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Characterization of induced matching extendable graphs with $2n$ vertices and $3n-1$ edges *

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Abstract

We say that a graph G is induced matching extendable, shortly IM-extendable, if every induced matching of G is included in a perfect matching of G. In J. J. Yuan, Induced Matching Extendable Graph, *J. Graph Theory*, **28**(1998), 203-213, it was shown that a connected IM-extendable graph on $2n$ vertices has at least $3n-2$ edges, and the only IM-extendable graph with $2n$ vertices and $3n-2$ edges is $T \times K_2$, where T is an arbitrary tree on n vertices. We show in this paper that the only IM-extendable graph with $2n \geq 6$ vertices and $3n-1$ edges is $T \times K_2 + e$, where T is an arbitrary tree on n vertices and e is any edge connecting two vertices that lie in different copies of T and have distance 3 between them in $T \times K_2$.

1 Introduction

Graphs considered in this paper are finite and simple. Let G be a graph with *vertex set* $V(G)$ and *edge set* $E(G)$. For any vertex set $S \subseteq V(G)$, set

$$E(S) = \{uv \in E(G) : \ u, v \in S\}.$$

For any edge set $M \subseteq E(G)$, set

$$V(M) = \{u \in V(G) : \text{ there is a vertex } v \text{ of } G \text{ such that } uv \in M\}.$$

For any vertex $v \in V(G)$, the *neighbor set* $N_G(v)$ of v is defined by

$$N_G(v) = \{u \in V(G) \backslash \{v\} : \text{ there is a vertex } u \text{ of } G \text{ such that } uv \in E(G)\}.$$

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If vertices u and v are connected in G, the *distance* between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. For any $v \in V(G)$, set

$$N_G^{(2)}(v) = \{u \in V(G) : \ d_G(u,v) = 2\}.$$

A set of edges $M \subseteq E(G)$ is called a *matching* of G if no two of them share a common endpoint. A matching is *perfect* if it covers all vertices of G. A matching M is *induced* [2] if $E(V(M)) = M$. We say that a graph G is *induced matching extendable* [10], shortly IM-extendable, if every induced matching M of G is included in a perfect matching of G. Clearly, an IM-extendable graph must have an even number of vertices. Researches on the IM-extendable graphs can be found in [3, 6 − 13].

The *product* of two graphs T and H, denoted by $T \times H$, is the graph with vertex set

$$V(T) \times V(H) = \{(u,v) : u \in V(T), v \in V(H)\}$$

such that two vertices $(u,v),(x,y) \in V(T) \times V(H)$ are adjacent in $T \times H$ if and only if either $[u = x$ and $vy \in E(H)]$ or $[v = y$ and $ux \in E(T)]$. When $H = K_2$, $T \times K_2$ can be defined by setting

$$V(T \times K_2) = \{(x,i) : x \in V(T), i = 1,2\}$$

and

$$E(T \times K_2) = \{(x,1)(x,2) : x \in V(T)\} \cup \{(x,i)(y,i) : xy \in E(T), i = 1,2\}.$$

For $i = 1,2$, the subgraph T_i of $T \times K_2$ induced by $\{(x,i) : x \in V(T)\}$ is called the i-th copy of T.

A graph isomorphic to $T \times K_2$ (where T is a tree) is said to be a *fat tree*. A graph isomorphic to $T \times K_2 + e$ where e is an edge connecting two vertices that lie in different copies of T and has distance 3 between them $T \times K_2$ is called a *braced fat tree*. A (braced) fat tree with $2n$ vertices is called a *braced fat n-tree*.

In [11], Yuan proved that a connected IM-extendable graph on $2n$ vertices has at least $3n-2$ edges, and the only IM-extendable graph with $2n$ vertices and $3n-2$ edges is a fat tree. We show in this paper that the only IM-extendable graph with $2n \geq 6$ vertices and $3n-1$ edges is a braced fat tree.

Terminology and notation not defined here can be found in [1].

2 Preliminaries

Lemma 2.1 [11] *If G is a connected IM-extendable graph with $|V(G)| \geq 4$, then $|V(G)|$ is even and G is 2-connected.*

Lemma 2.2 [11] *For every graph G and every positive integer r, $G \times K_{2r}$ is IM-extendable.*

Lemma 2.3 (Tutte's Theorem) [5] *A graph G has perfect matchings, if and only if, for every $S \subseteq V(G)$, $o(G-S) \leq |S|$.*

In Lemma 2.3, $o(H)$ is the number of odd components of Graph H.

Lemma 2.4 [11] *A graph G is IM-extendable, if and only if, for every induced matching M of G and every $S \subseteq V(G) \setminus V(M), o(G - V(M) - S) \leq |S|$.*

Let $u \in V(G)$ be a vertex of degree 2 and $N_G(u) = \{v, w\}$. We use $G^* ==$ $G \bullet \{u, v, w\}$ to denote a new graph obtained from G by identifying u, v, w to a new vertex u^*.

Lemma 2.5 [11] *If G is IM-extendable, then G^* is IM-extendable.*

Lemma 2.6 [11] *If G is IM-extendable, $uv \in E(G)$ such that $d(u) = d(v) = 2$, then $G - \{u, v\}$ is IM-extendable.*

Lemma 2.7 [11] *If G is a connected IM-extendable graph with $2n$ vertices, then $|E(G)| \geq 3n-2$; the equality holds if and only if G is a fat tree.*

The following lemma is implied in [11] (pp. 207–208).

Lemma 2.8 *Suppose that G is a connected IM-extendable graph on $2n$ vertices, where $n \geq 4$. If G has no adjacent vertices of degree 2 and every vertex of degree 2, if any, is not contained in any cycle of length 4, then $|E(G)| \geq 3n$.*

3 Main Results and Proofs

Denote by $K_{3,3}^-$ the graph obtained from $K_{3,3}$ by deleting an arbitrary edge. It is routine to check that the only braced fat tree with 6 vertices (up to isomorphism) is $K_{3,3}^-$. We can see that every braced fat tree must contain an induced subgraph isomorphic to $K_{3,3}^-$. We can also observe that every every braced fat tree must be bipartite.

Suppose that G is a connected IM-extendable graph with $2n$ vertices and $3n-1$ edges. Let $S = \{u : d_G(u) = 2\}$. It is trivial to check that there is no IM-extendable graph with 4 vertices and 5 edges. Hence, we suppose $n \geq 3$ in the sequel.

Proposition 3.1 *$\delta(G) = 2$ and $|S| \geq 2$. Furthermore, for every vertex $u \in S$, $N_G(u)$ is an independent set.*

Proof $\delta(G) \geq 2$ is implied in Lemma 2.1. $|S| \geq 2$ can be deduced from the facts $\delta(G) \geq 2, |V(G)| = 2n$ and $|E(G)| = 3n-1$. This further implies that $\delta(G) = 2$.

Now, let $u \in S$ and $N_G(u) = \{v, w\}$. If $vw \in E(G)$, then $\{vw\}$ is an induced matching of G such that u is an isolated vertex in $G - \{v, w\}$. So, $\{vw\}$ cannot be included in a perfect matching of G, a contradiction. Hence, for every vertex $u \in S$, $N_G(u)$ is an independent set. □

Proposition 3.2 *Let $u \in S, N_G(u) = \{v, w\}$. Then $|N_G(v) \cap N_G(w)| \leq 3$.*

Proof Otherwise, suppose $u \in S, N_G(u) = \{v, w\}$ such that $|N_G(v) \cap N_G(w)| \geq$

4. By Lemma 2.5, $G^* = G \bullet \{u, v, w\}$ is IM-extendable. But G^* has $2(n-1)$ vertices and less than $3(n-1)-2$ vertices, contradicting Lemma 2.7. □

Proposition 3.3 *Each IM-extendable graph G with 6 vertices and 8 edges is isomorphic to $K_{3,3}^-$.*

Proof If $E(S) \neq \emptyset$, let $uv \in E(S)$. Then by Lemma 2.6, $G^* = G - \{u, v\}$ is IM-extendable. We can see that G^* has 4 vertices and 5 edges, and there is no such IM-extendable graph. Then $E(S) = \emptyset$. If $|S| \geq 3$, let $S^* \subseteq S$ be such that $|S^*| = 3$. By the facts $|V(G)| = 6$ and $|E(G)| = 8$, $G - S$ must be a path of length 2, say $G - S = xyz$. Then $G - \{x, y\} - z$ has three isolated vertices, contradicting Lemma 2.4. Hence, we must have $|S| = 2$. Let $S = \{u, x\}$ and $N_G(u) = \{v, w\}$. If $N_G(u) = N_G(x)$, then there is $y \in V(G) \backslash \{u, x, v, w\}$ such that $vy \in E(G)$. But then, $\{vy\}$ is an induced matching of G such that $G - \{v, y\} - w$ has at least two isolated vertices u and x, contradicting Lemma 2.4. If $w \in N_G(x)$ and $v \notin N_G(x)$, we assume that $N_G(x) = \{w, y\}$. By the facts $|V(G)| = 6$ and $d_G(v), d_G(y) \geq 3$, vy must be an edge of G. Again, $\{vy\}$ is an induced matching of G such that $G - \{v, y\} - w$ has at least two isolated vertices u and x, contradicting Lemma 2.4. Hence, we must have $N_G(u) \cap N_G(x) = \emptyset$. By the facts $|V(G)| = 6$ and $|E(G)| = 8$ again, one can see that G is isomorphic to the graph obtained from K_4 by replacing two independent edges with two paths of length 2, respectively. Namely, G is isomorphic to $K_{3,3}^-$, which is indeed an IM-extendable graph. □

Proposition 3.4 *A connected IM-extendable graph with at least 6 vertices is a braced fat tree.*

Proof. We prove the theorem by induction. By Proposition 3.3, the result is true when $n = 3$.

Suppose that $n \geq 4$ and the result is true for any connected IM-extendable graph with $2m \leq 2n-2$ vertices and $3m-1$ edges. Let G be a connected graph with $2n$ vertices and $3n-1$ edges. By Proposition 3.1, we have $|S| \geq 2$. We first prove the following claim.

Claim 1: $E(S) \neq \emptyset$.

Otherwise, S is independent in G. By Proposition 3.1, for each $u \in S, N_G(u)$ is independent in G. By Proposition 3.2, for any $u \in S$ with $N_G(u) = \{v, w\}$, we have $|N_G(u) \cap N_G(w)| \leq 3$.

Suppose first that there is a vertex $u \in S$ with $N_G(u) = \{v, w\}$ such that $|N_G(v) \cap N_G(w)| = 3$, say, $N_G(v) \cap N_G(w) = \{u, v', w'\}$. By Lemma 2.5, $G^* = G \bullet \{u, v, w\}$ is IM-extendable. Since G^* has $2(n-1)$ vertices and $3(n-1)-2$ edges, from Lemma 2.7, G^* is isomorphic to $T^* \times K_2$, where T^* is a tree with $n-1$ vertices. We assume $G^* = T^* \times K_2$ so that the vertices of $T^* \times K_2$ correspond to that of G^*. It is easy to see that $T^* \times K_2$ has r disjoint pairs of adjacent vertices of degree 2, where $r \geq 2$ is the number of vertices of degree 1 in T^*. Let u^* be the new vertex of G^* obtained by identifying u, v, and w. Then $d_G(x) = d_{G^*}(x)$ for $x \in V(G^*) \backslash \{u^*, v', w'\}$; furthermore, there are at most two vertices in $\{u^*, v', w'\}$ being of degree 2 in G^*. Let $S^* = \{x \in V(G^*) : d_{G^*}(x) = 2\}$. Then $|S^* \cap \{u^*, v', w'\}| \leq 2$. Since G has no two adjacent vertices of degree 2, each pair of adjacent vertices of degree 2 in G^* contains at least one vertex in $S^* \cap \{u^*, v', w'\}$. Thus $r \le |S^* \cap \{u^*, v', w'\}| \le 2$. From the fact $r \ge 2$, we deduce $r = |S^* \cap \{u^*, v', w'\}| = 2$. It follows that T^* must be a path. By noting that two nonadjacent vertices of degree 2 in G^* are in $\{u^*, v', w'\}$ and $v'u^*w'$ is a path in G^*, the only possibility is that T^* is a path of length 2 and $v'u^*w'$ is one of the copies of T^* in $T^* \times K_2$. Let xyz be the other copy of T^* with $v'x, u^*y, w'z \in E(G^*)$. Then, exactly one of vy and wy, say, wy, belongs to $E(G)$. The structure of G is shown in Figure 1.

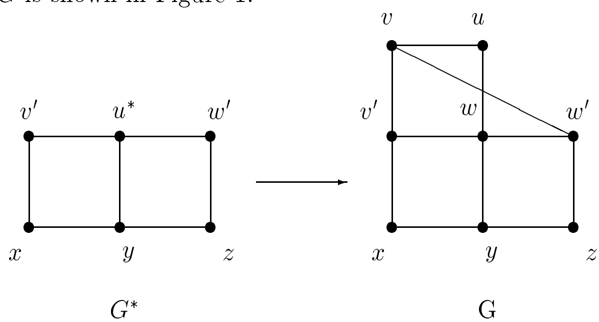


Figure 1.

It can be seen that $M = \{vw', xy\}$ is an induced matching of G such that $G - V(M)$ has an isolated vertex z, contradicting the assumption that G is IM-extendable. Hence, for every $u \in S$, say $N_G(u) = \{v, w\}$, we must have $|N_G(v) \cap N_G(w)| \le 2$.

Suppose $u \in S$ with $N_G(u) = \{v, w\}$ such that $|N_G(v) \cap N_G(w)| = 2$, say, $N_G(v) \cap N_G(w) = \{u, x\}$. Choose $v' \in N_G(v) \setminus \{u, x\}$ and $w' \in N_G(w) \setminus \{u, x\}$. Then $v'w' \in E(G)$, otherwise, $M = \{vv', ww'\}$ is an induced matching of G such that u is an isolated vertex of $G - V(M)$, contradicting Lemma 2.4. By Lemma 2.5, $G^* = G \bullet \{u, v, w\}$ is IM-extendable. Since G^* has $2(n-1)$ vertices and $3(n-1)-1$ edges, by the induction hypothesis, G^* is a braced fat tree. Since $v'w' \in E(G)$, we have $v'w' \in E(G^*)$ and then $u^*v'w'u^*$ is a 3-cycle of G^*, where u^* is the new vertex of G^* obtained by identifying u, v and w. This contradicts the fact that G^* is bipartite.

Thus, for every vertex $u \in S$, say, $N_G(u) = \{v, w\}$, we have $N_G(v) \cap N_G(w) = \{u\}$. Consequently, S is independent and any cycle of length 4 in G contains no vertices in S. By Lemma 2.8, $|E(G))| \ge 3n$. This is contrary to the assumption that $|E(G)| = 3n - 1$, and completes the proof of Claim 1.

Now, let $u, v \in S$ such that $uv \in E(G)$. By Lemma 2.6, $G^* = G - \{u, v\}$ is a connected IM-extendable graph. Since $|E(G^*)| = |E(G)| - 3 = 3(n-1) - 1$, then by induction hypothesis, G^* is a braced fat tree, i.e., $G^* \cong T^* \times K_2 + e$, where T^* is a tree with $n-1$ vertices and e is an edge connecting two vertices that lie in different copies of T and have distance 3 between them in $T \times K_2$. We reminder the reader that G^* is an IM-extendable bipartite graph.

Let $N_G(\{u, v\}) = \{x, y\}$ be such that $ux, vy \in E(G)$. Suppose, to the contrary, that $xy \notin E(G)$. If $N_{G^*}(x) \cap N_{G^*}(y) = \emptyset$, there must be a certain vertex $a \in N_G(x) \setminus \{u\}$ such that $a \notin N_G(y) \setminus \{v\}$. Then $M = \{xa, vy\}$ is an induced matching of G such that u is an isolated vertex in $G - V(M)$, contradicting the assumption that G is IM-extendable. Hence, we must have $N_{G^*}(x) \cap N_{G^*}(y) \neq \emptyset$. Let (A, B) be the bipartition of G^*. By the IM-extendability of G^*, we have $|A| = |B|$. From the above discussion, either $\{x, y\} \subseteq A$ or $\{x, y\} \subseteq B$. Suppose $\{x, y\} \subseteq A$. Then $N = \{ux\}$ is an induced matching of G such that $G - V(N) - A \setminus \{x\}$ has $|B| + 1 > |A \setminus \{x\}|$ isolated vertices, contradicting Lemma 2.4.

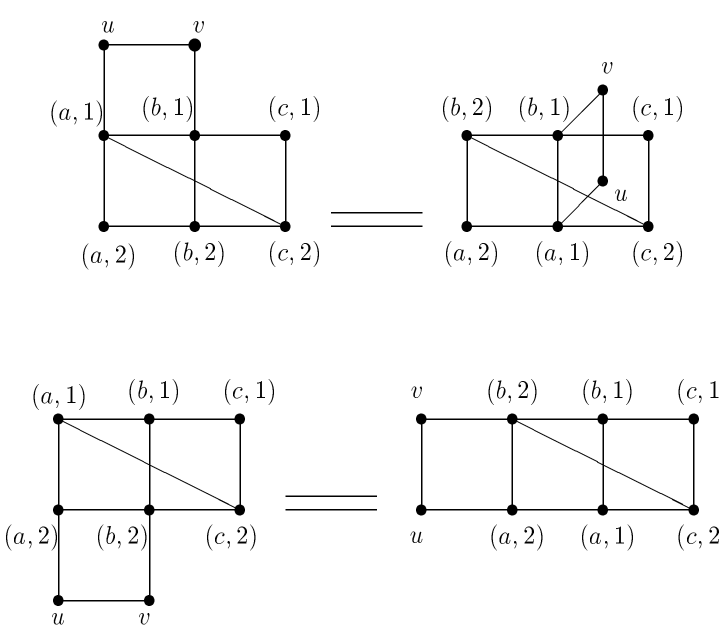
Now, we relabel the vertices and edges of G^* in the following way. Suppose $V(T^*)$ and $E(T^*)$ have been given. Write

$$V(G^*) = \{(x, i) : x \in V(T^*), i = 1, 2\},$$

$$E(G^*) = \{e\} \cup \{(x, 1)(x, 2) : x \in V(T^*)\} \cup \{(x, i)(y, i) : xy \in E(T^*), i = 1, 2\}.$$

Since G^* is a braced fat tree, there must be $a, b, c \in V(T^*)$ such that $ab, bc \in E(T^*)$, and either $e = (a, 1)(c, 2)$ or $e = (a, 2)(c, 1)$. By symmetry, we may assume that $e = (a, 1)(c, 2)$.

If $n = 4$, then G^* is isomorphic to $K_{3,3}^-$. It is routine to verify that G is indeed a braced fat tree. Three typical cases of $N_G(\{u, v\})$ are shown in Figure 2.



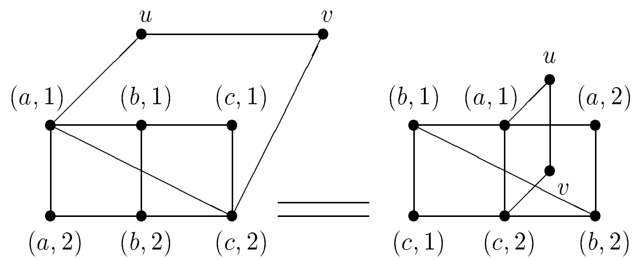


Figure 2.

In the following, we suppose $n \geq 5$. Then either $d_{T^*}(b) \geq 3$ or $\max\{d_{T^*}(a), d_{T^*}(c)\} \geq 2$. We distinguish the following four cases.

Case 1 $N_G(\{u,v\}) = \{(x,1),(x,2)\}$ for some $x \in V(T^*)$.

In this case, G is clearly a braced fat tree.

Case 2 $N_G(\{u,v\}) = \{(a,1),(c,2)\}$.

Suppose $u(a,1), v(c,2) \in E(G)$. If $d_{T^*}(b) \geq 3$, we pick $w \in N_{T^*}(b) \setminus \{a,c\}$. Then $M = \{u(a,1),(b,2)(w,2)\}$ is an induced matching of G such that $G - V(M)$ has odd components, contradicting Lemma 2.4. If $\max\{d_{T^*}(a), d_{T^*}(c)\} \geq 2$, say $d_{T^*}(c) \geq 2$, we pick $z \in N_{T^*}(c) \setminus \{b\}$. Then $N = \{u(a,1),(c,1)(z,1)\}$ is an induced matching of G such that $G - V(N) - (c,2)$ has three odd components, contradicting Lemma 2.4. Hence, Case 2 does not occur.

Based on the discussions of Case 1 and Case 2 we assume in the following that $N_G(\{u,v\}) = \{(x,i),(y,i)\}$ for some $xy \in E(T^*)$ and some i with $1 \leq i \leq 2$ such that $u(x,i), v(y,i) \in E(G)$.

Case 3 $|\{x,y\} \cap \{a,b,c\}| \leq 1$.

Since $|V(T^*)| = n - 1 \geq 4$, one of x and y, say y, must be of degree at least 2 in T^*. Pick $z \in N_{T^*}(y) \setminus \{x\}$. Then $M = \{u(x,i),(y,3-i)(z,3-i)\}$ is an induced matching of G such that $G - V(M)$ has odd components, contradicting Lemma 2.4. Hence, Case 3 does not occur.

Case 4 $\{x,y\} \subseteq \{a,b,c\}$.

In this case, either $\{x,y\} = \{a,b\}$ or $\{x,y\} = \{b,c\}$. Hence $b \in \{x,y\}$. We distinguish three subcases according to $d_{T^*}(b)$ and $\min\{d_{T^*}(x), d_{T^*}(y)\}$.

Case 4.1 $d_{T^*}(b) \geq 3$.

Suppose $u(x,i) \in E(G)$, $v(y,i) \in E(G)$, $y = b$ and $w \in N_{T^*}(b) \setminus \{a,c\}$. Then $M = \{u(x,i),(b,3-i)(w,3-i)\}$ is an induced matching of G such that $G - V(M) - (b,i)$ has three odd components, contradicting Lemma 2.4. Hence, this case does not occur.

Case 4.2 $d_{T^*}(b) = 2$ and $\min\{d_{T^*}(x), d_{T^*}(y)\} \geq 2$.

Suppose $u(x,i) \in E(G)$, $v(y,i) \in E(G)$, $y = b$ and $w \in N_{T^*}(x) \setminus \{b\}$. Then $M = \{(x, 3-i)(w, 3-i), v(b,i)\}$ is an induced matching of G such that $G - V(M) - (x,i)$ has three odd components, contradicting Lemma 2.4. Hence, this case does not occur.

Case 4.3 $d_{T^*}(b) = 2$ and $\min\{d_{T^*}(x), d_{T^*}(y)\} = 1$.

Suppose $x = a$ and $y = b$. Then $d_{T^*}(a) = 1$, $d_{T^*}(b) = 2$. Let $T' = T^* - \{a, b\}$. Then $Q = G^* - \{(a,1),(a,2),(b,1),(b,2)\}$ is isomorphic to $T' \times K_2$. If $i = 1$, $N_G(\{u,v\}) = \{(a,1),(b,1)\}$; as shown in Figure 3, G is a braced fat tree. If $i = 2$, $N_G(\{u,v\}) = \{(a,2),(b,2)\}$; as shown in Figure 4, G is a braced fat tree. This completes the proof of Proposition 3.4. □

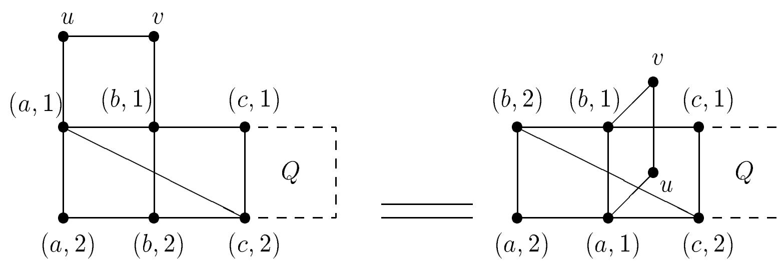


Figure 3.

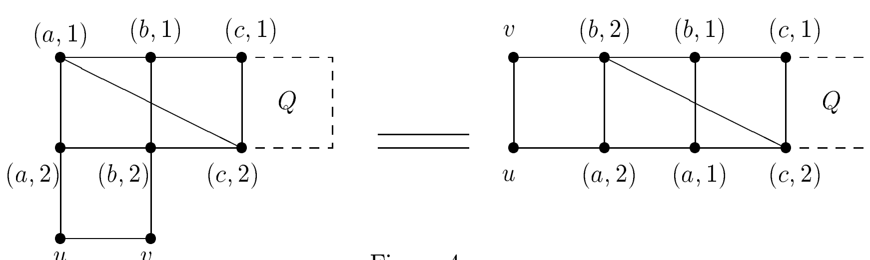


Figure 4.

Proposition 3.5 *Suppose G is a a braced fat tree with no less than 6 vertices. Then G is IM-extendable.*

Proof Let M be an induced matching of G. If $e \notin M$, then $N = \{x_i y_i : x_{3-i} y_{3-i} \in M\} \cup \{x_1 x_2 : \{x_1, x_2\} \cap V(M) = \emptyset\} \cup M$ is a perfect matching of G containing M. If $e \in M$, say, $e = x_i z_{3-i}$, $xy, yz \in E(T)$, then $N = \{x_i y_i : x_{3-i} y_{3-i} \in M\} \cup \{x_1 x_2 : \{x_1, x_2\} \cap V(M) = \emptyset\} \cup M \cup \{x_{3-i} y_{3-i}, y_i z_i\}$ is a perfect matching of G containing M. Hence, G is IM-extendable. □

Combining Proposition 3.4 and Proposition 3.5, we have

Theorem 3.6 *Suppose that G is a connected graph with $2n \geq 6$ vertices and $3n - 1$ edges. Then G is IM-extendable if and only if G is a braced fat tree.*

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