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Nesting kite and 4-cycle systems

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Abstract

The idea of nesting Steiner triple systems is generalized to kite and 4-cycle systems. A complete solution is given for both kite systems and 4-cycle systems.

1 Introduction

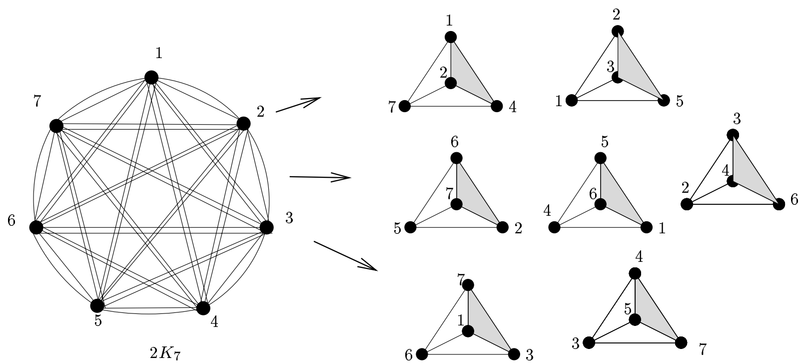
A *Steiner triple system* is a pair (S,T), where T is a collection of edge disjoint triangles which partitions the edge set of K_n (= the complete undirected graph on n vertices) with vertex set S. The number n is called the *order* of the Steiner triple system (S,T). It is well-known that the *spectrum* for Steiner triple systems is precisely the set of all $n \equiv 1$ or $3 \pmod 6$ [2] and that if (S,T) is a triple system $|T| = n(n-1)/6$. A 2-fold block design with block size 4 is a pair (S,B), where B is a collection of edge disjoint copies of K_4 (called *blocks*) which partitions the edge set of $2K_n$ (= 2 copies of K_n) with vertex set S. From now on "block design", unless stated otherwise, means block size 4. As with Steiner triple systems, it is well known [1] that the spectrum for 2-fold block designs is precisely the set of all $n \equiv 1 \pmod 3$, and that if (S,B) is a block design of order n the number of blocks is $|B| = n(n-1)/6$.

Now it doesn't take the wisdom of a saint to notice that if (S, T) is a Steiner triple system of order n and (S, B) is a 2-fold block design of order n, that $|T| = n(n-1)/6 = |B|$. Hence the following problem.

THE NESTING PROBLEM FOR STEINER TRIPLE SYSTEMS

Determine all n such that there exists a 2-fold block design (S, B) of order n with the property that a triangle can be selected from each block in B so that the resulting collection of triangles T is a Steiner triple system (S, T) of order n. The Steiner triple system (S, T) is said to be *nested* in the 2-fold block design (S, B).

Example 1.1 (Steiner triple system of order 7 nested in a 2-fold block design of order 7). The shaded triangles form a Steiner triple system of order 7.



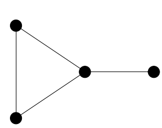
The nesting problem for Steiner triple systems was solved in 1985.

Theorem (D. R. Stinson [4]). *The spectrum for Steiner triple systems which can be nested in precisely the set of all* $n \equiv 1 \pmod 6$. □

The object of this paper is the generalization of the nesting of Steiner triple systems to kite systems and 4-cycle systems.

2 The nesting problem

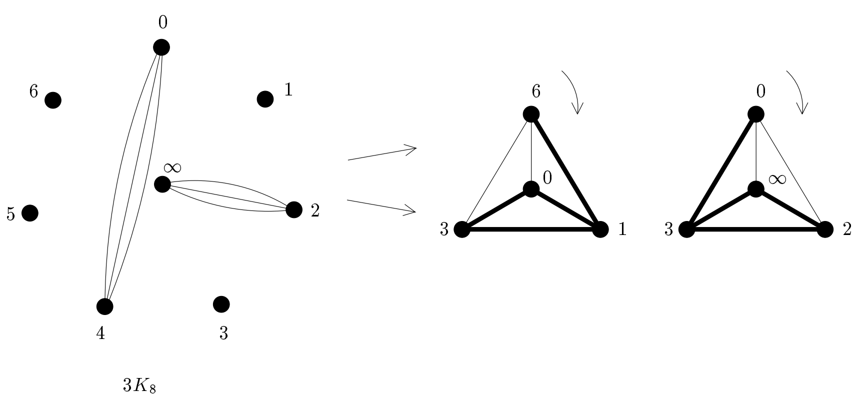
To begin with, a *kite* is a triangle with a tail consisting of one edge.



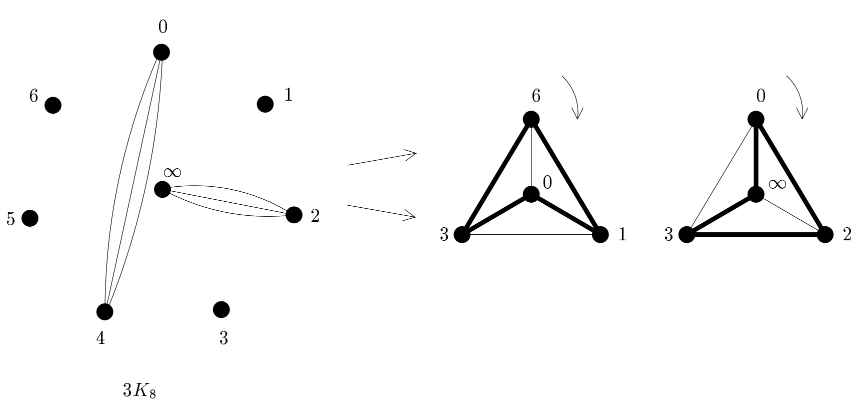
A λ-fold kite system is a pair (X, K), where K is a collection of edge disjoint kites which partitions the edge set of λK_n with vertex set X. A λ-fold 4-cycle system is a pair (X, C), where C is a collection of edge disjoint 4-cycles which partitions the edge set of λK_n with vertex set X. If we replace "2" with "λ" in the definition of a 2-fold block design in Section 1 we have the definition of a λ-fold block design.

Now if (X, B) is a 3-fold block design, (X, K) a 2-fold kite system, and (X, C) a 2-fold 4-cycle system of order n, $|B| = n(n-1)/4 = |K| = |C|$. Hence we can ask the same question for kites and 4-cycles as for triangles: for which n does there exist a 3-fold block design (X, B) with the property that a kite (4-cycle) can be selected from each block in B so that the resulting collection of kites K (4-cycles C) is a 2-fold kite system (X, K) (4-cycle system (X, C)) of order n? As with Steiner triple systems, the 2-fold kite system (4-cycle system) is said to be *nested* in the 3-fold block design (X, B).

Example 2.1 (2-fold kite system of order 8 nested in a 3-fold block design of order 8). Let $K_8 = \{\infty\} \cup Z_7$.



Example 2.2 (2-fold 4-cycle system of order 8 nested in a 3-fold block design of order 8). Let $K_8 = \{\infty\} \cup Z_7$.



In general we will say that the λ_1-fold kite system (X, K) is *nested* in the λ_2-fold block design (X, B) provided a kite can be selected from each block of B so that the resulting collection of kites is K. We have the same definition for the nesting of a λ_1-fold 4-cycle system (X, C). Now if the λ_1-fold kite system (X, K) (λ_1-fold 4-cycle system (X, C)) can be nested in the λ_2-fold block design (X, B), then $\lambda_2 n(n-1)/12 = |B| = |K| = |C| = \lambda_1 n(n-1)/8$ implies $\lambda_1 = 2k$ and $\lambda_2 = 3k$. Hence the following general problem.

THE NESTING PROBLEM FOR KITES AND 4-CYCLES. For each $\lambda = 2k$, determine the set of all n (= the spectrum) such that there exists a $2k$-fold kite system (4-cycle system) which can be nested in a $3k$-fold block design.

The object of this paper is the *complete* solution of the nesting problem for $2k$-fold kite (4-cycle) systems. It is easy to see that a solution for $2k \geq 6$ can be obtained by pasting together solutions for $\lambda = 2$ and $\lambda = 4$. Hence we will organize our results into the following sections: preliminaries, $\lambda = 2$, $\lambda = 4$, and a summary.

3 Preliminaries

We will need the following two results on Pairwise Balanced Designs (PBD) in order to obtain the nestings in the following sections.

Lemma 3.1 *If $n \equiv 0$ or $1 (mod\ 4) \geq 13$, there exists a PBD of order n with block sizes belonging to the set $\{4, 5, 8\}$.*

Proof: If we widen the modulus to 12, then $n \equiv 0, 1, 4, 5, 8$ or $9 \pmod{12}$. It is well-known [1] that the spectrum for block designs (remember that without quantification block design means block size 4) is precisely the set of all $n \equiv 1$ or $4 \pmod{12}$. This leaves $n \equiv 0, 5, 8$, or $9 \pmod{12} \geq 17$. It is also well-known [3] that there exists a *resolvable* block design of every order $12k + 4$. If (S, B) is a resolvable block design of order $12k + 4$ the number of parallel classes in B is $4k + 1$. If $k = 1$, B contains 5 parallel classes and if $k \geq 2$, $0 \leq 9 \leq 4k + 1$, B contains at least 9 parallel classes.

Suppose $n \equiv 0, 5, 8$, or $9 \pmod{12} \geq 17$ and write $n = (12k + 4) + r$, where $r \in \{1, 4, 5, 8\}$. Then in every case, EXCEPT $n = 24$, there is a resolvable block design of order $12k + 4$ with r parallel classes. Let $\pi_1, \pi_2, \ldots, \pi_r$ be any r parallel classes in B and let $\infty = \{\infty_1, \infty_2, \ldots, \infty_r\}$ be a set of size r, $S \cap \infty = \emptyset$, and define a collection of blocks B^* on $S^* = S \cup \infty$ as follows:

(1) $b \in B^*$, for all $b \notin \pi_i, i = 1, 2, \ldots, r$;

(2) $b \cup \{\infty_i\} \in B^*$, if $b \in \pi_i$; and

(3) $\infty = \{\infty_1, \infty_2, \infty_3, \ldots, \infty_r\} \in B^*$.

Then (S^*, B^*) is a PBD of order $n = (12k + 4) + r$ with block sizes belonging to $\{4, 5, 8\}$.

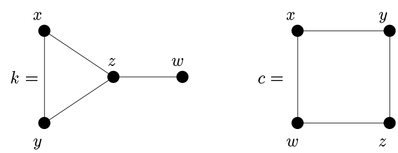
If $n = 24$, delete a point from the affine plane of order 5 (= block design of order 25 with block size 5). The result is a PBD with block sizes 4 and 5. □

Lemma 3.2 *If $n \in \{34, 35, 46, 47, 50, 51\} \cup \{x \equiv 2$ or $3(mod\ 4) \geq 58\}$ there exists a PBD of order n with block sizes belonging to the set $\{4, 5, 6, 7, 10, 11, 14, 15\}$.*

Proof: Write $34 = 28 + 6$, $35 = 28 + 7$, $46 = 40 + 6$, $47 = 40 + 7$, $50 = 40 + 10$, $51 = 40 + 11$, and $x = (12k + 4) + r$, where $k \geq 4$ and $r \in \{6, 7, 10, 11, 14, 15\}$. Then the resolvable block designs of orders $28, 40$, and $12k+4 (k \geq 4)$ have enough parallel classes to use the construction in Lemma 3.1. □

4 Nesting 2-fold kite and 4-cycle systems

In what follows we will denote the kite k and 4-cycle c by:



$k = (x, y, z) - w$ or $(y, x, z) - w$ and $c =$ any cyclic shift of (x, y, z, w) or (y, x, w, z).

We begin with some examples.

Example 4.1 (n=4) $B = \{\{1, 2, 3, 4\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4\}\}$; $K = \{(1, 3, 2) - 4, (1, 2, 4) - 3, (1, 4, 3) - 2\}$; *and* $C = \{(1, 2, 3, 4), (1, 3, 2, 4), (1, 3, 4, 2)\}$.

Example 4.2 (n=5) $B = \{\{i, 1 + i, 4 + i, 2 + i\} \mid i \in Z_5\}$; $K = \{(i, 1 + i, 4 + i) = 2 + i \mid i \in Z_5\}$; *and* $C = \{(i, 2 + i, 1 + i, 4 + i) \mid i \in Z_5\}$.

Example 4.3 (n=8) $B = \{\{3 + i, i, 1 + i, 6 + i\}, \{\infty, i, 3 + i, 2 + i\} \mid i \in Z_7\}$; $K = \{(3+i, i, 1+i) - (6+i), (\infty, i, 3+i) - (2+i) \mid i \in Z_7\}$; *and* $C = \{(i, 1+i, 6+i, 3+i), (\infty, i, 2+i, 3+i) \mid i \in Z_7\}$.

Example 4.4 (n=9 for kites) $B = \{\{i, 4 + i, 1 + i, 3 + i\}, \{i, 6 + i, 7 + i, 2 + i\} \mid i \in Z_9$; $K = \{(i, 4 + i, 1 + i) - (3 + i), (i, 6 + i, 7 + i) - (2 + i) \mid i \in Z_9\}$.

Example 4.5 (n=9 for 4-cycles) For each block $\{a, b, c, d\}$ in B, the 4-cycle is (a, b, c, d).

$B = \{\{1, 2, 5, 4\}, \{2, 3, 6, 5\}, \{1, 3, 6, 4\}, \{4, 5, 8, 7\}, \{5, 6, 9, 8\}, \{4, 6, 9, 7\}, \{1, 3, 9, 7\}, \{1, 2, 8, 7\}, \{2, 3, 9, 8\}, \{2, 7, 3, 4\}, \{1, 5, 7, 6\}, \{1, 9, 4, 8\}, \{1, 5, 3, 8\}, \{2, 4, 8, 6\}, \{7, 2, 9, 5\}, \{1, 6, 2, 9\}, \{3, 4, 9, 5\}, \{3, 7, 6, 8\}$.

Example 4.6 (n=12) $B = \{\{i, 3+i, 1+i, 8+i\}, \{i, 1+i, 5+i, 2+i\}, \{\infty, 6+i, i, 2+i\} \mid i \in Z_{11}\}$; $K = \{(i, 3+i, 1+i)-(8+i), (i, 1+i, 5+i)-(2+i), (\infty, 6+i, i)-(2+i) \mid i \in Z_{11}\}$; *and* $C = \{(i, 1+i, 8+i, 3+i), (i, 1+i, 5+i, 2+i), (\infty, 6+i, i, 2+i) \mid i \in Z_{11}\}$.

Lemma 4.7 *The spectrum for 2-fold kite systems (4-cycle systems) that can be nested in a 3-fold block design is precisely the set of all $n \equiv 0$ or $1 \pmod 4$.*

Proof: Examples 4.1, 4.2, 4.3, 4.4, 4.5 and 4.6 take care of $n = 4, 5, 8, 9$, and 12. If $n \equiv 0$ or 1 (mod 4) ≥ 13 let (X, P) be a PBD of order n with block sizes belonging to $\{4, 5, 8\}$ (Lemma 3.1). Place a copy of Example 4.1 on each block of size 4, a copy of Example 4.2 on each block of size 5, and a copy of Example 4.3 on each block of size 8. This will give either a nesting of a 2-fold kite system or a 2-fold 4-cycle system of order n into a 3-fold block design of order n. □

5 Nesting 4-fold kite and 4-cycle systems

Since the spectrum for 2-fold kite and 2-fold 2-cycle systems which can be nested in precisely the set of all $n \equiv 0$ or 1 (mod 4), we need concern ourselves with $n \equiv 2$ or 3 (mod 4) only, since we can double the nesting for a 2-fold kite or 2-fold 4-cycle system to obtain a nesting for a 4-fold kite and a 4-fold 4-cycle system.

We begin this section with a very important lemma.

Lemma 5.1 *If $p \geq 5$ is a prime, then there is a 4-fold kite system (4-cycle system) of order p and $p+1$ which can be nested in a 6-fold block design.*

Proof: Let $p \geq 5$ be a prime. Let (Z_p, B) be the cyclic 6-fold block design with base blocks $\{0, i, 2i, p-i\}$, $i = 1, 2, \ldots, (p-1)/2$. Then (Z_p, K) is a cyclic 4-fold kite system of order p with base blocks $(i, p-i, 2i)-0$, $i = 1, 2, \ldots, (p-1)/2$; and (Z_p, C) is a 4-fold 4-cycle system of order p with base blocks $(0, 2i, i, p-i)$, $i = 1, 2, \ldots, (p-1)/2$. (Note that in each case the kite (cycle) is taken from the corresponding block of B. Now let $(Z_p \cup \{\infty\}, B^*)$ be the cyclic 6-fold block design with base blocks $\{0, i, 2i, p-i\}$, $i = 2, 3, \ldots, (p-1)/2$, $\{0, 1, 2, \infty\}$, and $\{0, 1, 3, \infty\}$. Then $(Z_p \cup \{\infty\}, K^*)$ is a cyclic 4-fold kite system of order $p+1$ with base blocks $(i, p-i, 2i)-0$, $i = 2, 3, 4, \ldots, (p-1)/2$, $(0, 2, \infty) - 1$, and $(1, 3, 0) - 8$; and $(Z_p \cup \{\infty\}, C^*)$ is a 4-fold 4-cycle system of order $p+1$ with base blocks $(0, 2i, i, p-i)$, $i = 2, 3, \ldots, (p-1)/2$, $(\infty, 2, 0, 1)$, and $(\infty, 3, 1, 0)$. (Note that in this case i begins with 2 and NOT 1.) □

Example 5.2 (n = 10,15,22,26,27,39,55) We will handle this case by case:

n = 10. Put a copy of Example 4.1 on each block of a 2-fold block design of order 10.

n = 15. Let (Z_{15}, B) be the cyclic 6-fold block design with base blocks $\{0, 1, 4, 6\}$, $\{0, 2, 8, 12\}, \{0, 4, 1, 9\}, \{0, 8, 2, 3\}, \{0, 1, 5, 7\}, \{0, 2, 10, 14\}$, and $\{0, 1, 3, 7\}$. Then (Z_{15}, K) is a 4-fold kite system with base blocks $(0, 4, 6) - 1, (0, 8, 12) - 2, (0, 1, 9) - 4, (0, 2, 3) - 8, (0, 1, 7) - 5, (0, 2, 14) - 10$, and $(0, 3, 7) - 1$; and (Z_{15}, C) is a 4-fold 4-cycle system with base blocks $(0, 4, 1, 6), (0, 8, 2, 12), (0, 1, 4, 9), (0, 2, 8, 3), (0, 1, 5, 7)$, $(0, 2, 10, 14)$, and $(0, 1, 3, 7)$.

n = 22. Put a copy of Example 4.1 on each block of a 2-fold block design of order 22.

n = 26. Add a point to each block of a parallel class of the affine plane of order 5. This gives a PBD with block sizes 5 and 6. Use Lemma 5.1 on each block.

n = 27. Delete 4 points, no 3 of which are collinear from the projective plane of order 5. This gives a PBD with block sizes 4, 5, and 6. Use Examples 4.1 and Lemma 5.1.

$\mathbf{n = 39}$. Delete a block and 3 points from a parallel block (for a total of 10 points) from the affine plane of order 7. This gives a PBD with block sizes 4, 5, 6, and 7. Use Examples 4.1 and Lemma 5.1.

$\mathbf{n = 55}$. Delete 2 points from the projective plane of order 7. This gives a PBD with blocks of size 6, 7, and 8. Use Lemma 5.1.

Lemma 5.3 *The spectrum for* 4*-fold kite systems* (4*-fold* 4*-cycle systems*) *that can be nested in a* 6*-fold block design is the set of all* $n \geq 4$.

Proof: As previously noted, doubling up Lemma 4.7 takes care of all $n \equiv 0$ or 1 (mod 4). So we need to consider only $n \equiv 2$ or 3 (mod 4). Lemma 5.1 and Example 5.2 take care of all of the cases not covered by Lemma 3.2. Placing the appropriate examples on each block having size belonging to the set $\{4, 5, 6, 7, 10, 11, 14, 15\}$ completes the proof. □

6 Nesting 2k-fold kite and 4-cycle systems

To begin with, the spectrum for $2k$-fold kite (4-cycle) systems is $n \equiv 0$ or $1 \pmod 4$ if k is odd and all $n \geq 4$ if k is even. Since we have solutions for $2k = 2$ and 4 we can paste these solutions together for all of the remaining values of $2k \geq 6$ as follows. If k is odd, paste together k solutions of 2-fold kite (4-cycle) systems. If $k = 2t$ is even, so $2k = 4t$, paste together t solutions of 4-fold kite (4-cycle) systems. This gives the following general theorem.

Theorem 6.1 *The necessary and sufficient conditions to nest a* λ_1*-fold kite* (4*-cycle*) *system in a* λ_2*-fold block design are* $\lambda_1 = 2k$ *and* $\lambda_2 = 3k$. *The spectrum for* $2k$*-fold kite* (4*-cycle*) *systems which can be nested is* $n \equiv 0$ *or* 1 (*mod* 4) *if* k *is odd and all* $n \geq 4$ *if* k *is even.* □

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[3] D. K. Ray-Chaudhuri and R. M. Wilson, The existence of resolvable block designs, in: *A Survey of Combinatorial Theory* (Srivastava et.al., eds), North-Holland, Amsterdam, (1973), 361–375.

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Note Added in Proof

The authors have recently been informed that Lemma 4.7 appears in C. J. Colbourn and D. R. Stinson, Edge-coloured designs with block size four, *Aequationes Mathematicae* 36 (1988), 230–245.

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