e is the geometry of rank 4 defined as follows (see Subsection 2.1):

Take $\{0, 1, 2, 3\}$ as the set of types. The 0-elements of $\mathrm{Af}_e(\mathcal{I}^*)$ are the points of $AG(4, q)$. For $1 \leq i \leq 3$ and an i-dimensional affine subspace X of $AG(4, q)$, let X^∞ be the point, line or plane at infinity of X (according to whether i is 1, 2 or 3). We take X as an i-element of $\mathrm{Af}_e(\mathcal{I}^*)$ if and only if X^∞ is an element of the image $e(\mathcal{I}^*)$

of $\mathcal{I}^*$. The incidence relation of $\mathrm{Af}_e(\mathcal{I}^*)$ is inherited from $AG(4,q)$.

$\mathrm{Af}_e(\mathcal{I}^*)$ is a residually connected geometry belonging to diagram $Af.Af^*.c^*$. Clearly, $\mathrm{Af}_e(\mathcal{I}^*)$ is flag-transitive if and only if $\mathcal{I}$ is flag-transitive. It is well known that $\mathcal{I} = \mathcal{I}(O)$ is flag-transitive if and only if O is classical (see Delandtsheer [11]; also [9]). Suppose that O is classical (which is always the case when q is odd). Then the stabilizer $G_O \cong P\Gamma O^-(4,q)$ of O in $P\Gamma L(4,q)$ induces on $\mathcal{I}$ its full automorphism group. Accordingly, denoted by T the translation group of $AG(4,q)$, we have $\mathrm{Aut}(\mathrm{Af}_e(\mathcal{I}^*)) \cong T{:}\Gamma O^-(4,q)$ ($< A\Gamma L(4,q)$; the symbol : stands for *split extension*, as in [6]). Moreover, every flag-transitive subgroup of $\mathrm{Aut}(\mathrm{Af}_e(\mathcal{I}^*))$ contains $T{:}SO^-(4,q)$ (Delandtsheer [11]).

We recall that if $\mathcal{I}$ is classical then, up to automorphisms of $PG(V)$, $\mathcal{I}$ admits a unique embedding as $\mathcal{I}(O)$ in $PG(V)$ with O a classical ovoid. Accordingly, the embedding $e : \mathcal{I}^* \to PG(V^*)$ is uniquely determined up to automorphisms of $PG(V^*)$. We call it the *natural* embedding of $\mathcal{I}^*$.

As shown by Coxeter [7], the Steiner systems $\Sigma_1 := S(11,5,4)$ and $\Sigma_2 := S(12,6,5)$ also admit embeddings in $PG(V_1)$ and $PG(V_2)$ respectively, where $V_1 := V(5,3)$ and $V_2 := V(6,3)$ (see Section 2 for more details). These embeddings are uniquely determined up to automorphisms of $PG(V_1)$ and $PG(V_2)$ (Theorem 2.2). We call them the *natural embeddings* of Σ_1 and Σ_2.

For $i = 1,2$, let $f_i : \Sigma_i \to PG(V_i)$ be the natural embedding of Σ_i and Δ_i be the dual of Σ_i. By composing f_i with a correlation of $PG(V_i)$ we obtain a projective embedding $e_i : \Delta_i \to PG(V_i^*)$, which we call the *natural embeding* of Δ_i. We can define the *affine expansion* $\mathrm{Af}_{e_i}(\Delta_i)$ of Δ_i by e_i in the same way as we have done for $\mathrm{Af}_e(\mathcal{I}^*)$. Thus, we obtain flag-transitive geometries of rank 4 and 5, belonging to the diagrams $Af.Af^*.c^{*2}$ and $Af.Af^*.c^{*3}$ and with orders $2,3,2,1,1$ and $2,3,2,1,1,1$ respectively. Their automorphism groups are as follows:

$$\mathrm{Aut}(\mathrm{Af}_{e_1}(\Delta_1)) = 3^5{:}(2 \times M_{11}), \qquad \mathrm{Aut}(\mathrm{Af}_{e_2}(\Delta_2)) = 3^6{:}(2^{\cdot}M_{12}).$$

(The symbol $^{\cdot}$ stands for *non-split extension*, as in [6].) $\mathrm{Aut}(\mathrm{Af}_{e_2}(\Delta_2))$ is the full automorphism group of the 6-dimensional ternary Golay code $\mathcal{C}_6(3)$ (see Section 2 for more details). Clearly, the translation subgroup T of $AG(6,3)$ is the maximal normal 3-subgroup of $\mathrm{Aut}(\mathrm{Af}_{e_2}(\Delta_2))$. Its elements may be regarded as the words of $\mathcal{C}_6(3)$. The parallelism relation of $AG(6,3)$ induces an equivalence relation on the set of 1-elements of $\mathrm{Af}_{e_2}(\Delta_2)$. The words of $\mathcal{C}_6(3)$ of weight 6 correspond to the elements of T that elementwise stabilize a parallel class of 1-elements of $\mathrm{Af}_{e_2}(\Delta_2)$.

We are now ready to state our main theorem. For the sake of uniformity, we denote by Σ_0 the inversive plane of order 3 arising from a quadric of $PG(V_0)$, where $V_0 = V(4,3)$. The dual of Σ_0 will be denoted by Δ_0 and e_0 is the natural embedding of Δ_0 in $PG(V_0^*)$.

THEOREM 1 (1) *Let Γ be a flag-transitive geometry belonging to diagram $Af.Af^*.c^*$ with orders $2,3,2,1$. Then $\Gamma \cong \mathrm{Af}_{e_0}(\Delta_0)$, where Δ_0 and e_0 are as above.*

(2) *Let Γ be a flag-transitive geometry belonging to diagram $Af.Af^*.c^{*2}$ with orders $2,3,2,1,1$. Then $\Gamma \cong \mathrm{Af}_{e_1}(\Delta_1)$ where Δ_1 is the dual of $\Sigma_1 = S(11,5,4)$ and e_1 is the natural embedding of Δ_1.*

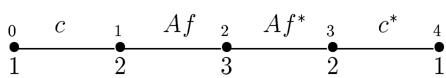
(3) *Let* Γ *be a flag-transitive geometry belonging to diagram* $Af.Af^*.c^{*3}$ *with orders* $2, 3, 2, 1, 1, 1$. *Then* $\Gamma \cong \mathrm{Af}_{e_2}(\Delta_2)$ *where* Δ_2 *is the dual of* $\Sigma_2 = S(12, 6, 5)$ *and* e_2 *is the natural embedding of* Δ_2.

Theorem 1 will be proved in Section 5. In Section 2 we shall discuss the natural embeddings of Σ_1, Δ_1, Σ_2 and Δ_2. Section 3 contains a survey of examples and properties of $Af.Af^*$-geometries, to be used in Section 4, where we will study $Af.Af^*.c^*$-geometries, eventually focusing on the flag-transitive case. Claim (1) of Theorem 1 will be obtained as a corollary from the final theorem of Section 4. The results of Section 4 may be regarded as contributions to a possible proof of the following conjecture:

Conjecture 1 *Every flag-transitive* $Af.Af^*.c^*$*-geometry is the affine expansion of the dual of a classical inversive plane by its natural projective embedding.*

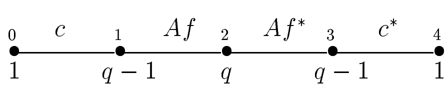
For the sake of completeness, we also mention the following theorem, proved in [20], where a two-sided extension of the $Af.Af^*$-diagram is considered.

THEOREM 2 *No flag-transitive geometry exists with diagram and orders as follows:*



The next is plausible:

Conjecture 2 *No flag-transitive geometry exists with diagram as follows, where* $q > 2$*:*



The restriction $q > 2$ is essential in the above conjecture. Indeed, there exists at least one flag-transitive geometry for the above diagram with $q = 2$. It is obtained by truncating a Coxeter complex of type E_6.

2 Embeddings of $S(11, 5, 4)$ and $S(12, 6, 5)$ and their duals

2.1 Preliminaries

Embeddings and affine expansion have already been mentioned in Section 1, but we shall fix these notions more formally here. A general theory of embeddings and expansions is developed in [19], but we do not need it in this paper. The definitions we shall state are special cases of those of [19].

Let Σ be a geometry belonging to a string diagram of rank n, with the integers $0, 1, ..., n-1$ as types, labelling the nodes of the diagram in increasing order from left to right, as usual. To make things easier, we also assume that Σ satisfies the Intersection Property IP (see [18, Chapter 6]). We denote the set of 0-elements of Σ by P and, for an element x of Σ, we denote by $P(x)$ the set of 0-elements of Σ incident to x.

For a vector space V, let $f : P \to PG(V)$ be an injective mapping from P to the set of points of the projective geometry $PG(V)$ of linear subspaces of V such that $f(P)$ spans $PG(V)$. For an element x of Σ of type $t(x) > 0$, let $f(x) := \langle f(P(x)) \rangle$ be the span of $f(P(x))$ in $PG(V)$. In this way, f is extended to a mapping from the whole of Σ to the set of subspaces of $PG(V)$. Assume the following:

(E1) $f(x)$ is a line for every 1-element x of Γ;

(E2) for $p \in P$ and an element x of Σ of type $t(x) > 0$, we have $f(p) \in f(x)$ only if $p \in P(x)$.

Then we call f a *projective embedding* of Σ. Note that when $t(x) = 1$ the set $f(P(x))$ might not be a line of $PG(V)$ (even if it spans a line, by (E1)). That is, $f(P(x))$ might be properly contained in $f(x) = \langle f(P(x)) \rangle$. If $f(P(x)) = f(x)$ for every 1-element x of Γ then we say that the embedding f is *full*. Following Van Maldeghem [22], if f is non-full then we say it is *lax*. Property (E2) on f and IP on Σ imply the following:

(E3) For any two elements x, y of Σ of type $t(x), t(y) > 0$, we have $f(x) \subseteq f(y)$ if and only if x and y are incident in Σ and $t(x) \leq t(y)$. In particular, $f(x) = f(y)$ only if $x = y$.

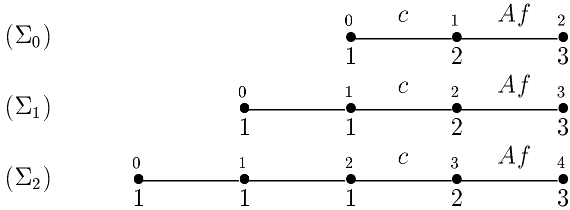
Affine expansions. The *affine expansion* $\mathrm{Af}_f(\Sigma)$ of Σ by f is defined as follows: Take $\{0, 1, ..., n\}$ as type-set for $\mathrm{Af}_f(\Sigma)$. The 0-elements of $\mathrm{Af}_f(\Sigma)$ are the points of the affine geometry $AG(V)$. Regarding $PG(V)$ as the geometry at infinity of $AG(V)$, the 1-elements of $\mathrm{Af}_f(\Sigma)$ are the lines L of $AG(V)$ with point at infinity $L^\infty \in e(P)$. For $i > 1$, the i-elements of $\mathrm{Af}_f(\Sigma)$ are the affine subspaces X of $AG(V)$ with space at infinity $X^\infty = f(x)$ for an $(i-1)$-element x of Σ. The incidence relation is the natural one, namely inclusion. The structure $\mathrm{Af}_f(\Sigma)$ is indeed a geometry (in particular, it is residually connected [19]) and the residues of its 0-elements are isomorphic to Σ. In view of (E1), the lower residues of the 2-elements of $\mathrm{Af}_f(\Sigma)$ are nets. In particular, when f is full those residues are affine planes.

Remark. A number of authors (as De Clerck and Van Maldeghem [8], for instance) call affine expansions *linear representations*.

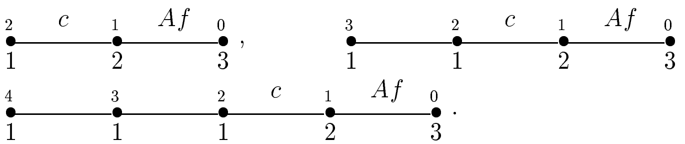
Isomorphisms of embeddings. Given two embeddings $f : \Sigma \to PG(V)$ and $g : \Sigma \to PG(W)$, if $g = hf$ for an isomorphism h from $PG(V)$ to $PG(W)$ then we say that f and g are *isomorphic* and we write $f \cong g$. Given a class $\mathcal{C}$ of projective embeddings of Σ, if $f \cong g$ for any two embeddings $f, g \in \mathcal{C}$, then we say that $\mathcal{C}$ *contains a unique* embedding. (This is a linguistic abuse, but it is harmless.)

2.2 The natural projective embeddings of $S(11,5,4)$ and $S(12,6,5)$ and their duals

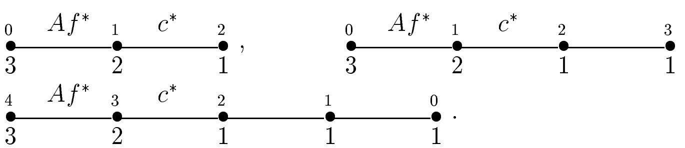
For $i = 0,1,2$, let Σ_i be the Steiner system $S(10+i, 4+i, 3+i)$, regarded as $(3+i)$-dimensional matroid. So, Σ_0 is the unique inversive plane of order 3, Σ_1 is the Steiner system for M_{11} and Σ_2 that for M_{12}. We take $\{0,1,2\}$, $\{0,1,2,3\}$ and $\{0,1,2,3,4\}$ as sets of types for Σ_0, Σ_1 and Σ_2:



For $i = 0,1,2$, we denote by Δ_i the dual of Σ_i. Namely, Δ_i is the same thing as Σ_i, except that types are permuted as follows:



The above diagrams are usually drawn as follows:



As recalled in the introduction of this paper, Σ_0 admits a lax embedding in $PG(3,3)$. As shown by Coxeter [7], the Steiner systems Σ_1 and Σ_2 also admit lax embeddings in $PG(4,3)$ and $PG(5,3)$ respectively. We shall describe these embeddings here. In view of this, we need to recall some properties of the 6-dimensional ternary Golay code $\mathcal{C}_6(3)$ and its dual $\mathcal{C}_6^*(3)$. We refer to [6, page 31] (also [3, 11.3]) for a description of $\mathcal{C}_6(3)$. We warn that $\mathcal{C}_6(3)$ is called 'extended ternary Golay code' in [3], but simply 'ternary Golay code' in [6]. In this paper we follow [6].

We recall that the code $C := \mathcal{C}_6(3)$, regarded as a linear subspace of $\widehat{V} = V(12,3)$, is 6-dimensional and the non-zero vectors of C have weight 6, 9 and 12 with respect to B. (We recall that the *weight* of a vector $v = (\lambda_i)_{i=1}^n$ of $V(n,q)$ is the number of entries $\lambda_i \neq 0$ and the set $S(v) := \{i \in \{1,2,...,n\} | \lambda_i \neq 0\}$ is called the *support* of v.) For every $i = 1,2,...,12$, let C_i be the set of vectors $v \in C$ with $i \notin S(v)$. It is well known that C_i is a hyperplane of C (see [3], where C_i is called 'perfect ternary Golay code'). Thus, we get 24 non-zero vectors of the dual C^* of C, partioned in 12 pairs of mutually opposite vectors. (These vectors are the 24 words of weight 1 of the cocode $C^* = \mathcal{C}_6^*(3)$). Accordingly, we have obtained a set S of 12 points of

$PG(V)$, where $V := C^* \cong V(6,3)$. As the non-zero vectors of C have weight 6, 9 or 12, the set S satisfies the following property:

($*$) every hyperplane of $PG(V)$ meets S in 6, 3 or 0 points.

Moreover, for every subset X of $\{1,2,...,12\}$ of size 5, there is exactly one 1-dimensional linear subspace $\{0, v, -v\}$ of C ($= V^*$, dual space of $V = C^*$) such that $S(v) \cap X = \emptyset$. Therefore,

($**$) any five points of S span a hyperplane of $PG(V)$.

For every point $p \in S$, let w_p be one of the two vectors $w \in V$ such that $\langle w \rangle = p$. Then C is the kernel of the linear transformation $\varphi : \widehat{V} \to V = C^*$ mapping $v = (\lambda_i)_{i=1}^{12} \in \widehat{V}$ to $\varphi(v) = \sum_{p \in S} \lambda_p w_p \in V$. The usual definition of $\mathcal{C}_6^*(3)$ as the quotient $\widehat{V}/C$ of $\widehat{V}$ by $C = \mathcal{C}_6(3)$ is implicit in the natural isomorphism from $\widehat{V}/V^* = \widehat{V}/\mathrm{Ker}(\varphi)$ to $V = \mathrm{Im}(\varphi)$.

Turning to Σ_2 and with S as above, we can take S as the set of 0-elements of Σ_2. The lines, planes, 3-spaces and hyperplanes of $PG(5,3)$ that meet S in 2, 3, 4 and, respectively, 6 points will be taken as elements of type 1, 2, 3 and 4. Thus, we obtain a lax embedding $f_2 : \Sigma_2 \to PG(V) \cong PG(5,3)$. Clearly, f_2 induces lax embeddings $f_1 : \Sigma_1 \to PG(4,3)$ and $f_0 : \Sigma_0 \to PG(3,3)$. The latter embedding is the unique embedding of the inversive plane Σ_0 in $PG(3,3)$.

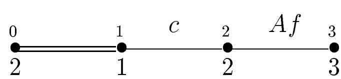
For $i = 0,1,2$, let $V_i = V(4+i,3)$ be the underlying vector space of the projective space $PG(V_i) = PG(3+i,3)$ in which Σ_i is embedded by f_i and let V_i^* be its dual. (In particular, V_2 and V_2^* are the spaces previously called V and V^*.) The embedding $f_i : \Sigma_i \to PG(V_i)$ induces a full embedding e_i of Δ_i in $PG(V_i^*)$ and we can consider the affine expansion $\mathrm{Af}_{e_i}(\Delta_i)$. As noticed in the introduction of this paper, $\mathrm{Af}_{e_i}(\Delta_i)$ belongs to $Af.Af^*.c^{*(i+1)}$ with orders $2,3,1,...,1$ and it is flag-transitive. Moreover,

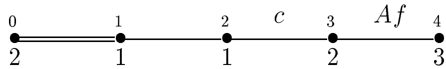
$$\begin{array}{rcl}
\mathrm{Aut}(\mathrm{Af}_{e_0}(\Delta_0)) & = & 3^4{:}\Gamma O^-(4,3), \\
\mathrm{Aut}(\mathrm{Af}_{e_1}(\Delta_1)) & = & 3^5{:}(2 \times M_{11}), \\
\mathrm{Aut}(\mathrm{Af}_{e_2}(\Delta_2)) & = & 3^6{:}(2^{\cdot}M_{11}).
\end{array}$$

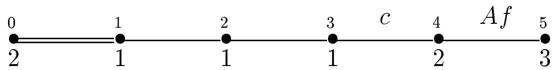
Clearly, $\mathrm{Af}_{e_0}(\Delta_0)$ is a residue of $\mathrm{Af}_{e_1}(\Delta_1)$ and the latter is a residue of $\mathrm{Af}_{e_2}(\Delta_2)$. The collinearity graph of $\mathrm{Af}_{e_2}(\Delta_2)$ is the minimal distance graph of $\mathcal{C}_6(3)$. That is, two vectors $v_1, v_2 \in \mathcal{C}_6(3)$ are collinear as points of $\mathrm{Af}_{e_2}(\Delta_2)$ if and only if $v_1 - v_2$ has weight 6. Similarly, the collinearity graph of $\mathrm{Af}_{e_1}(\Delta_1)$ is the minimal distance graph of the 5-dimensional ternary Golay code $\mathcal{C}_5(3)$ ('perfect Golay code' in [3]) and the collinearity graph of $\mathrm{Af}_{e_0}(\Delta_0)$ is the minimal distance graph of the code $\mathcal{C}_4(3)$ (called the 'truncated Golay code' in [3]).

Remark. We can also consider the affine expansions $\mathrm{Af}_{f_0}(\Sigma_0)$, $\mathrm{Af}_{f_1}(\Sigma_1)$ and $\mathrm{Af}_{f_2}(\Sigma_2)$. Their diagrams are as follows:

($\mathrm{Af}_{f_0}(\Sigma_0)$)



$(\mathrm{Af}_{f_1}(\Sigma_1))$ 

$(\mathrm{Af}_{f_2}(\Sigma_2))$ 

In particular, the point-line geometry of 0- and 1-elements of $\mathrm{Af}_{f_2}(\Sigma_2)$ is a well known near-hexagon, discovered by Shult and Yanushka [21] and characterized by Brouwer [2] (see also [3, 11.3.A]). Its collinearity graph is the coset graph of $\mathcal{C}_6(3)$. Similarly, the collinearity graphs of $\mathrm{Af}_{f_1}(\Sigma_1)$ and $\mathrm{Af}_{f_0}(\Sigma_0)$ are the coset graphs of $\mathcal{C}_5(3)$ and $\mathcal{C}_4(3)$, respectively.

2.3 Uniqueness of the embeddings f_0, f_1, f_2

We keep the notation of the previous subsection. For $i = 0, 1, 2$, let P_i be the set of 0-elements of Σ_i. The lax embeddings f_0, f_1, f_2 satisfy the following properties (compare $(*)$ and $(**)$ of the previous subsection):

(S0) every triple of points of $f_0(P_0)$ spans a plane of $PG(V_0)$ and every plane of $PG(V_0)$ meets $f_0(P_0)$ in either 1 or 4 points. (That is, $f_0(P_0)$ is an ovoid.)

(S1) every quadruple of points of $f_1(P_1)$ spans a hyperplane of $PG(V_1)$ and every hyperplane of $PG(V_1)$ meets $f_1(P_1)$ in either 2 or 5 points.

(S2) any five points $f_2(P_2)$ span a hyperpane of $PG(V_2)$ and every hyperplane of $PG(V_2)$ meets $f_2(P_2)$ in either 0, 3 or 6 points.

Lemma 2.1 *For $i = 0, 1, 2$, let f be an embedding of Σ_i in $PG(V_i)$. Then f satisfies* (Si).

Proof. We shall only prove the lemma for $i = 2$, leaving the remaining cases to the reader.

In view of (E3), for $j = 0, 1, 2, 3, 4$ the embedding f maps the j-elements of Σ_2 onto j-dimensional subspaces of $PG(V_2)$. As any five 0-elements of Σ_2 are contained in a unique 4-element, any five points of $f(P_2)$ span a hyperplane of $PG(V_2)$. On the other hand, every quadruple of 0-elements of Σ_2 is contained in four 4-elements, the latters are mapped by f onto four hyperplanes of Σ_2 and each of these hyperplanes meets $f(P_2)$ in six points. Therefore, if a hyperplane of $PG(V_2)$ contains four points of $f(P_2)$, then it meets $f(P_2)$ in six points.

Every triple X of points of $f(P_2)$ is contained in 13 hyperplanes of $PG(V_2)$. As every triple of 0-elements of Σ_2 is contained in exactly twelve 4-elements, exactly one of those hyperplanes meets $f(P_2)$ in 3 points. It follows that $PG(V_2)$ contains exactly $\binom{12}{3} = 220$ hyperplanes that meet $f(P_2)$ in 3 points. Every point $p \in f(P_2)$ is contained in $(3^5 - 1)/2 = 121$ hyperplanes. As every 0-element of Σ_2 belongs to exactly 66 elements of type 4, eaxctly 66 of those 121 hyperplanes meet $f(P_2)$ in 6 points. Moreover, p is contained in $\binom{11}{2} = 55$ triples of points of $f(P_2)$ and each of these triples is contained in exactly one hyperplane meeting $f(P_2)$ in 3 points. As $66 + 55 = 121$, every hyperplane containing p meets $f(P_2)$ in either 6 or 3 points. □

In the next theorem the word ‘unique’ means ‘unique up to isomorphisms’, as stated at the end of Subsection 2.1.

Theorem 2.2 *For $i = 0, 1, 2$, Σ_i admits a unique projective embedding in $PG(V_i)$.*

Proof. Given f_i as in the previous subsection, put $f := f_i$ and let $g : \Sigma_i \to PG(V_i)$ be another embedding of Σ_i. We shall prove the following:

(A) $g = hf$ for an automorphism h of $PG(V_i)$.

It is well known that (A) holds true when $i = 0$. So, in order to prove (A) for $i = 1$ and $i = 2$ we only must prove the following, where $i \in \{1, 2\}$:

(Bi) if (A) holds for $i - 1$, then (A) holds for i too.

We shall only prove (B2). Claim (B1) can be proved by a similar but easier argument, which we leave for the reader.

In view of (E3) and Lemma 2.1, the images $\Sigma_f := f(\Sigma_2)$ and $\Sigma_g := g(\Sigma_2)$ of Σ_2 by f and g are isomorphic to Σ_2. Accordingly, there exist an abstract isomorphism $\omega : \Sigma_f \to \Sigma_g$. Let $P_f := f(P_2)$ and $P_g := g(P_2)$ be the sets of 0-elements of Σ_f and Σ_g, $P_f = \{a_1, ..., a_{12}\}$ and $P_g = \{b_1, ..., b_{12}\}$ say. We may assume to have chosen indices in such a way that $\omega(a_i) = b_i$ for $i = 1, 2, ..., 7$, but $\omega(a_i)$ might be different from b_i when $i > 7$. Put $A = \{a_1, a_2, ..., a_7\}$ and $B = \{b_1, b_2, ..., b_7\} = \omega(A)$.

(1) $|A \cap H| = 6$ for exactly one hyperplane H.

(Proof of (1).) $|H \cap A| \leq 6$ for every hyperplane H. If $|A \cap H| = |A \cap H'| = 6$ for two hyperplanes H and H', then $|A \cap H \cap H'| \geq 5$, which forces $H = H'$. Suppose that $|A \cap H| < 6$ for every hyperplane H. Then distinct 5-subsets of A are contained in distinct hyperplanes. Let $S_5(A)$ be the family of 5-subsets of A and, for $X \in S_5(A)$, let H_X be the hyperplane containing X and a_X be the point of $H_X \cap (P_f \setminus A)$. For two distinct 5-subsets $X, Y \in S_5(A)$, we have $a_X = a_Y$ only if $|X \cap Y| = 3$. Moreover, it is not difficult to see that, given $X, Y \in S_5(A)$ with $|X \cap Y| = 3$, there exists exactly one $Z \in S_5(A)$ such that $|X \cap Z| = |Y \cap Z| = 3$. Therefore, the function α sending $X \in S_5(A)$ to $\alpha(X) = a_X$ has fibers of size at most 3. Consequently, the image $Im(\alpha)$ of α contains at least $|S_5(A)|/3 = 21/3 = 7$ elements. However, $Im(\alpha) \subseteq P_f \setminus A$ and $|P_f \setminus A| = 5$. We have reached a contradiction. Claim (1) is proved.

We may assume to have chosen indices in such a way that $\{a_1, ..., a_6\}$ is the unique 6-subset of A contained in a hyperplane. Clearly, we may also assume that $a_1 = b_1 = p$, say. For $i = 2, ...12$, let L_i be the line of $PG(V_2)$ through p and a_i, and M_i be the line through p and b_i. Thus $(L_2, ..., L_{12})$ and $(M_2, ..., M_{12})$ yield embeddings of Σ_1 in the star of p. Both these embeddings satisfy (S1). Therefore, by (A) for $i = 1$, there exists an automorphism of $PG(V_2)$ that fixes p and maps $\{M_2, ..., M_{12}\}$ onto $\{L_2, ..., L_{12}\}$. So,

(2) We may also assume that $b_i \in L_i$ for $i = 2, 3, ..., 12$.

As $\cup_{i=2}^{6} L_i$ is contained in a hyperplane, $\{b_1, ..., b_6\}$ is the unique 6-subset of B contained in a hyperplane (compare (1)). Let L be the line through a_6 and a_7, let a'_6 be one of the two points of $L \setminus \{a_6, a_7\}$ and put $A' = \{p, a_2..., a_5, a'_6, a_7\}$. Then $|H \cap A'| \leq 5$ for every hyperplane H of $PG(V_2)$. Accordingly, we can take $(p, a_2, ..., a_5, a'_6, a_7)$ as a coordinate system, where $p, a_2, ..., a_5, a'_6$ form the basis and a_7 is the unit point. Similarly, denoted by M the line through b_6 and b_7 and chosen a point $b'_6 \in M \setminus \{b_6, b_7\}$, the sequence $(p, b_2, ..., b_5, b'_6, b_7)$ is a coordinate system, where $(p, b_2, ..., b_5, b'_6)$ is the basis and b_7 is the unit point. Consequently, there exists a linear mapping h of V_2 fixing p and sending b_i to a_i for $i = 2, 3, 4, 5, 7$ and b'_6 to a'_6. Clearly, h maps M onto L and stabilizes the hyperplane $H_0 := \langle L_1, L_2, ..., L_5 \rangle$ (which is the unique hyperplane meeting A in 6 points). However, $b_6 = M \cap H_0$ and $a_6 = L \cap H_0$. Hence $h(b_6) = a_6$.

(3) We may assume that $b_i = a_i$ for $i = 1, 2, ..., 7$ and $b_i \in L_i$ for $i = 8, 9, ..., 12$.

(Proof of (3).) Modulo applying a linear transformation h as in the previous paragraph, we may assume that $b_i = a_i$ for $i = 1, 2, ..., 7$. If h induces the identity mapping on the star of p, then we are done. Otherwise, h induces a non-trivial homology h_p on the star of p. The center of h_p is the line L_7 and the axis of h_p is the set of lines of H_0 through p. However, in this case, we can consider the non-trivial homology h_0 of $PG(V_2)$ with center a_7 and axis H_0. The composition $h_0 h$ induces the identity on the star of p and maps $b_2, b_3, ..., b_7$ on $a_2, a_3, ..., a_7$ respectively. Claim (3) is proved.

We now turn back to the isomorphism $\omega : \Sigma_f \to \Sigma_g$ considered at the beginning of the proof. Modulo composing ω with the linear transformations considered in the previous paragraphs, ω stabilizes $a_i = b_i$ for $i = 1, 2, .., 7$.

(4) $\omega(a_i) = b_i$ for $i = 8, 7, ..., 12$.

(Proof of (4).) Let π be the projection of $PG(V_2) \setminus \{p\}$ onto the star of p. By claim (3), π maps both Σ_f and Σ_g onto a copy Σ_p of Σ_1 with $\{L_2, L_3, ..., L_{12}\}$ as the point-set. It is also clear that there exist a unique automorphism ω_p of Σ_p such that $\pi\omega = \omega_p \pi$. As ω fixes $a_i = b_i$ for $i = 1, 2, .., 7$, ω_p fixes six points of Σ_p, namely $L_2, L_3, ..., L_7$. This forces ω_p to be the identity. Hence $\omega(a_i) = b_i$ for $i = 8, 9, ..., 12$, as claimed in (4).

The next claim finishes the proof of (B2).

(5) $b_i = a_i$ for $i = 1, 2, ..., 12$.

(Proof of (5).) In view of (3), we only must prove that $b_i = a_i$ for $i > 7$. The set $A \setminus \{p\}$ contains 6 subsets of size 5. Just one of them is contained in H_0. So, denoted by $\mathcal{X}$ the set of 5-subsets of A that are not contained in H_0, we have $|\mathcal{X}| = 5$. Given $X \in \mathcal{X}$, let H_X be the hyperplane of $PG(V_2)$ spanned by X. Then H_X contains exactly one of the points $a_8, a_9, ..., a_{12}$ and exactly one of $b_8, b_9, ..., b_{12}$. Moreover,

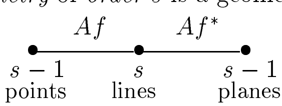
$p \notin H_X$, as $H_X \neq H_0$ and H_0 is the unique hyperplane of $PG(V_2)$ that contains six points of A. For a given $X \in \mathcal{X}$, two indices $i(X), j(X) \in \{8, 9, ..., 12\}$ are uniquely determined such that $a_{i(X)}$ and $b_{j(X)}$ are the points of $(H_X \cap P_f) \backslash A$ and $(H_X \cap P_g) \backslash A$ respectively. (Recall that $A = \{a_1, a_2, ..., a_7\} = \{b_1, b_2, ..., b_7\} = B$, by claim (3).) We have $\omega(X) = X$ by claim (4). Hence $H_X = \omega(H_X)$. Consequently, $b_{j(X)} = \omega(a_{i(X)})$. Therefore $j(X) = i(X) = k$, say, by (4). Accordingly, $\{b_k, a_k\} \subseteq H_X \cap L_k$. However, H_X meets L_k in precisely one point, as $p \notin H_X$. Therefore $b_k = a_k$. On the other hand, the function mapping $X \in \mathcal{X}$ onto $i(X)$ $(= j(X))$ is a bijection from $\mathcal{X}$ to $\{8, 9, ..., 12\}$. Therefore, $b_k = a_k$ for every $k = 8, 9, ..., 12$. □

Corollary 2.3 *For $i = 0, 1, 2$, Δ_i admits a unique projective embedding in $PG(V_i^*)$.* □

3 A survey of $Af.Af^*$-geometries

An *$Af.Af^*$-geometry* of *order* s is a geometry with diagram and orders as follows:

$(Af.Af^*)$



The elements of an $Af.Af^*$-geometry are called *points*, *lines* and *planes*, as indicated in the above picture. The *diameter* of an $Af.Af^*$-geometry is the diameter of its collinearity graph. In this paper we are only interested in finite $Af.Af^*$-geometries. Accordingly, s is assumed to be finite. Note that the finiteness of s implies the finiteness of the geometry, as it follows from the next proposition.

Proposition 3.1 (Del Fra and Pasini [12, 4.7]) *Every $Af.Af^*$-geometry has diameter $d \leq 2$.* □

A few classes of $Af.Af^*$-geometries are described in the next three subsections. We will turn to general properties of $Af.Af^*$-geometries in Subsection 3.4.

3.1 Bi-affine geometries and their quotients

Bi-affine geometries can be defined for any rank $n \geq 3$, but we are only interested in the rank 3 case here. Given a prime power q, a *bi-affine* geometry of order q (and rank 3) is the induced subgeometry $\Sigma(p_0, \pi_0)$ of a projective geometry $\Sigma = PG(3, q)$ obtained by removing a distinguished point p_0 of Σ (called the *pole at infinity* of $\Sigma(p_0, \pi_0)$), a distinguished plane π_0 of Σ (called the *plane at infinity* of $\Sigma(p_0, \pi_0)$), all lines and planes of Σ through p_0 and all points and lines of π_0. We say that $\Sigma(p_0, \pi_0)$ is of *flag*-type or *non-flag*-type according to whether $p_0 \in \pi_0$ or $p_0 \notin \pi_0$.

Clearly, $\Sigma(p_0, \pi_0)$ is an $Af.Af^*$-geometry of order q and it is flag-transitive, with automorphism group isomorphic to the stabilizer of p_0 and π_0 in $P\Gamma L(4, q) = \mathrm{Aut}(\Sigma)$. The subgroup $\mathrm{Aut}_{\mathrm{lin}}(\Sigma(p_0, \pi_0))$ of $\mathrm{Aut}(\Sigma(p_0, \pi_0))$ induced by the stabilizer of p_0 and π_0 in $PGL(4, q)$ also acts flag-transitively on $\Sigma(p_0, \pi_0)$. For the rest of this subsection Z stands for the center of $\mathrm{Aut}_{\mathrm{lin}}(\Sigma(p_0, \pi_0))$.

Flag-transitive quotients of $\Sigma(p_0, \pi_0)$ are obtained by factorizing by subgroups of Z. The quotient $\Sigma(p_0, \pi_0)/Z$ is the *minimal* one. In the flag-type case (namely $p_0 \in \pi_0$) the group Z has order q and is induced by the group of all elations of Σ with axis π_0 and center p_0. In this case the minimal quotient $\Sigma(p_0, \pi_0)/Z$ is isomorphic to the canonical gluing of two copies of $AG(2, q)$ (see the next subsection). On the other hand, if $p_0 \notin \pi_0$ then Z has order $q - 1$ and $\Sigma(p_0, \pi_0)/Z$ is isomorphic to the anti-flag geometry of the projective plane $\pi_0 \cong PG(2, q)$ (see Subsection 3.3).

Bi-affine geometries are simply connected, as it follows from Proposition 3.3 of Subsection 3.4. Accordingly, all quotients of $\Sigma(p_0, \pi_0)$ are obtained by factorizing by suitable subgroups of $\mathrm{Aut}(\Sigma(p_0, \pi_0))$ (see [18, Theorem 12.56]). Moreover, if $X < \mathrm{Aut}(\Sigma(p_0, \pi_0))$ defines a quotient of $\Sigma(p_0, \pi_0)$, then no two collinear points of $\Sigma(p_0, \pi_0)$ belong to the same orbit of X and, if two planes of $\Sigma(p_0, \pi_0)$ belong to the same orbit of X, then they meet trivially in $\Sigma(p_0, \pi_0)$. It follows that X, regarded as a subgroup of $\mathrm{Aut}(\Sigma)$, fixes all lines through p_0 and all points of π_0. Namely, $X \leq Z$. So, we have proved the following:

Proposition 3.2 *All quotients of $\Sigma(p_0, \pi_0)$ are obtained by factorizing by subgroups of the center Z of $\mathrm{Aut}_{\mathrm{lin}}(\Sigma(p_0, \pi_0))$. In particular, all quotients of $\Sigma(p_0, \pi_0)$ are flag-transitive.* □

The following is also worth to be mentioned. Suppose that $p_0 \in \pi_0$. Then, regarding π_0 as the plane at infinity of $AG(3, q)$, $\Sigma(p_0, \pi_0)$ is the induced subgeometry of $AG(3, q)$ obtained by removing all lines with p_0 as the point at infinity and every plane the line at infinity of which contains p_0. In other words, $\Sigma(p_0, \pi_0)$ is the affine expansion of the punctured projective plane obtained by removing from $\pi_0 \cong PG(2, q)$ a point p_0 and all lines through it.

We finish this survey of bi-affine geometries with a remark on collinearity graphs. The collinearity graph of $\Sigma(p_0, \pi_0)$ is a complete $(q^2 + \varepsilon(q + 1))$-partite graph, with classes of size $q - \varepsilon$, where ε stands for 0 or 1 according to whether p_0 belongs to π_0 or not. Clearly, given a subgroup $X \leq Z$ of order $\lambda = |X|$, the collinearity graph of the quotient $\Sigma(p_0, \pi_0)/X$ is a complete $(q^2 + \varepsilon(q + 1))$-partite graph with classes of size $(q - \varepsilon)/\lambda$. In particular, when $X = Z$ that graph is a complete graph with $q^2 + \varepsilon(q + 1)$ vertices.

3.2 Gluings

Gluings have been introduced by Del Fra, Pasini and Shpectorov [13], in view of a classification of $Af.A_{n-2}.Af^*$-geometries. Later, a general theory of gluings has been developed by Buekenhout, Huybrechts and Pasini [4]. However, we will only consider gluings of two affine planes in this paper.

Given two affine planes $\mathcal{A}_1$ and $\mathcal{A}_2$ of the same order s, with lines at infinity $\mathcal{A}_1^\infty$ and $\mathcal{A}_1^\infty$ and a bijection α from $\mathcal{A}_1^\infty$ to $\mathcal{A}_2^\infty$, the *gluing* $\mathrm{Gl}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$ of $\mathcal{A}_1$ with $\mathcal{A}_2$ by α is the $Af.Af^*$-geometry defined as follows: the points of $\mathcal{A}_1$ and $\mathcal{A}_2$ are taken as points and planes, respectively; the lines of $\mathrm{Gl}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$ are the pairs (L_1, L_2) of lines of $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\alpha(L_1^\infty) = L_2^\infty$, where L_i^∞ is the point at infinity of L_i. Every point of $\mathrm{Gl}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$ is declared to be incident with all planes. A point p_1 (a plane p_2) and a line (L_1, L_2) of $\mathrm{Gl}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$ are incident precisely when $p_1 \in L_1$ (respectively, $p_2 \in L_2$). When $\mathcal{A}_1 \cong \mathcal{A}_2 \cong AG(2, q)$ and α is induced by an isomorphism from $\mathcal{A}_1$ to $\mathcal{A}_2$, then the gluing $\mathrm{Gl}_\alpha(\mathcal{A}_1, \mathcal{A}_2)$ is said to be *canonical*.

Up to isomorphism, there is only one canonical gluing of two copies of $AG(2, q)$. That gluing is flag-transitive and it is isomorphic to the minimal quotient of a bi-affine geometry of order q and flag-type (Del Fra, Pasini and Shpectorov [13]; see also Del Fra and Pasini [12, 2.1]).

More generally, every canonical gluing of two copies of the same flag-transitive affine plane is flag-transitive. Many flag-transitive non-canonical gluings also exist. A classification of flag-transitive non-canonical gluings of two copies of $AG(2, q)$ has been obtained by Baumeister and Stroth [1].

3.3 Anti-flag geometries

Given a projective plane $\mathcal{P}$ of order s, let $\Delta(\mathcal{P})$ be the geometry of rank 3 defined as follows: the points and the planes of $\Delta(\mathcal{P})$ are the points and the lines of $\mathcal{P}$, whereas the lines of $\Delta(\mathcal{P})$ are the flags of $\mathcal{P}$. We say that a point p and a line L of $\mathcal{P}$ are incident in $\Delta(\mathcal{P})$ when $p \notin L$. A flag (p_1, L_1) and a point p_2 (a line L_2) of $\mathcal{P}$ are incident in $\Delta(\mathcal{P})$ precisely when $p_2 \in L_1$ but $p_2 \neq p_1$ (respectively, $p_1 \in L_2$ but $L_2 \neq L_1$). It is not difficult to see that $\Delta(\mathcal{P})$ is an $Af.Af^*$-geometry of order s. We call it the *anti-flag geometry* of $\mathcal{P}$. (Note that the point-plane flags of $\Delta(\mathcal{P})$ are just the anti-flags of $\mathcal{P}$.)

When $\mathcal{P}$ is classical, then $\Delta(\mathcal{P})$ is isomorphic to the minimal quotient of the bi-affine geometry of non-flag-type (Del Fra, Pasini and Shpectorov [13]; also Del Fra and Pasini [12, 2.1]). In view of Kantor [15], an anti-flag geometry $\Delta(\mathcal{P})$ is flag-transitive if and only if $\mathcal{P}$ is classical.

3.4 A few properties of $Af.Af^*$-geometries

All $Af.Af^*$-geometries obtained as gluings are *flat*, namely all points are incident to all planes. In a flat $Af.Af^*$-geometry of order s, every pair of points is incident with exactly s common lines. A similar situation occurs in anti-flag geometries: if Δ is an anti-flag geometry of order s, then Δ has diameter $d = 1$ and every pair of points of Δ is incident with $s - 1$ common lines. A situation completely different from the above is described below:

(LL) no two distinct points are incident with two common lines.

This property characterizes bi-affine geometries. Indeed:

Proposition 3.3 (Levefre & Van Nypelseer [17]) *An $Af.Af^*$-geometry is bi-affine if and only if it satisfies* (LL). □

In the general case, the following holds:

Proposition 3.4 (Del Fra and Pasini [12, 4.6]) *Let Δ be an $Af.Af^*$-geometry of finite order s. Then there exists a positive integer $\lambda \leq s$ such that:*

(1) *if p_1 and p_2 are distinct collinear points, then there are exactly λ lines incident with both p_1 and p_2;*

(2) *if l_1 and l_2 are distinct lines with at least two points in common (whence $\lambda > 1$), then l_1 and l_2 have exactly λ points in common;*

(3) *if π_1 and π_2 are distinct planes with at least one line in common, then there are exactly λ lines incident with both π_1 and π_2;*

(4) *if l_1 and l_2 are distinct lines incident with at least two common planes (hence $\lambda > 1$), then l_1 and l_2 are incident with exactly λ common planes.*

Moreover, λ divides $s(s-1)$. □

We call λ the *index* of the $Af.Af^*$-geometry Δ. Clearly, $\lambda = 1$ if and only if Δ satisfies (LL), namely Δ is bi-affine (Proposition 3.3). Opposite situations are considered in the next proposition:

Proposition 3.5 (Del Fra and Pasini [12, 4.14, 5.1]) *Let Δ be an $Af.Af^*$-geometry of finite order s and index λ. Then:*
(1) *Δ has diameter $d = 1$ if and only if $s - 1 \leq \lambda \leq s$*
(2) *Δ is flat if and only if $\lambda = s$.* □

Proposition 3.6 (Del Fra and Pasini [12, 5.4, 5.6]) *Let Δ be a flag-transitive $Af.Af^*$-geometry of order s and index $\lambda \in \{s-1, s\}$.*
(1) *If $\lambda = s$ then Δ is a gluing of two affine planes.*
(2) *If $\lambda = s - 1$ then Δ is an anti-flag geometry.* □

Turning back to the case of diameter $d = 2$, we mention the following:

Proposition 3.7 (Del Fra and Pasini [12, 4.15]) *Let Δ be an $Af.Af^*$-geometry of diameter $d = 2$ and $\mathcal{G}(\Delta)$ be its collinearity graph. Then $\mathcal{G}(\Delta)$ is a complete n-partite graph, for a suitable integer $n \geq s^2$. Moreover, $\lambda|C| \leq q$ for every class C of the n-partition of $\mathcal{G}(\Delta)$.* □

3.5 Classical and nearly classical $Af.Af^*$-geometries

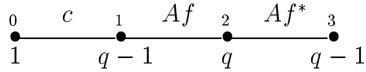
We say that an $Af.Af^*$-geometry Δ of prime power order q is *classical* if $\Delta = \widetilde{\Delta}/X$ for a bi-affine geometry $\widetilde{\Delta} = \Sigma(p_0, \pi_0)$ and a subgroup X of the center of $\mathrm{Aut}_{\mathrm{lin}}(\Sigma(p_0, \pi_0))$.

Let Δ be an $Af.Af^*$-geometry of order s and index λ. For $\varepsilon \in \{0, 1\}$, we say that Δ is *nearly classical* of ε-type if s is a prime power, λ divides $s - \varepsilon$, $(s^3 - \varepsilon)/\lambda$ is the number of points as well as the number of planes of Δ and the collinearity graph of Δ is a complete $(s^2 + \varepsilon(s+1))$-partite graph with all classes of size $(s-\varepsilon)/\lambda$ (possibly, a complete graph with $s^2 + \varepsilon(s+1)$ vertices, when $\lambda = s - \varepsilon$). In short, Δ has the same parameters as a classical $Af.Af^*$-geometry. For instance, if s is a prime power, all flat $Af.Af^*$-geometries of order s are nearly classical of 0-type and all anti-flag geometries of order s are nearly classical of 1-type, but not all of these geometries are classical.

4 c-extensions of $Af.Af^*$-geometries

This section is devoted to $Af.Af^*.c^*$-geometries, but we prefer to focus on their duals. So, troughout this section Γ is a geometry with diagram as follows and orders $1, q-1, q, q-1$, where q is a prime power, say $q = p^n$ for a prime p and a positive integer n.

$(c.Af.Af^*)$



Note that, given a 0-element x of Γ, the elements of Γ incident to x of type 1, 2 and 3 are respectively the points, lines and planes of the $Af.Af^*$-geometry $\mathrm{Res}(x)$. On the other hand, the residues of the 3-elements of Γ are $c.Af^*$-geometries. As recalled in the introduction of this paper, every $c.Af^*$-geometry is an inversive plane. Therefore the residues of the 3-elements of Γ are inversive planes.

Given a 1-element e and a 0-element x of Γ, if x is incident with e then we say that x belongs e, also that e lies on x, or that it passes through x. We denote by $\mathcal{G}(\Gamma)$ the collinearity graph of Γ, where the elements of type 0 and 1 are taken as points and lines, respectively. The adjacency relation of $\mathcal{G}(\Gamma)$ will be denoted by $\sim$. When we say that two 0-elements x, y are adjacent (or that they have distance $d(x, y) = 2$) we mean that they are adjacent (respectively, at distance 2) in $\mathcal{G}(\Gamma)$. Given a 0-element x, we denote by $x^\perp$ the set of 0-elements adjacent with x or equal to x. The *multiplicity* $\mu(x, y)$ of an edge $\{x, y\}$ of $\mathcal{G}(\Gamma)$ is the number of 1-elements that are incident with $\{x, y\}$. We say that Γ admits *uniform multiplicity* μ if $\mu(x, y) = \mu$ for every edge $\{x, y\}$ of $\mathcal{G}(\Gamma)$.

We denote by $\mathcal{S}(\Gamma)$ be the point-line geometry with the 3-elements of Γ as points and the 2-elements as lines. We also denote the collinearity graph of $\mathcal{S}(\Gamma)$ by $\mathcal{G}^*(\Gamma)$ and its adjacency relation by $\sim^*$.

Given a type i and an element x of type $t(x) \neq i$, we denote by $\sigma_i(x)$ the set of i-elements incident with x. Also, $\sigma_i(x, y) := \sigma_i(x) \cap \sigma_i(y)$.

Lemma 4.1 *Assume the following:*

(A1) *for* $\varepsilon \in \{0, 1\}$ *and a given divisor* λ *of* $q - \varepsilon$, $\mathrm{Res}(x)$ *is nearly classical of* ε*-type and index* λ*, for every 0-element* x *of* Γ*;*

(A2) Γ *admits uniform multiplicity* μ.

Then all the following hold:

(B1) $\varepsilon = 0$ *and* $\lambda = 1$*, namely* $\mathrm{Res}(x)$ *is isomorphic to the bi-affine geometry of order* q *and flag-type, for every* 0*-element* x.

(B2) μ *divides* q *and is smaller than* q.

(B3) $\mathcal{G}(\Gamma)$ *is a complete* (q^2+1)*-partite graph with classes of size* q/μ. (*In particular, if* $\mu = q$ *then* $\mathcal{G}(\Gamma)$ *is a complete graph with* $q^2 + 1$ *vertices.*) *Accordingly, if* N_i *is the number of* i*-elements of* Γ*, then*
$N_0 = (q^2+1)q/\mu, \quad N_1 = (q^2+1)q^4/2\mu, \quad N_2 = (q^2+1)q^4/\mu, \quad N_3 = q^4/\mu.$

(B4) *For any three distinct* 0*-elements* x, y, z *with* $y, z \in x^{\perp}$ *and for any choice of* $e \in \sigma_1(x,y)$ *and* $f \in \sigma_1(x,z)$*,* e *and* f *are coplanar as points of* $\mathrm{Res}(x)$ *if and only if* $y \sim z$*.*

(B5) *The graph* $\mathcal{G}^*(\Gamma)$ *has diameter* $d^* \leq 2$*.*

Proof. Let k be the valency of $\mathcal{G}(\Gamma)$. Clearly,

(1) $k = (q^3 - \varepsilon)/\mu\lambda$.

Note also that, given two 0-elements x, y, the μ elements of $\sigma_1(x,y)$ form a coclique in the collinearity graph of the $Af.Af^*$-geometry $\mathrm{Res}(x)$. Each of the maximal cocliques of the collinearity graph of $\mathrm{Res}(x)$ is partitioned in $(q-\varepsilon)/\mu$ cocliques as above. Hence μ divides $(q-\varepsilon)/\lambda$, namely

(2) $\lambda\mu$ divides $q - \varepsilon$.

We shall now prove the following:

(3) the graph $\mathcal{G}(\Gamma)$ has diameter $d \leq 2$ and, if $d = 2$ and x, y are 0-elements at distance 2, then $|x^{\perp} \cap y^{\perp}| = (q^3 - \varepsilon)/\lambda\mu = k$.

Suppose $d \geq 2$. Given two 0-elements x, y at distance 2, pick $z \in x^{\perp} \cap y^{\perp}$. If $e_1 \in \sigma_1(x,z)$ and $e_2 \in \sigma_1(y,z)$ with $e_1 \neq e_2$, then e_1 and e_2 belong to the same maximal coclique of the collinearity graph of $\mathrm{Res}(z)$. Therefore, given $e \in \sigma_1(z)$ and regarded e, e_1, e_2 as points of $\mathrm{Res}(x)$, e is collinear with e_1 in $\mathrm{Res}(x)$ if and only if it is collinear with e_2. As the common neighbourhood of two non-collinear points of $\mathrm{Res}(z)$ has size $(q^3 - q)/\lambda$, we obtain that

$$(*) \quad |x^{\perp} \cap z^{\perp} \cap y^{\perp}| \geq (q^3 - q)/\lambda\mu.$$

With e as above, suppose that e is coplanar with e_1 and e_2 in $\mathrm{Res}(x)$. Let u be the 0-element of e different from z and v a 0-element adjacent with y but at distance 2 from u. Let $f_1 \in \sigma_1(y,u)$ and $f_2 \in \sigma_1(y,v)$. As $d(z,v) = 2$, f_1 and f_2 belong to the same maximal coclique of the collinearity graph of $\mathrm{Res}(y)$. On the other hand, the 0-elements z, y and u are incident with a common 2-element. Hence f_1 and f_2 are collinear as points of $\mathrm{Res}(y)$. Therefore $\sigma_2(f_1, f_2) \neq \emptyset$. So, at least $(q^3 - q)/\lambda\mu$ elements of $z^{\perp} \cap y^{\perp}$ belong to $v^{\perp}$. This implies that $d = 2$.

The equality $|x^{\perp} \cap y^{\perp}| = k$ remains to be proved. With z, e and u as above, let $f_1 \in \sigma_1(u,x)$ and $f_2 \in \sigma_1(u,y)$. Let $f \in \sigma_1(u)$ be such that f, regarded as a point of $\mathrm{Res}(u)$, is not collinear with e. Then f is collinear with either of f_1 and f_2. Therefore the 0-element of f different from u belongs to $x^{\perp} \cap y^{\perp} \cap u^{\perp}$, but not to $z^{\perp}$. As the maximal coclique of $\mathrm{Res}(u)$ containing e contains $(q-\varepsilon)/\lambda - 1$ points of $\mathrm{Res}(u)$ different from e, $|(u^{\perp} \cap x^{\perp} \cap y^{\perp}) \setminus z^{\perp}| \geq (q-\varepsilon)/\lambda\mu - 1$. By this inequality and $(*)$, $|x^{\perp} \cap y^{\perp}| \geq (q^3 - q)/\lambda\mu + (q-\varepsilon)/\lambda\mu = (q^3 - \varepsilon)/\lambda\mu$. However, $(q^3 - \varepsilon)/\lambda\mu$ is just the valency of $\mathcal{G}(\Gamma)$, by (1). Therefore $|x^{\perp} \cap y^{\perp}| = (q^3 - q)/\lambda\mu$. Claim (3) is proved.

(4) Suppose that $\mathcal{G}(\Gamma)$ has diameter $d = 2$. Then $\varepsilon = 0$, $\lambda = 1$, $\mu < q$ and $\mathcal{G}(\Gamma)$ is a complete (q^2+1)-partite graph with all classes of size q/μ.

(Proof of (4).) By (3), the number of common neighbours of two 0-elements at distance 2 is equal to the valency of $\mathcal{G}(\Gamma)$. Hence $\mathcal{G}(\Gamma)$ is a complete N-partite graph, for some positive integer N. Moreover, $\mathcal{G}(\Gamma)$ is regular. Hence all classes of $\mathcal{G}(\Gamma)$ have the same size, say h. So, Γ has $N_0 := h + (q^3 - \varepsilon)/\lambda\mu$ 0-elements and h divides $(q^3 - \varepsilon)/\lambda\mu$. Given two adjacent 0-elements x, y and a 1-element $e \in \sigma_1(x, y)$, the 1-elements on x that contain 0-elements in the same class as y belong to the maximal coclique of $\mathrm{Res}(x)$ containing e. Hence $h \le (q-\varepsilon)/\lambda\mu$, namely $h\lambda\mu \le q - \varepsilon$. Also, $\lambda\mu < q - \varepsilon$ as $h > 1$ (note that $d = 2$, by assumption). Clearly, $N_0(q^3-\varepsilon)/\lambda = N_3(q^2+1)$. Accordingly, q^2+1 divides $(q^3-\varepsilon)(q^3-\varepsilon+h\lambda\mu)$. This forces q^2+1 to divide $(q+\varepsilon)(q+\varepsilon-h\lambda\mu) = q^2+2q\varepsilon+\varepsilon^2-(q+\varepsilon)h\lambda\mu$. Therefore, q^2+1 divides $(q+\varepsilon)h\lambda\mu + 1 - 2q\varepsilon - \varepsilon^2$. Assume first $\varepsilon = 1$. Then q^2+1 divides $(q+1)h\lambda\mu - 2q$. However, this contradicts the inequality $h\lambda\mu \le q$. Therefore $\varepsilon = 0$. Hence q^2+1 divides $qh\lambda\mu+1$. This implies that $h\lambda\mu = q$. As $h > 1$, we have $\lambda\mu < q$. Also, $N_0 = q(q^2+1)/\lambda\mu$.

The equality $\lambda = 1$ remains to be proved. Since $\mathcal{G}(\Gamma)$ is a complete (q^2+1)-partite graph with classes of size $q/\lambda\mu$ and Γ has multiplicity μ, we obtain that

$$N_1 = \frac{q^4(q^2+1)}{2\lambda^2\mu}.$$

By counting $\{1,2\}$-flags in two ways, we obtain that $N_1(q^2+q) = N_2\binom{q+1}{2}$. Hence

$$N_2 = \frac{q^4(q^2+1)}{\lambda^2\mu}.$$

However, we can also compute N_2 by counting $\{0,2\}$-flags. In this way, recalling that $N_0 = q(q^2+1)/\lambda\mu$ and that $(q^3/\lambda)((q^3-q)/\lambda)/q(q-1)$ is the number of 2-elements on a given 0-element of Γ, we obtain that

$$N_2 = \frac{q^4(q^2+1)}{\lambda^3\mu}.$$

Comparing the two expressions obtained for N_2 we see that $\lambda = 1$. All claims of (4) are proved.

(5) Suppose that $\mathcal{G}(\Gamma)$ has diameter $d = 1$. Then $\varepsilon = 0$, $\lambda = 1$, $\mu = q$ and Γ is flat, namely every 0-element of Γ is incident with all 3-elements (hence Γ has q^2+1 elements of type 0 and q^3 elements of type 3).

(Proof of (5).) As $d = 1$, Γ has exactly $N_0 = 1 + (q^3 - \varepsilon)\lambda\mu$ 0-elements. As $N_0(q^3-\varepsilon)/\lambda = N_3(1+q^2)$, we obtain that

$$(1 + \frac{q^3-\varepsilon}{\lambda\mu})\frac{q^3-\varepsilon}{\lambda} = N_3(1+q^2).$$

This forces $1+q^2$ to divide $2q\varepsilon+\varepsilon^2-\lambda\mu(q+\varepsilon)-1$. If $\varepsilon=0$, then $\lambda\mu$ is a divisor of q and the fact that $1+q^2$ divides $2q\varepsilon+\varepsilon^2-\lambda\mu(q+\varepsilon)-1$ forces $\lambda\mu=q$. On the other hand, if $\varepsilon=1$ then $\lambda\mu$ divides $q-1$ and $2q\varepsilon+\varepsilon^2-\lambda\mu(q+\varepsilon)-1=2q-\lambda\mu(q+1)$. It is not difficult to see that, for any divisor δ of $q-1$, $1+q^2$ does not divide $2q-\delta(q+1)$. So, $\lambda\mu=q$ and $\varepsilon=0$. Therefore Γ has q^2+1 elements of type 0 and q^3/λ elements of type 3. Hence Γ is flat.

Counting $\{0,1\}$-flags in two ways, one can see that Γ has $N_1=q^3(q^2+1)/2\lambda=q^2(q^2+1)\mu/2$ elements of type 1. As every 1-element is in q^2+q elements of type 2 and every 2-element contains $(q+1)q/2$ elements of type 1, the number of 2-elements of Γ is $N_2=q^3(q^2+1)/\lambda$. However, we can compute that number also by counting the number of $\{0,2\}$-flags in two ways, thus obtaining that

$$(q^2+1)\frac{q^3\lambda(q^3/\lambda-q/\lambda)}{q(q-1)}=N_2(q+1).$$

As $N_2=q^3(q^2+1)/\lambda$, the above implies $\lambda=1$. All claims contained in (5) are proved.

(6) $\mu<q$ (hence $d=2$).

(Proof of (6).) Suppose to the contrary that $\mu=q$. By (4) and (5), Γ is flat. Hence Γ has exactly q^3 elements of type 3. For a pair $\{\alpha,\beta\}$ of 3-elements, let $\nu(\alpha,\beta):=|\sigma_2(\alpha,\beta)|$. Suppose that $\sigma_2(\alpha,\beta)\neq\emptyset$. Let $X,Y\in\sigma_2(\alpha,\beta)$ be incident with a common 0-element, say x. As $\lambda=1$, $\mathrm{Res}(x)$ is a bi-affine geometry. The 2-elements X and Y are lines of that bi-affine geometry. However, they are contained in two distinct planes of $\mathrm{Res}(x)$, namely α and β. This is impossible, unless $X=Y$. Therefore, if $X\neq Y$ then $\sigma_0(X,Y)=\emptyset$. Accordingly, $\sigma_2(\alpha,\beta)$ is a set of mutually disjoint blocks of the inversive plane $\mathrm{Res}(\alpha)$. Every block of $\mathrm{Res}(\alpha)$ has $q+1$ points and $\mathrm{Res}(\alpha)$ has q^2+1 points. Hence $\nu(\alpha,\beta)(q+1)\leq q^2+1$. This forces $\nu(\alpha,\beta)\leq q-1$. As $\mathrm{Res}(\alpha)$ contains $(q^2+1)q$ elements of type 2 and each of them is in $q-1$ elements of type 3 different from α, the number of neighbours of α in $\mathcal{G}^*(\Gamma)$ is at least $(q^2+1)q$. Accordingly, Γ admits at least $1+(q^2+1)q=q^3+q+1>q^3$ elements of type 3. This is a contradiction, since Γ contains exactly q^3 elements of type 3. Hence $\mu<q$, as claimed in (6).

Claims (B1)-(B4) of the lemma follow from (1)-(6). Claim (B5) remains to be proved. Given two 3-elements α and β of Γ, let $x\in\sigma_0(\alpha)$ and $y\in\sigma_0(\beta)$. $\mathcal{G}(\Gamma)$ is a complete (q^2+1)-partite graph, by (B3). Hence we can choose x and y in such a way that $x\sim y$. Let $e\in\sigma_1(x,y)$. Then $|\sigma_3(e)|=q^2$. By (B1), $\mathrm{Res}(x)$ is a bi-affine geometry of flag-type. In the bi-affine geometry $\mathrm{Res}(x)$, but with 3- and 2-elements regarded as points and lines, we see that α is collinear with at least q^2-1 points of $\mathrm{Res}(x)$. Hence α is adjacent in $\mathcal{G}^*(\Gamma)$ with at least q^2-1 elements of $\sigma_3(e)$. Similarly, β is adjacent with at least q^2-1 elements of $\sigma_3(e)$. As $q^2-2>0$, at least one of the 3-elements on e is adjacent with either of α and β. Therefore, $d^*\leq 2$. □

Lemma 4.2 *Under the hypotheses of Lemma* 4.1, *we have* $\mu=1$ *if and only* $\mathcal{S}(\Gamma)$ *is a semi-linear space.*

Proof. Suppose that $\mathcal{S}(\Gamma)$ is a semi-linear space. Then no two 3-elements are incident with the same pair of distinct 2-elements. It follows that Γ admits at least $N_3 = 1+(q^2+1)q(q-1)$ elements of type 3. However, $N_3 = q^4/\mu$ by (B3) of Lemma 4.1. Hence $1+(q^2+1)q(q-1) \leq q^4/\mu$. If $\mu > 1$, the previous inequality implies that $q^2+1 \leq q$, which is impossible. Hence $\mu = 1$.

Conversely, let $\mu = 1$. We recall that, by (B1) of Lemma 4.1, $\mathrm{Res}(x) \cong \Delta$ for a given bi-affine geometry Δ of flag-type, for every 0-element x. In particular, the Intersection Property holds in $\mathrm{Res}(x)$. This fact and the hypothesis $\mu = 1$, combined with Lemma 7.25 of [18], imply that Γ satisfies the Intersection Property. Hence $\mathcal{S}(\Gamma)$ is semi-linear. □

In the next lemma, $\mathcal{A}(\Gamma)$ is the point-line geometry obtained from $\mathcal{S}(\Gamma)$ by keeping all points and lines of $\mathcal{S}(\Gamma)$ but adding the maximal cocliques of the collinearity graph of $\mathrm{Res}(x)$ as additional lines, where x ranges in the set of 0-elements of Γ and the 3- and 2-elements of Γ incident with x are regarded as points and lines of $\mathrm{Res}(x)$. The lines of $\mathcal{A}(\Gamma)$ defined in this latter way will be called *new lines*, whereas the lines of $\mathcal{S}(\Gamma)$ will be called *old lines* of $\mathcal{A}(\Gamma)$.

Lemma 4.3 *Under the hypotheses of Lemma* 4.1*, suppose that* $\mu = 1$*. Then* Γ *satisfies the Intersection Property,* $\mathcal{G}^*(\Gamma)$ *has diameter* $d^* = 2$ *and* $\mathcal{A}(\Gamma)$ *is a linear space with* q^4 *points and the same parameters as* $AG(4,q)$*, namely* q *points on each line and* q^3+q^2+q+1 *lines on each point.*

Proof. As remarked in the second part of the proof of Lemma 4.2, Γ satisfies the Intersection Property (IP for short) and $\mathrm{Res}(x) \cong \Delta$ for a given bi-affine geometry Δ of flag-type, for every 0-element x. With N_0, N_1, N_2 and N_3 as in (B3) of Lemma 4.1, the hypothesis $\mu = 1$ implies the following:

(1) $N_0 = (q^2+1)q$, $N_1 = (q^2+1)q^4/2$, $N_2 = (q^2+1)q^4$ and $N_3 = q^4$.

(2) The graph $\mathcal{G}^*(\Gamma)$ has valency $k = (q^2+1)q(q-1)$.

(Proof of (2).) Given a 3-element α, the inversive plane $\mathrm{Res}(\alpha)$ contains exactly $(q^2+1)q$ elements of type 2. Each of these 2-elements is incident with $q-1$ elements of type 3 different from α. As IP holds in Γ, no two 3-elements are incident with the same pair of distinct 2-elements. Hence $k = (q^2+1)q(q-1)$, as claimed in (2).

As $1+(q^2+1)q(q-1) < q^4 = N_3$, $\mathcal{G}^*(\Gamma)$ is not a complete graph. Hence $\mathcal{G}^*(\Gamma)$ has diameter $d^* = 2$, by (B5) of Lemma 4.1.

(3) Given two 3-elements α and β, if $\alpha \sim^* \beta$ then $|\sigma_0(\alpha,\beta)| = q+1$, otherwise $|\sigma_0(\alpha,\beta)| = 1$.

(Proof of (3).) Suppose first that $\alpha \sim^* \beta$ and let A be the 2-element of $\sigma_2(\alpha,\beta)$. By IP, A is unique and $\sigma_0(\alpha,\beta) = \sigma_0(A)$. Hence $|\sigma_0(\alpha,\beta)| = q+1$, as $|\sigma_0(A)| = q+1$. Property IP also implies that, if $|\sigma_0(\alpha,\beta)| > 1$, then $\alpha \sim^* \beta$.

We now pick a 3-element α. By (2), α is adjacent with $k = (q^2+1)q(q-1)$ elements of type 3. Let h be the number of 3-elements β such that α and β have exactly one 0-element in common. We have $|\sigma_0(\alpha)| = q^2+1$ and, if $x \in \sigma_0(\alpha)$, then the bi-affine geometry $\mathrm{Res}(x)$ (which is of flag-type, by (B1) of Lemma 4.1) contains exactly $q-1$ elements of type 3 non-adjacent with α in $\mathcal{G}^*(\Gamma)$. In view of the previous paragraph, these 3-elements have distance 2 from α in $\mathcal{G}^*(\Gamma)$. Also, none of them can be contributed by two different 0-elements of α. Therefore, $h = (q^2+1)(q-1)$. Hence

$$k + h = (q^2+1)q(q-1) + (q^2+1)(q-1) = (q^2+1)(q^2-1) = q^4 - 1.$$

As q^4 is the number of 3-elements of Γ (see Lemma 4.1, (B3)), every 3-element at distance 2 from α in $\mathcal{G}^*(\Gamma)$ shares a 0-element with α. Claim (3) is proved.

By claim (3) and the definition of $\mathcal{A}(\Gamma)$, any two 3-elements of Γ are collinear in $\mathcal{A}(\Gamma)$. Moreover, if $|\sigma_0(\alpha,\beta)| > 1$, then $\alpha \sim^* \beta$. Hence $\mathcal{A}(\Gamma)$ is a linear space. Clearly, it has the same parameters as $AG(4,q)$. □

Lemma 4.4 *Assume the following:*

(A1) *for every* 0*-element* x *of* Γ*, the* $Af.Af^*$*-geometry* $\mathrm{Res}(x)$ *is nearly classical;*

(A2) $\mathrm{Aut}(\Gamma)$ *is flag-transitive.*

Then:

(B1) *For every* 0*-element* x*,* $\mathrm{Res}(x)$ *is isomorphic to the bi-affine geometry of flag-type and order* q*.*

(B2) Γ *satisfies the Intersection Property. In particular, it admits uniform multiplicity* $\mu = 1$*.*

(B3) $\mathcal{G}(\Gamma)$ *is a complete* (q^2+1)*-partite graph with all classes of size* q*.*

(B4) *Let* H *be kernel of the action of* $\mathrm{Aut}(\Gamma)$ *on the set of classes of the* (q^2+1)*-partition of* $\mathcal{G}(\Gamma)$*. Then* $|H| = q^4\gamma$ *for a divisor* γ *of* $q-1$ *(possibly,* $\gamma = 1$*) and* H *admits a normal subgroup* T *of order* q^4 *acting regularly on the set of* 3*-elements of* Γ*.*

Proof. As $\mathrm{Aut}(\Gamma)$ is flag-transitive and $\mathrm{Res}(x)$ is nearly classical for every 0-element x, Γ satisfies the hypotheses of Lemma 4.1. In particular, it admits uniform multiplicity μ. Therefore, by Lemma 4.1, $\mathrm{Res}(x) \cong \Delta$ where Δ is the bi-affine geometry of flag-type and order q, as claimed in (B1).

Henceforth, given an element x of Γ, we denote by G_x its stabilizer in $G := \mathrm{Aut}(\Gamma)$ and by K_x the elementwise stabilizer of $\mathrm{Res}(x)$ in G_x. So, G_x/K_x is the group induced by G_x in $\mathrm{Res}(x)$. If $x, y, z, ...$ are elements of Γ, we put $G_{x,y} := G_x \cap G_y$, $G_{x,y,z} := G_x \cap G_y \cap G_z$, etc. By Delandtsheer [11] (see also [9] and [10]) we have the following:

(1) For every 3-element α of Γ, $\mathrm{Res}(\alpha)$ is isomorphic to the inversive plane associated with the elliptic quadric $Q_3^-(q)$, and $PSL(2,q^2) \le G_\alpha/K_\alpha \le P\Gamma L(2,q^2)$.

Consequently,

(2) $PSL(2,q) \leq G_{\alpha,A}/K_\alpha \leq P\Gamma L(2,q)^{\cdot} Z_n$ for every $\{2,3\}$-flag $\{A,\alpha\}$, where Z_n is a cyclic group of order n. (Recall that, according to the conventions stated at the beginning of this section, n is the exponent of $q = p^n$ as a power of p.)

(3) $\mu = 1$.

(Proof of (3).) Given two 3-elements α and β with $\alpha \sim^* \beta$, put $\nu := |\sigma_3(\alpha,\beta)|$. No two elements $A, B \in \sigma_3(\alpha,\beta)$ can have any 0-element in common. Indeed, if $x \in \sigma_0(A,B)$, then A and B are distinct lines of the bi-affine geometry $\mathrm{Res}(x)$ contained in two distinct planes α, β of $\mathrm{Res}(x)$, which is impossible. Since no two elements of $\sigma_3(\alpha,\beta)$ have any 0-element in common, $\nu \leq q-1$. Given $A \in \sigma_3(\alpha,\beta)$, we have:

$$(i) \quad |G_{\alpha,A} : G_{\alpha,\beta,A}| \leq q-1.$$

(Indeed $q-1$ is the number of 3-elements on A different from α.) However, it is well known that $PSL(2,q)$ does not admit any proper subgroup of index less than q (see Huppert [14]). Hence Claim (2) and (i) force the group $X := G_{\alpha,\beta,A}K_\alpha/K_\alpha$ to contain a copy L of $PSL(2,q)$. On the other hand, $|G_{\alpha,\beta} : G_{\alpha,\beta,A}| \leq \nu \leq q-1$. Hence X has index at most ν in $Y := G_{\alpha,\beta}K_\alpha/K_\alpha$. Therefore the subgroup $L \cong PSL(2,q)$ of X has index at most $\nu\delta$ in Y, where δ is the index of L in X. In view of claim (2), $\delta \leq (q-1)n^2/\theta$, where $\theta := |G_{\alpha,A} : G_{\alpha,\beta,A}|$. As $\nu \leq q-1$,

$$(ii) \quad |Y : L| \leq (q-1)^2 n^2/\theta.$$

Consequently, Y does not contain $PSL(2,q^2)$. On the other hand, L is maximal in $PSL(2,q^2)$. Hence

$$(iii) \quad Y \cap PSL(2,q^2) = L.$$

By (ii) and (iii), Y is a subgroup of the stabilizer $P\Gamma L(2,q)^{\cdot} Z_n$ of A in $P\Gamma L(2,q^2) = \mathrm{Aut}(\mathrm{Res}(\alpha))$. Therefore Y stabilizes A. The same conclusion holds if we replace A with any other element of $\sigma_3(\alpha,\beta)$. Hence Y stabilizes every element of $\sigma_3(\alpha,\beta)$. As $L \leq Y$, the same holds for L. However, $L \cong PSL(2,q)$ stabilizes exactly one block of the inversive plane $\mathrm{Res}(\alpha) \cong \mathcal{I}(O)$. In order to avoid a contradiction, we must conclude that $\nu = 1$, namely $|\sigma_3(\alpha,\beta)| = 1$. So, $\mathcal{S}(\Gamma)$ is a semi-linear space. By Lemma 4.2, $\mu = 1$.

As $\mu = 1$, we obtain (B3) from Lemma 4.1 and Γ satisfies the Intersection Property IP (as remarked in the proof of Lemma 4.3). As $\mu = 1$, every 1-element is uniquely determined by its pair of 0-elements. If x, y are the two 0-elements of a 1-element e, we write $e = xy$.

(4) $K_x = 1$ for every 0-element x.

(Proof of (4).) K_x stabilizes all 0-elements of $\mathcal{G}(\Gamma)$ except possibly those that belong to the same class as x. Therefore, and since $\mu = 1$, given $y \in x^\perp$, K_xK_y/K_y is seen to stabilize all points of the biaffine geometry $\mathrm{Res}(y)$, except possibly those that belong to a distinguished maximal coclique of the collinearity graph of $\mathrm{Res}(y)$. Clearly, this forces K_xK_y/K_y to fix all points of $\mathrm{Res}(y)$. Hence $K_x \leq K_y$. By symmetry, $K_y \leq K_x$. Therefore, $K_x = K_y$. The connectedness of $\mathcal{G}(\Gamma)$ now implies that $K_x = 1$.

With x as above, we have $\mathrm{Aut}(\mathrm{Res}(x)) = [U_x{:}(Z_{q-1} \times PGL(2,q))]Z_n$, where U_x is a group of order q^5, isomorphic to the multiplicative group formed by the following matrices:

$$\begin{bmatrix} 1 & r_1 & r_2 & t \\ 0 & 1 & 0 & s_1 \\ 0 & 0 & 1 & s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (r_1, r_2, s_1, s_2, t \in GF(q))$$

The center $Z(U_x)$ of U_x is elementary abelian of order q and $U_x/Z(U_x)$ is elementary abelian of order q^4. It is not difficult to see that every flag-transitive subgroup of $\mathrm{Aut}(\mathrm{Res}(x))$ contains U_x and a complement $L_x \cong SL(2,q)$ of U_x. By (6), G_x acts faithfully in $\mathrm{Res}(x)$. Therefore,

(5) $U_x{:}L_x \leq G_x \leq [U_x{:}(Z_{q-1} \times GL(2,q))]Z_n$.

Let $\mathcal{C}_x$ be the set of maximal cocliques of the collinearity graph of $\mathrm{Res}(x)$ and H_x be the elementwise stabilizer of $\mathcal{C}_x$. Put $T_x := U_x \cap T_x$. The following is straightforward:

(6) H_x is a Frobenius group with T_x as the Frobenius kernel and cyclic complements of order a divisor of $q-1$. Moreover:

(6.1) T_x is elementary abelian of order q^3 and acts regularly on the set of planes of $\mathrm{Res}(x)$.

(6.2) $Z(U_x)$ is contained in T_x and acts regularly on each member of $\mathcal{C}_x$.

(6.3) Every complement C of T_x in H_x stabilizes a unique plane α_C of $\mathrm{Res}(x)$. The group C fixes α_C elementwise and acts semi-regularly on the set of points of $\mathrm{Res}(x)$ not in α_C and on the set of planes of $\mathrm{Res}(x)$ different from α_C. Every plane of $\mathrm{Res}(x)$ is stabilized by a unique complement of T_x.

(6.4) For a point e of $\mathrm{Res}(x)$, put $H_{x,e} := H_x \cap G_e$ and $T_{x,e} := T_x \cap G_e$. Then $T_x = T_{x,e} \times Z(U_x)$ and $H_{x,e} = T_{x,e}{:}C$ for a complement C of T_x. The set of points of $\mathrm{Res}(x)$ fixed by $T_{x,e}$ is the union of q members of $\mathcal{C}_x$ and meets every plane of $\mathrm{Res}(x)$ in a line.

Let H be the elementwise stabilizer of the set of classes of the (q^2+1)-partition of $\mathcal{G}(\Gamma)$. Clearly, $H_x = H \cap G_x$ for every 0-element x. Hence H contains every complement C of T_x in H_x. Put $\gamma := |C|$. We recall that γ is a divisor of $q-1$.

(7) H has order $q^4\gamma$ and acts transitively on every class of the (q^2+1)-partition of $\mathcal{G}(\Gamma)$.

(Proof of (7).) $T_x \leq H$ for every 0-element x. As T_x contains $Z(U_x)$, which acts regularly on every member of $\mathcal{C}_x$, H is transitive on every class of $\mathcal{G}(\Gamma)$, except possibly the class containing x. However, x is an arbitrary 0-element of Γ. Hence H is transitive on every class of $\mathcal{G}(\Gamma)$. Therefore $|H| = q^4\gamma$, since $|H_x| = q^3\gamma$ and every class of $\mathcal{G}(\Gamma)$ has size q. Claim (7) is proved.

(8) H admits a normal subgroup T of order q^4 and $H = T{:}C$ for any complement C of T_x in H_x and every 0-element x. Accordingly, $T = O_p(H)$.

(Proof of (8).) Let T be a Sylow p-subgroup of H. We have $|T| = q^4$ by (7). Moreover, T contains T_x for some 0-element x. Let y be a 0-element adjacent to x. By (6.4), $|T_x \cap T_y| = q^2$. Therefore $\langle T_x, T_y\rangle$ has order at least q^4. Suppose first that we can choose y in such a way that $T_y \leq T$. Then, as $|\langle T_x, T_y\rangle| \geq q^4 = |T|$, we have $T = \langle T_x, T_y\rangle$. Pick a 3-element $\alpha \in \sigma_3(x,y)$ and put $C := G_\alpha \cap H$. Then $C \in G_z$ for every $z \in \sigma_0(\alpha)$. In particular, $C \leq H_x \cap H_y$. In view of (6.3), C is a complement of T_x in H_x as well as a complement of T_y in H_y. Hence C normalizes either of T_x and T_y. Consequently, C normalizes T. Hence T is normal in H and we have $H = TC$. Suppose now that $T_y \not\leq T$ for every 0-element $y \in x^\perp$. Then, given $\alpha \in \sigma_3(x)$, no conjugate of T contains two subgroups T_y and T_z for two distinct $y, z \in \sigma_0(\alpha)$. Therefore T admits at least $q^2 + 1$ conjugates in H. However, this is impossible, as $|H : T| = \gamma \leq q - 1$. Claim (8) is proved.

(9) The subgroup $T = O_p(H)$ acts regularly on the set of 3-elements of Γ.

Indeed, T has order q^4, which is the number of 3-elements of Γ, and $T \cap G_\alpha = 1$ for every 3-element α, by (6). Claim (9) finishes the proof of the lemma. □

Theorem 4.5 *Assume the following:*

(A) *for every* 0*-element* x *of* Γ*, the* $Af.Af^*$*-geometry* $\mathrm{Res}(x)$ *is nearly classical;*

(B) $\mathrm{Aut}(\Gamma)$ *is flag-transitive;*

(C) q *is prime.*

Then Γ *is isomorphic to the dual of the affine expansion* $\mathrm{Af}_e(\mathcal{I}^*)$ *of the dual* $\mathcal{I}^*$ *of a classical inversive plane* $\mathcal{I}$*, where* e *is the projective embedding of* $\mathcal{I}^*$ *in* $PG(3,q)$.

Proof. We have $\mu = 1$ by Lemma 4.4. We keep the notation used in the proof of Lemma 4.4. In particular, T_x, C and T are defined as in claims (6) and (8) of that proof. We first prove that T is elementary abelian.

(1) $T_x \trianglelefteq T$ for any 0-element x.

Indeed, as T is a p group, $N_T(T_x) > T_x$. However, $|T : T_x| = q$, which is assumed to be prime. Hence $N_T(T_x) = T$.

(2) T is elementary abelian.

Indeed, by (1) and the commutativity of T_x and T_y, the commutator subgroup T' of T is contained in $T_x \cap T_y$, for any two adjacent 0-elements x and y. Therefore $T' \leq T_x$ for any 0-element x. This forces $T' = 1$. Moreover, $T = \langle T_x, T_y\rangle$ and T_x and T_y are elementary abelian. Therefore T is elementary abelian.

(3) $T_x = T_y$ for any two non-adjacent 0-elements x and y.

Indeed, denoted by X be the class of the (q^2+1)-partition of $\mathcal{G}(\Gamma)$ containing x, T_x acts trivially on $X \setminus \{x\}$, since every orbit of T_x has order at least q (which is prime), whereas $|X \setminus \{x\}| = q - 1$.

In view of (3), given a 3-element α and a 0-element x, $T_x = T_{x_\alpha}$ for a unique 0-element x_α of α. Therefore

$$(4) \quad \{T_x\}_{x\in(\Gamma)_0} = \{T_x\}_{x\in\sigma_0(\alpha)},$$

where $(\Gamma)_0$ stands for the set of 0-elements of Γ. For every $x \in \sigma_0(\alpha)$, let $\mathcal{S}_x$ be the family of subgroups of T_x of order q. In view of claim (6.4) of the proof of Lemma 4.4, $Z(U_x)$ is the unique member of $\mathcal{S}_x$ that acts semi-regularly on the point-set of $\mathrm{Res}(x)$, whereas each the remaining members of $\mathcal{S}_x$ fixes all points of a line of $\mathrm{Res}(x)$ contained in α and moves each of the remaining $q^2 - q$ points of $\mathrm{Res}(x)$ beloning to α. We call $Z(U_x)$ a *special* subgroup of T.

$$(5) \quad T = \cup_{x\in\sigma_0(\alpha)} T_x.$$

(Proof of (5).) With $\mathcal{S}_x$ defined as above, we have $|\mathcal{S}_x| = q^2 + q + 1$, since T_x is elementary abelian of order q^3. The special subgroup of T_x is the unique member of $\mathcal{S}_x$ that is not contained in T_y for any $y \in \sigma_0(\alpha) \setminus \{x\}$. Each of the $q^2 + q$ remaining members of $\mathcal{S}_x$ is contained in T_y for q choices of y in $\sigma_0(\alpha)\setminus\{x\}$. Therefore $\cup_{x\in\sigma_0(\alpha)}\mathcal{S}_x$ contains exactly $(q^2+1)+(q^2+1)(q^2+q)/(q+1) = q^3+q^2+q+1$ subgroups. However, as q is prime, q^3+q^2+q+1 is just the number of subgroups of T of order q. Equality (5) follows.

We now turn to $\mathcal{A}(\Gamma)$, which is a linear space by Lemma 4.3 (indeed $\mu = 1$). We denote the point-set of $\mathcal{A}(\Gamma)$ by P. Namely, P is the set of 3-elements of Γ. By Lemma 4.4, T acts regularly on P. Thus, given a point $\alpha \in P$, a bijection τ is established from T to P, sending every $t \in T$ to the image α^t of α by t. Moreover T, being elementary abelian of order q^4, can be regarded as the additive group of $V = V(4, q)$.

(6) τ induces an isomorphism from the affine space $AG(V) = AG(4, q)$ to $\mathcal{A}(\Gamma)$.

To see this, we only must prove that, for every 1-dimensional linear subspace S of V, the orbit α^S of α by S is a line of $\mathcal{A}(\Gamma)$. By (5), $S \leq T_x$ for at least one 0-element x of α. Turning to $\mathrm{Res}(x)$, we easily see that claim (6) holds true (compare (6.4) of the proof of Lemma 4.4). The following is implicit in the proof of claim (5):

(7) For a 1-dimensional linear space S of V, the line $\tau(S)$ of $\mathcal{A}(\Gamma)$ is new if and only if S, regarded as a subgroup of T, is special.

As remarked in the proof of claim (5), $q^2 + 1$ is the number of special subgroups of T. Therefore,

(8) $\mathcal{A}(\Gamma)$ contains $(q^2 + 1)q^3$ new lines, forming $q^2 + 1$ bundles of parallel lines.

We say that a new line L belongs to a 0-element x if all points of L, regarded as 3-elements of Γ, are incident with x. The following is clear:

(9) every bundle of parallel new lines of $\mathcal{A}(\Gamma)$ is contained in exactly one 0-element x and each such element contains exactly one bundle of parallel new lines.

In particular, every new line on the distinguished point α belongs to a unique 0-element of α and each of these 0-elements contains exactly one new line through α. Accordingly, the action of G_α on the set of new lines through α is isomorphic to its action on $\sigma_0(\alpha)$. Let $\widehat{\mathcal{A}}_3(\Gamma)$ be $\mathcal{A}(\Gamma)$ enriched with its planes and 3-subspaces.

(10) No three new lines on α are coplanar in $\widehat{\mathcal{A}}_3(\Gamma)$.

(Proof of (10).) Suppose the contrary. Then, by (8), (9) and claim (1) of the proof of Lemma 4.4, every plane of $\widehat{\mathcal{A}}_3(\Gamma)$ on α contains 0, 1 or s new lines, where either $s = q+1$ or $s = 2+(q-1)/2 = (q+3)/2$. Therefore, the q^2+1 new lines on α form a linear space with all lines of size s. It is easily seen that no such linear space can exist. Claim (10) is proved.

The residue of α in $\widehat{\mathcal{A}}_3(\Gamma)$ is a 3-dimensional projective space. We shall denote it by $\mathcal{P}_\alpha$. In view of (11), the new lines on α form an ovoid O^* in $\mathcal{P}_\alpha$. As q is prime, the ovoid O^* is classical and the set $O := \sigma_0(\alpha)$ is its dual. It is now clear that Γ is isomorphic to the dual of the affine expansion of the dual of the inversive plane $\mathcal{I}(O)$ associated to O. □

The next Corollary is claim (1) of Theorem 1.

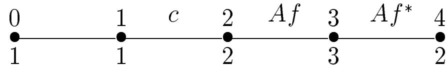
Corollary 4.6 *Assume $q = 3$ and suppose that* Aut(Γ) *is flag-transitive. Then Γ is isomorphic to the dual of the affine expansion* $\mathrm{Af}_e(\mathcal{I}^*)$ *of the dual $\mathcal{I}^*$ of the inversive plane $\mathcal{I}$ of order* 3*, where e is the projective embedding of $\mathcal{I}^*$ in $PG(3,3)$.*

Proof. As Γ is flag-transitive, Γ has uniform multiplicity μ. If we prove that the residues of the 0-elements of Γ are nearly classical, then the conclusion follows from Theorem 4.5. Let $\Delta = \mathrm{Res}(x)$ for a 0-element x of Γ and λ be the index of Δ. By Proposition 3.4, $\lambda \in \{1,2,3\}$. Hence Δ is nearly classical, by propositions 3.3 and 3.6. (In fact, Δ is classical.) □

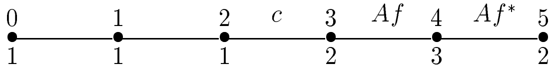
5 Proof of Theorem 1

As claim (1) of Theorem 1 has been settled by Corollary 4.6, we only must prove claims (2) and (3) of that theorem. So, in this section we consider a flag-transitive $Af.Af^*.c^{*n}$-geometry with $n = 2$ and orders 2, 3, 2, 1, 1 or $n = 3$ and orders 2, 3, 2, 1, 1, 1. However, we prefer to focus on the dual of such a geometry. Thus, henceforth Γ is a flag-transitive geometry of rank $n+3$ with diagram, orders and types as follows:

$(n = 2)$



$(n = 3)$



For an element x of Γ and a type $i = 0, 1, ..., n-1$ we denote by $\sigma_i(x)$ the set of i-elements incident with x. We also put $\sigma_i(x, y) := \sigma_i(x) \cap \sigma_i(y)$ and $\sigma_i(x, y, z) := \sigma_i(x) \cap \sigma_i(y) \cap \sigma_i(z)$.

As in Section 4, $\mathcal{G}(\Gamma)$ is the collinearity graph of Γ where the elements of type 0 and 1 are taken as points and lines respectively, and $\sim$ is the adjacency relation of $\mathcal{G}(\Gamma)$. As Γ is assumed to be flag-transitive, all edges of $\mathcal{G}(\Gamma)$ are incident with the same number μ of 1-elements, namely $\mu = |\sigma_1(x, y)|$ for every edge $\{x, y\}$ of $\mathcal{G}(\Gamma)$. We call μ the *multiplicity* of Γ.

We denote by $\mathcal{S}(\Gamma)$ the point-line geometry with the $(n+2)$-elements of Γ as points and the $(n+1)$-elements as lines. $\mathcal{G}^*(\Gamma)$ is the collinearity graph of $\mathcal{S}(\Gamma)$ and $\sim^*$ is the adjacency relation of $\mathcal{G}^*(\Gamma)$. For an edge $\{\alpha, \beta\}$ of $\mathcal{G}^*(\Gamma)$, we put $\mu^*(\alpha, \beta) := |\sigma_{n+1}(\alpha, \beta)|$.

For the rest of this section G is a given flag-transitive subgroup of $\mathrm{Aut}(\Gamma)$. For an element x of Γ, G_x is the stabilizer of x in G and K_x is the elementwise stabilizer of $\mathrm{Res}(x)$ (compare the notation used in the proof of Lemma 4.4).

5.1 The case of $n = 2$

In this subsection, $n = 2$. So, Γ has rank $n + 3 = 5$. By Delandtsheer [11],

Lemma 5.1 *Let α be a 4-element of Γ. Then $\mathrm{Res}(\alpha)$ is isomorphic to the Steiner system $S(11, 5, 4)$ for the Mathieu group M_{11}, where the points, duads, triples and blocks correspond to the elements of $\mathrm{Res}(\alpha)$ of type 0, 1, 2 and 3, respectively. Furthermore, $G_\alpha/K_\alpha = \mathrm{Aut}(\mathrm{Res}(\alpha)) \cong M_{11}$.* □

Moreover, by Corollary 4.6,

Lemma 5.2 *The residues of the 0-elements of Γ are isomorphic to the dual of the affine expansion $\mathrm{Af}_e(\mathcal{I}^*)$ of the dual $\mathcal{I}^*$ of the classical inversive plane $\mathcal{I}$ or order 3, where e is the projective embedding of $\mathcal{I}^*$ in $PG(3, 3)$.* □

Lemma 5.3 $\mu = 1$.

Proof. Let $\{x, y\}$ be an edge of $\mathcal{G}(\Gamma)$. The μ elements of $\sigma_1(x, y)$ form a coclique of the graph $\mathcal{G}(\mathrm{Res}(x))$. Moreover, given a maximal coclique C of $\mathcal{G}(\mathrm{Res}(x))$, the relation 'being incident to the same pair of 0-elements' defined on the set of 1-elements of Γ induces an equivalence relation on C and all classes of that induced equivalence relation have size μ. On the other hand, by Lemma 5.2, the maximal cocliques of $\mathcal{G}(\mathrm{Res}(x))$ have size 3. Hence $\mu \in \{1, 3\}$.

Suppose $\mu = 3$. Then every 4-element incident with x is also incident with one of the three elements of $\sigma_1(x, y)$. Hence $\sigma_4(x) = \sigma_4(y)$. By the connectedness of $\mathcal{G}(\Gamma)$, all 4-elements are incident with all 0-elements, namely Γ is flat. Consequently, denoted by N_i the number of i-elements of Γ, we have:

$$N_0 = 11, \quad N_1 = N_0 \cdot 10 \cdot 3/2 = 3 \cdot 5 \cdot 11, \quad N_2 = N_1 \cdot 3^3/3 = 3^3 \cdot 5 \cdot 11,$$
$$N_3 = N_2(3^2 + 3)/10 = 2 \cdot 3^4 \cdot 11, \quad N_4 = 3^4.$$

Let $\{\alpha, \beta\}$ be an edge of $\mathcal{G}^*(\Gamma)$. By Lemma 5.2, the residues of the 0-elements of Γ satisfy the Intersection Property (IP). Therefore,

(1) $\sigma_0(A, B) = \emptyset$ for any two distinct 3-elements $A, B \in \sigma_3(\alpha, \beta)$.

Consequently:

(2) $\mu^*(\alpha, \beta) \leq 2$.

For $i = 1, 2$, let $S_i(\alpha)$ be the set of 4-elements β adjacent with α in $\mathcal{G}^*(\Gamma)$ and such that $\mu^*(\alpha, \beta) = i$. By the flag-transitivity of Γ, the number $|\sigma_4(A) \cap S_i(\alpha)|$ does not depend on the choice of the 3-element $A \in \sigma_3(\alpha)$. We must examine the following three cases:

Case 1. $|\sigma_4(A) \cap S_1(\alpha)| = 2$ and $\sigma_4(A) \cap S_2(\alpha) = \emptyset$.
Case 2. $|\sigma_4(A) \cap S_1(\alpha)| = |\sigma_4(A) \cap S_2(\alpha)| = 1$.
Case 3. $|\sigma_4(A) \cap S_2(\alpha)| = 2$ and $\sigma_4(A) \cap S_1(\alpha) = \emptyset$.

(3) Cases 1 and 2 are impossible.

(Proof of (3).) In Case 1, $|S_1(\alpha)| = 2|\sigma_3(\alpha)|$. However, $|\sigma_3(\alpha)| = 66$, which is the number of blocks of $S(11, 5, 4)$ (compare Lemma 5.1). So, $|S_1(\alpha)| = 2 \cdot 66 > 81 = N_4$. This is a contradiction.

Suppose we have Case 2. Let β and γ be the 4-elements of $S_1(\alpha) \cap \sigma_4(A)$ and $S_2(\alpha) \cap \sigma_4(A)$, respectively. By replacing α with γ, we have $\alpha \in S_2(\gamma) \cap \sigma_4(A)$. Hence $\beta \in S_1(\gamma) \cap \sigma_4(A)$, since Case 2 also occurs for the flag $\{A, \gamma\}$ (recall that Γ is flag-transitive). However, $\alpha, \gamma \in S_1(\beta) \cap \sigma_4(A)$. This is a contradiction with the fact that, since Γ is flag-transitive, Case 2 also occurs for the flag $\{A, \beta\}$. Claim (3) is proved.

So, Case 3 holds. Given a 3-element $A \in \sigma_3(\alpha)$, let β and γ be the two 4-elements of $\sigma_4(A) \setminus \{\alpha\}$. One of the following occurs:

Case 3.1. $\sigma_3(\alpha, \beta) = \sigma_3(\alpha, \gamma)$.
Case 3.2. $\sigma_3(\alpha, \beta) \neq \sigma_3(\alpha, \gamma)$.

We shall prove that neither of the above two cases is possible, thus finishing the proof of the lemma. By Lemma 5.1, we have $G_{\alpha,A}/K_\alpha \cong \mathrm{Sym}(5)$, acting 2-transitively as $PGL(2, 5)$ on the complement $\sigma_0(\alpha) \setminus \sigma_0(A)$ of the block $\sigma_0(A)$ of $\mathrm{Res}(\alpha)$, which has size 6. On the other hand, $G_{\alpha,A}$ stabilizes the pair $\{\beta, \gamma\} = \sigma_4(A) \setminus \{\alpha\}$. In Case 3.1 we have $\sigma_3(\alpha, \beta) = \sigma_3(\alpha, \gamma) = \{A, B\}$ and $G_{\alpha,A}/K_\alpha$ stabilizes B. As $\sigma_0(B)$ is disjoint from $\sigma_0(\alpha)$ (by (1)) and $|\sigma_0(B)| = 5$, $G_{\alpha,A}$ cannot induce a transitive action on $\sigma_0(\alpha) \setminus \sigma_0(A)$, contrary to what we have said in the previous paragraph. On the other hand, in Case 3.2 we have $\sigma_3(\alpha, \beta) = \{A, B\}$ and $\sigma_3(\alpha, \gamma) = \{A, C\}$ where $B \neq C$ and $\sigma_0(A, B) = \sigma_0(A, C) = \emptyset$. Therefore $G_{\alpha,A}$ stabilizes $\sigma_0(B, C) \subset \sigma_0(\alpha) \setminus \sigma_0(A)$. Consequently, it cannot act transitively on $\sigma_0(\alpha) \setminus \sigma_0(A)$. In any case, we have obtained a contradiction. Hence Case 3 is impossible, too. □

Corollary 5.4 Γ *satisfies the Intersection Property* (IP)*. In particular,* $\mathcal{S}(\Gamma)$ *is a semi-linear space.*

Proof. This follows from Lemmas 5.2 and 5.3 via [18, Lemma 7.25]. □

Given an edge $\{x, y\}$ of $\mathcal{G}(\Gamma)$, we denote by xy the 1-element of $\sigma_1(x, y)$ (unique by lemma 5.3). Note that, in view of property IP (which holds in Γ by Corollary 5.4), given a 0-element x and two edges $\{x, y\}$ and $\{x, z\}$ of $\mathcal{G}(\Gamma)$ on x, we have $\sigma_2(xy) \cap \sigma_2(xz) = \sigma_2(x, y, z)$. The next lemma can be proved by arguments similar to those used to prove claims (B3) and (B4) of Lemma 4.1. We leave the details for the reader.

Lemma 5.5 *The graph* $\mathcal{G}(\Gamma)$ *is a complete* 11*-partite graph with classes of size* 3*. Accordingly, if* N_i *is the number of* i*-elements of* Γ $(i = 0, 1, 2, 3, 4)$*, then*

$$N_0 = 3 \cdot 11, \quad N_1 = 3^2 \cdot 5 \cdot 11, \quad N_2 = 3^4 \cdot 5 \cdot 11, \quad N_3 = 2 \cdot 3^5 \cdot 11, \quad N_4 = 3^5.$$

Moreover:
(1) *for every* 4*-element* α*,* $\sigma_0(\alpha)$ *meets each class of the* 11*-partition of* $\mathcal{G}(\Gamma)$*;*
(2) *for every* 0*-element* x *and any two edges* $\{x, y\}$*,* $\{x, z\}$ *of* $\mathcal{G}(\Gamma)$ *on* x*, we have* $\sigma_2(x, y, z) \neq \emptyset$ *if and only if* y *and* z *are adjacent in* $\mathcal{G}(\Gamma)$*.* □

We define the point-line geometry $\mathcal{A}(\Gamma)$ as follows: The points of $\mathcal{A}(\Gamma)$ are those of $\mathcal{S}(\Gamma)$. The lines of $\mathcal{A}(\Gamma)$ are the lines of $\mathcal{S}(\Gamma)$ and the new lines of the affine space $\mathcal{A}(\mathrm{Res}(x))$, for any 0-element x of Γ. The latter lines will be called *new lines* of $\mathcal{A}(\Gamma)$, whereas the lines of $\mathcal{S}(\Gamma)$ are the *old lines* of $\mathcal{A}(\Gamma)$.

Lemma 5.6 *For every new line* L *of* $\mathcal{A}(\Gamma)$*, we have* $|L \cap \sigma_4(e)| > 1$ *for exactly one* 1*-element* e*. Moreover, if* e *is the* 1*-element such that* $|L \cap \sigma_4(e)| > 1$*, then* $L \subset \sigma_4(e)$*.*

Proof. A 1-element e with $L \subset \sigma_4(e)$ exists by definition. Let $|L \cap \sigma_4(f)| > 1$ for a 1-element f. We shall prove that $f = e$. Suppose to the contrary that $f \neq e$. Then, by IP, $\sigma_4(f) \cap L \subseteq \sigma_4(X)$ for an element X of type 2 or 3, incident with both e and f. In particular, $|L \cap \sigma_4(Y)| > 1$ for a 2-element $Y \in \sigma_2(e)$. However, by definition, L is a new line of $\mathcal{A}(\mathrm{Res}(x))$ for a 0-element x. With no loss, we may assume $x \in \sigma_0(e)$. By the definition of $\mathcal{A}(\mathrm{Res}(x))$, no new line of $\mathcal{A}(\mathrm{Res}(x))$ meets $\sigma_4(Y)$ in two elements, for any $Y \in \sigma_2(x)$. We have reached a contradiction. □

We can now imitate the proof of Lemma 4.3, obtaining the following:

Corollary 5.7 *The geometry $\mathcal{A}(\Gamma)$ is a linear space with 3^5 points and the same parameters as $AG(5,3)$.* □

The next lemma can be proved by an argument similar to that exploited for claim (4) in the proof of Lemma 4.4.

Lemma 5.8 $K_x = 1$ *for every* 0*-element* x. □

Let H be the elementwise stabilizer of the set of classes of the 11-partition of $\mathcal{G}(\Gamma)$. For a 0-element x, we set $H_x := H \cap G_x$. The structure of H_x is clear by Lemmas 5.8 and 5.2: H_x is a Frobenius group with elementary abelian Kernel T_x of order 3^4 and complement C of size $|C| \leq 2$. We can now imitate the arguments used to prove claims (7)-(9) of the proof of Lemma 4.4 and (2), (3) of the proof of Theorem 4.5, obtaining the following:

Lemma 5.9 *We have $H = T{:}C$ where $T \trianglelefteq H$ is elementary abelian of order 3^5 and C is a complement of T_x in G_x, for a* 0*-element x of Γ. In particular, $T = O_3(H)$ and $|C| \leq 2$. Moreover:*
(1) *H acts transitively on every class of the* 11*-partition of $\mathcal{G}(\Gamma)$.*
(2) *The subgroup $T = O_p(H)$ acts regularly on the set of* 4*-elements of Γ, whereas C stabilizes a* 4*-element.*
(3) *$T = \langle T_x, T_y \rangle$ for any two adjacent* 0*-elements x and y.*
(4) *$T_x = T_y$ for any two non-adjacent* 0*-elements x and y. Consequently, $\{T_x\}_{x \in (\Gamma)_0} = \{T_x\}_{x \in \sigma_0(\alpha)}$ for every* 4*-element α, where $(\Gamma)_0$ stands for the set of* 0*-elements of Γ.* □

For a 0-element x we denote by $\mathcal{S}_x$ the family of minimal subgroups of T_x, namely subgroups of order 3. Given a 4-element α, for every $x \in \sigma_0(\alpha)$ we put $\mathcal{S} := \cup_{x \in \sigma_0(\alpha)} \mathcal{S}_x$ $(= \cup_{x \in (\Gamma)_0} \mathcal{S}_x$, by (3) of Lemma 5.9). By definition, every line of $\mathcal{A}(\Gamma)$ through α belongs to $\mathcal{A}(\mathrm{Res}(x))$ for a suitable $x \in \sigma_0(\alpha)$. Hence every such line is stabilized by a member of $\mathcal{S}$ (uniquely determined, as T is regular on the point-set of $\mathcal{A}(\Gamma)$). On the other hand, $3^4 + 3^3 + 3^2 + 3 + 1$ is the number of lines of $\mathcal{A}(\Gamma)$ through α. Hence $\mathcal{S}$ is the family of all 3-subgroups of T, since $3^4 + 3^3 + 3^2 + 3 + 1$ is also the number of minimal subgroups of T. The following is now clear:

Lemma 5.10 *We have $\mathcal{A}(\Gamma) \cong AG(V)$, where $V = V(5,3)$ and T is the additive group of V.* □

We are now ready to finish the proof of claim (2) of Theorem 1. The isomorphism $\mathcal{A}(\Gamma) \cong AG(V)$ induces an embedding e of the dual $\mathrm{Res}(\alpha)^*$ of $\mathrm{Res}(\alpha)$ in the projective geometry $PG(V)$, where $PG(V)$ is regarded as the projective space of lines and planes of $\mathcal{A}(\Gamma)$ through the distinguished point α. Accordingly, Γ is the affine expansion of $\mathrm{Res}(\alpha)^*$ embedded in $PG(V)$ via e.

5.2 The case of $n = 3$

Assume $n = 3$. By Delandtsheer [11],

Lemma 5.11 *Let α be a 5-element of Γ. Then $\mathrm{Res}(\alpha)$ is isomorphic to the Steiner system $S(12,6,5)$ for the Mathieu group M_{12}, where the points, duads, triples and blocks correspond to the elements of $\mathrm{Res}(\alpha)$ of type 0, 1, 2 and 3, respectively. Furthermore, $G_\alpha/K_\alpha = \mathrm{Aut}(\mathrm{Res}(\alpha)) \cong M_{12}$.* □

Moreover, by claim (2) of Theorem 1,

Lemma 5.12 *Let x be a 0-element of Γ. Then $\mathrm{Res}(x)$ is isomorphic to the dual of the affine expansion $\mathrm{Af}_e(\Delta)$ of the dual Δ of $\Sigma = S(12,5,4)$, where e is the (unique) embedding of Δ in $PG(4,3)$.* □

With Δ and e as above, we have $\mathrm{Aut}(\mathrm{Af}_e(\Delta)) = 3^5{:}(2 \times M_{11})$ and $3^5{:}M_{11}$ is the minimal flag-transitive automorphism group of $\mathrm{Af}_e(\Delta)$. Therefore,

Corollary 5.13 *We have $3^5{:}M_{11} \leq G_x/K_x \leq 3^5{:}(2 \times M_{11})$ for every 0-element x of Γ.* □

The geometry $\mathrm{Af}_e(\Delta)$ can be recovered from its point-line system $\mathcal{S}(\mathrm{Af}_e(\Delta))$. Accordingly, the residue $\mathrm{Res}(x)$ of a 0-element x can be recovered from the semi-linear space $\mathcal{S}(\mathrm{Res}(x))$ of 5- and 4-elements incident with x. Consequently:

Corollary 5.14 *We have $\mathrm{Aut}(\mathcal{S}(\mathrm{Res}(x))) = \mathrm{Aut}(\mathrm{Res}(x)) = 3^5{:}(2 \times M_{11})$ for every 0-element x of Γ.* □

Lemma 5.15 *$\mu = 1$.*

Proof. We have $\mu \in \{1,3\}$, as in the proof of Lemma 5.3. Suppose $\mu = 3$. Then Γ is flat, as in the proof of Lemma 5.3. Consequently, denoted by N_i the number of i-elements of Γ, we have:
$N_0 = 12$, $N_1 = N_0 \cdot 11 \cdot 3/2 = 2 \cdot 3^2 \cdot 11$, $N_2 = N_1 \cdot 10 \cdot 3/3 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$,
$N_3 = N_2 \cdot 3^3/4 = 3^5 \cdot 5 \cdot 11$, $N_4 = N_3(3^2+3)/15 = 2^2 \cdot 3^5 \cdot 11$, $N_5 = 3^5$.

Let $\{\alpha,\beta\}$ be an edge of the graph $\mathcal{G}^*(\Gamma)$. As in the proof of Lemma 5.3, we obtain that $\sigma_0(A,B) = \emptyset$ for any two distinct 4-elements $A, B \in \sigma_4(\alpha,\beta)$. Hence $\mu^*(\alpha,\beta) \leq 2$. For $i = 1,2$, let $S_i(\alpha)$ be the set of 5-elements β adjacent with α in $\mathcal{G}^*(\Gamma)$ and such that $\mu^*(\alpha,\beta) = i$. By the flag-transitivity of Γ, the number $|\sigma_5(A) \cap S_i(\alpha)|$ does not depend on the choice of the 4-element $A \in \sigma_3(\alpha)$. One of the following occurs:

Case 1. $|\sigma_5(A) \cap S_1(\alpha)| = 2$ and $\sigma_5(A) \cap S_2(\alpha) = \emptyset$.

Case 2. $|\sigma_5(A) \cap S_1(\alpha)| = |\sigma_5(A) \cap S_2(\alpha)| = 1$.

Case 3. $|\sigma_5(A) \cap S_2(\alpha)| = 2$ and $\sigma_5(A) \cap S_1(\alpha) = \emptyset$.

Cases 1 and 2 can be ruled out by arguments similar to those used in the proof of Lemma 5.3 for the cases analogous to these. So, Case 3 holds. Given a 4-element $A \in \sigma_4(\alpha)$, let β and γ be the two 5-elements of $\sigma_5(A) \setminus \{\alpha\}$. Let B and C be the elements different from A in $\sigma_4(\alpha,\beta)$ and $\sigma_4(\alpha,\gamma)$ respectively. Then $\sigma_0(A)\cap\sigma_0(B) = \sigma_0(A)\cap\sigma_0(C) = \emptyset$. It follows that $\sigma_0(B) = \sigma_0(C)$, whence $B = C$. Accordingly, for every 5-element α, the sets $\sigma_5(A)$ for $A \in \sigma_4(\alpha)$ bijectively correspond to the partition of the design $\mathrm{Res}(\alpha)$ in two disjoint hexads. Therefore,

(1) For any two 4-elements A and B, we have $\sigma_5(A) = \sigma_5(B)$ if and only if $\sigma_5(A)\cap\sigma_5(B) \neq \emptyset$ and $\{\sigma_0(A),\sigma_0(B)\}$ is a partition of the set of 0-elements of Γ.

If $\sigma_5(A) = \sigma_5(B)$ for two 4-elements A and B, then we write $A \equiv B$. Clearly, $\equiv$ is an equivalence relation on the set of 4-elements of Γ and all classes of $\equiv$ have size 2. Moreover, by (1), for every 0-element x, $\sigma_4(x)$ meets every class of $\equiv$ in at most one element. On the other hand, $|\sigma_4(x)| = 66 = N_4/2$. Therefore,

(2) For every 0-element x, $\sigma_4(x)$ meets every class of $\equiv$ in exactly one element.

We define a semi-linear space $\widetilde{\mathcal{S}}(\Gamma)$ on the set of 5-elements of Γ by taking the classes of $\equiv$ as lines, with the convention that such a class $\{A,B\}$ and a 5-element α are incident precisely when $\alpha \in \sigma_5(A) = \sigma_5(B)$. Claim (2) implies the following:

(3) $\widetilde{\mathcal{S}}(\Gamma) \cong \mathcal{S}(\mathrm{Res}(x))$ for every 0-element x.

Let U be the elementwise stabilizer of $\widetilde{\mathcal{S}}(\Gamma)$. Then (3) and Corollaries 5.13 and 5.14 imply the following:

(4) $3^5{:}M_{11} \leq G/U \leq 3^5{:}(2\times M_{11})$.

(5) $U\cap G_x = K_x = 1$ for every 0-element x.

(Proof of (5).) We have $U\cap G_x \leq K_x$ since $U\cap G_x$ stabilizes all elements of $\mathcal{S}(\mathrm{Res}(x))$ and $\mathrm{Res}(x)$ can be recovered from $\mathcal{S}(\mathrm{Res}(x))$. The equality $K_x = 1$ remains to be proved. Clearly, K_x fixes all 0-elements. Given a 0-element $y \neq x$, Lemma 5.12 implies that $\mathcal{G}(\mathrm{Res}(y))$ is a complete 11-partite graph with all classes of size 3. One of those classes, say C, contains the three elements of $\sigma_1(x,y)$. The group K_xK_y/K_y fixes all classes of the 11-partition of $\mathcal{G}(\mathrm{Res}(y))$ and all elements of $\mathrm{Res}(y)$ incident with any of the 1-elements of C. This forces K_xK_y/K_y to be trivial. Hence $K_x \leq K_y$. By symmetry, $K_x = K_y$. Finally $K_x = 1$, since y is an arbitrary 0-element of Γ. Claim (5) is proved.

By (5), G_x acts faithfully in $\mathrm{Res}(x)$ and G contains a semi-direct product $\widehat{G} = U{:}G_x$. By (4) and Corollary 5.13 and 5.14, $\widehat{G}$ has index at most 2 in G. Moreover, $U \leq G_\alpha$, for every 5-element α. It follows that $G_\alpha = U{:}G_{x,\alpha}$ for $x \in \sigma_0(\alpha)$. On the other hand, G_α induces M_{12} on $\sigma_0(\alpha)$, by Lemma 5.11. This does not fit with the description of G_α as $U{:}G_{x,\alpha}$ (recall that $G_{x,\alpha}$ is isomorphic to either M_{11} or $2\times M_{11}$). We have reached a final contradiction. □

The proof now can be continued as in the case of $n = 2$. We only recall its main steps, leaving all proofs for the reader:

Corollary 5.16 Γ *satisfies the Intersection Property* (IP)*. In particular,* $\mathcal{S}(\Gamma)$ *is a semi-linear space.* □

Lemma 5.17 *The graph* $\mathcal{G}(\Gamma)$ *is a complete* 12*-partite graph with classes of size* 3*. Accordingly, if* N_i *is the number of* i*-elements of* Γ $(i = 0, 1, 2, 3, 4)$*, then*

$$N_0 = 12 \cdot 3 = 2^2 \cdot 3^3, \quad N_1 = 2 \cdot 3^3 \cdot 11, \quad N_2 = 2^2 \cdot 3^3 \cdot 5 \cdot 11,$$
$$N_3 = 3^6 \cdot 5 \cdot 11, \quad N_4 = 2^2 \cdot 3^6 \cdot 11, \quad N_5 = 3^6.$$

Moreover:
(1) *for every* 5*-element* α*,* $\sigma_0(\alpha)$ *meets each class of the* 12*-partition of* $\mathcal{G}(\Gamma)$*;*
(2) *for every* 0*-element* x *and two edges* $\{x, y\}$*,* $\{x, z\}$ *of* $\mathcal{G}(\Gamma)$ *on* x*, we have* $\sigma_2(x, y, z) \neq \emptyset$ *if and only if* y *and* z *are adjacent in* $\mathcal{G}(\Gamma)$*.* □

The point-line geometry $\mathcal{A}(\Gamma)$ is defined as in the case of $n = 2$: The points of $\mathcal{A}(\Gamma)$ are those of $\mathcal{S}(\Gamma)$. The lines of $\mathcal{A}(\Gamma)$ are the lines of $\mathcal{S}(\Gamma)$ and the new lines of the affine space $\mathcal{A}(\mathrm{Res}(x))$, for any 0-element x of Γ. The latter lines are the *new lines* of $\mathcal{A}(\Gamma)$, whereas the lines of $\mathcal{S}(\Gamma)$ are the *old lines* of $\mathcal{A}(\Gamma)$.

Lemma 5.18 *For every new line* L *of* $\mathcal{A}(\Gamma)$*, we have* $|L \cap \sigma_5(e)| > 1$ *for exactly one* 2*-element* e*. Moreover, if* e *is the* 2*-element such that* $|L \cap \sigma_5(e)| > 1$*, then* $L \subset \sigma_5(e)$*.* □

Corollary 5.19 $\mathcal{A}(\Gamma)$ *is a linear space with* 3^6 *points and the same parameters as* $AG(6, 3)$*.* □

Lemma 5.20 $K_x = 1$ *for every* 0*-element* x*.* □

Let H be the elementwise stabilizer of the set of classes of the 12-partition of $\mathcal{G}(\Gamma)$. For a 0-element x, we set $H_x := H \cap G_x$. Then H_x is a Frobenius group with elementary abelian Kernel T_x of order 3^5 and complement C of size $|C| \leq 2$.

Lemma 5.21 *We have* $H = T{:}C$ *where* $T \trianglelefteq H$ *is elementary abelian of order* 3^6 *and* C *is a complement of* T_x *in* G_x*, for a* 0*-element* x *of* Γ*. In particular,* $T = O_3(H)$ *and* $|C| \leq 2$*. Moreover:*
(1) H *acts transitively on every class of the* 12*-partition of* $\mathcal{G}(\Gamma)$*.*
(2) *The subgroup* $T = O_p(H)$ *acts regularly on the set of* 5*-elements of* Γ*, whereas* C *stabilizes a* 5*-element.*
(3) $T = \langle T_x, T_y \rangle$ *for any two adjacent* 0*-elements* x *and* y*.*
(4) $T_x = T_y$ *for any two non-adjacent* 0*-elements* x *and* y*. Consequently,* $\{T_x\}_{x \in (\Gamma)_0} = \{T_x\}_{x \in \sigma_0(\alpha)}$ *for every* 5*-element* α*, where* $(\Gamma)_0$ *stands for the set of* 0*-elements of* Γ*.* □

Lemma 5.22 *We have* $\mathcal{A}(\Gamma) \cong AG(V)$*, where* $V = V(6, 3)$ *and* T *is the additive group of* V*.* □

The isomorphism $\mathcal{A}(\Gamma) \cong AG(V)$ induces an embedding e of the dual $\mathrm{Res}(\alpha)^*$ of $\mathrm{Res}(\alpha)$ in the projective geometry $PG(V)$, where $PG(V)$ is regarded as the projective space of lines and planes of $\mathcal{A}(\Gamma)$ through the distinguished point α. Accordingly, Γ is the affine expansion of $\mathrm{Res}(\alpha)^*$ embedded in $PG(V)$ via the embedding e. This finishes the proof of Theorem 1.

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