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Total domination of random regular graphs

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Abstract

A total dominating set $\mathcal{T}$ of a graph G is a subset of the vertices of G such that every vertex of G is adjacent to a vertex in $\mathcal{T}$. When G represents a communication network, a total dominating set corresponds to a collection of servers having a certain desirable backup property, namely, that every server is adjacent to some other server. Total dominating sets of small cardinality are therefore of interest. We refer to the size of a smallest total dominating set of a graph G as the *total domination number* of G. In this paper we present upper bounds on the total domination number of random regular graphs. This is achieved by analysing the performance of a randomised greedy algorithm on random regular graphs using differential equations.

1 Introduction

Throughout this paper we consider simple graphs that are undirected, unweighted and contain no loops or multiple edges. A graph G is said to be d-regular if every vertex in $V(G)$ has degree d (i.e. each vertex is adjacent to precisely d other vertices in G). When discussing any graph G, we let n denote the cardinality of $V(G)$ and note that for d-regular graphs on n vertices, dn must be even. For other basic graph-theoretical definitions we refer the reader to Diestel [3].

A total dominating set $\mathcal{T}$ of a graph G is a subset of the vertices of G such that every vertex of G is adjacent to a vertex in $\mathcal{T}$. When G represents a communication network, a total dominating set corresponds to a collection of servers having a certain desirable backup property, namely, that every server is adjacent to some other server [6]. Total dominating sets of small cardinality are therefore of interest.

We refer to the size of a smallest total dominating set of a graph G as the *total domination number* of G and denote this using $\gamma_{\mathcal{T}}(G)$. For an arbitrary graph G the problem of determining $\gamma_{\mathcal{T}}(G)$ is known to be NP-hard [5]. Determining $\gamma_{\mathcal{T}}(G)$ remains NP-hard for bipartite graphs, split graphs [2] and circle graphs [8]. On the positive side, determining $\gamma_{\mathcal{T}}(G)$ may be achieved in polynomial time for other

classes of graphs [7]. As far as the author is aware, no previous results were known regarding bounds on the total domination number of regular graphs.

We consider random d-regular graphs that are generated *uniformly at random* (u.a.r.) and need some associated notation. We say that a property $\mathcal{B} = \mathcal{B}_n$ of a random graph holds asymptotically almost surely (a.a.s.) if the probability that $\mathcal{B}$ holds tends to 1 as n tends to infinity. When considering d-regular graphs, this is modified so that n is restricted to even numbers if d is odd. For other basic random graph theory definitions we refer the reader to Bollobás [1].

In this paper we analyse the average-case performance of a simple heuristic, which is a randomised greedy algorithm, that gives upper bounds on $\gamma_{\mathcal{T}}(G)$ when G is a random d-regular graph. In the following section we give a description of our algorithm and in Section 3 we outline the method used for its analysis. Our analysis uses a theorem of Wormald [12] which we restate in Section 3. The results of this paper are encompassed by the following theorem, the proof of which is given in Section 4.

Theorem 1 *Let $d \geq 3$ be fixed. Then, for a random d-regular graph G on n vertices, the size of a minimum total dominating set is asymptotically almost surely less than $\mathcal{T}_u(d)n$ where, for $3 \leq d \leq 10$, the constants $\mathcal{T}_u(d)$ are given in Table 1.*

Table 1: Bounds on $\gamma_{\mathcal{T}}(G)$ when G is a random d-regular graph on n vertices.

d	$\mathcal{T}_u(d)n$
03	$0.3901n$
04	$0.3272n$
05	$0.2855n$
06	$0.2552n$
07	$0.2319n$
08	$0.2132n$
09	$0.1978n$
10	$0.1849n$

The constants $\mathcal{T}_u(d)$ referred to in Theorem 1 arise from the solution of particular sets of differential equations.

2 Prioritising choices

Consider the following algorithm that greedily finds a total dominating set of a graph G. Repeatedly choose an edge uv randomly from G and add u and v to a set $\mathcal{T}$.

After each edge is chosen, remove u, v and all their neighbours from G along with all edges incident with neighbours of u and v. For any non-dominated vertex w that attains degree zero, add to $\mathcal{T}$ a vertex that was a neighbour of w that was also a neighbour of u or v and remove w from G. Once no vertices remain, the set $\mathcal{T}$ is a total dominating set in G as any vertices that were removed from G either (i) became part of $\mathcal{T}$ at the same time as one of their neighbours also became part of $\mathcal{T}$, or (ii) they were totally dominated by adding a pair of vertices to $\mathcal{T}$, or (iii) they were totally dominated by adding an additional neighbour of one of a pair of vertices that were added to $\mathcal{T}$.

We modify this algorithm slightly by the way in which each subsequent pair of vertices are chosen to be part of $\mathcal{T}$. We assign a priority to vertices of current minimum degree. Our algorithm is presented in Figure 1. It takes a d-regular n-vertex graph G as input and returns a total dominating set $\mathcal{T}$ for G. We use the notation $deg(v)$ to denote the current degree of the vertex v in G. Let $N(v)$ denote the set of neighbours of a vertex v and in the case that $deg(v) = 1$, by a slight abuse of notation, we also use $N(v)$ to denote the unique neighbour of v. Also, we use V_i to denote the set of vertices of current degree i in G, $0 \leq i \leq d$.

while ($|V(G)| > 0$)
do
  $k \leftarrow \texttt{MIN}\,[deg(p) \mid p \in V(G)]$;
  select u u.a.r. from V_k;
  if ($k = 1 \wedge deg(N(u)) = 1$)
  then
    $v \leftarrow u$; $w \leftarrow N(u)$;
  else
    $k \leftarrow \texttt{MAX}\,[deg(p) \mid p \in N(u)]$;
    select v u.a.r. from $\{N(u) \cap V_k\}$;
    $k \leftarrow \texttt{MAX}\,[deg(p) \mid p \in \{N(v) \setminus u\}]$;
    select w u.a.r. from $\{\{N(v) \setminus u\} \cap V_k\}$;
  endif
  $\mathcal{T} \leftarrow \mathcal{T} \cup \{v, w\}$;
  remove all edges incident with vertices in $N(v)$ from $E(G)$;
  remove all edges incident with vertices in $N(w)$ from $E(G)$;
  add at most one vertex to $\mathcal{T}$ for each non-dominated vertex in V_0 created;
  remove all vertices in V_0 from $V(G)$.
enddo

Figure 1: Total dominating set algorithm

The algorithm proceeds in a series of *operations*. In each operation, a pair of vertices are added to $\mathcal{T}$ along with zero or more additional vertices and all edges

incident with the neighbours of the pair of vertices are removed from G. All operations start by selecting a vertex u of current minimum degree in G. If the degree of both u and $N(u)$ is 1, this edge is isolated from the rest of the edges in G and we simply assign v to be u and w to be $N(u)$. In all other cases, a neighbour v of u is selected u.a.r. from those vertices of current maximum degree in $N(u)$. Then a neighbour w of v (not u) is selected u.a.r. from those vertices of current maximum degree in the set $\{N(v) \setminus u\}$. Once v and w have been chosen, v and w are added to $\mathcal{T}$ and all edges incident with the neighbours of these vertices are removed from G. For each non-dominated vertex that attains degree zero, at most one vertex is added to $\mathcal{T}$ and the vertices of degree zero are removed from G. These additional vertices are chosen arbitrarily from what were the neighbours of v and w so as to dominate the vertices that become isolated. These steps are repeated until no vertices remain in G and once this stage is reached, the set $\mathcal{T}$ forms a total dominating set for G.

We refer to algorithms such as the one above as *prioritised algorithms*. The analysis of prioritised algorithms requires complex arguments involving branching processes and large deviation inequalities and the justifications of those require checking complex conditions regarding derivatives (see [4], for example). In order to reduce the number of conditions that are required to be checked, the analysis of the prioritised algorithm presented in this section will be carried out using a technique introduced by Wormald [12]. This approach approximates the performance of a prioritised algorithm by analysing associated *deprioritised* algorithms. These algorithms entirely avoid prioritising by using a randomised mixture of operations. The particular mixture used for any sequence of operations is prescribed in advance but changes over the course of the algorithm in order to approximate the prioritised algorithm.

3 Use of deprioritised algorithms

The operations and priorities described in the prioritised algorithm given in Section 2 may be analysed using [12, Theorem 1]. This provides us with a set of differential equations whose solution describes the state of a deprioritised version of the algorithm during its execution. From this, we deduce asymptotically almost sure bounds on the size of the total dominating set at the end of the algorithm.

What is now considered to be the standard model for random d-regular graphs is as follows. Take a set of dn points in n buckets labelled $1, 2, \ldots, n$, with d points in each bucket, and choose u.a.r. a pairing $P = p_1, \ldots, p_{dn/2}$ of the points such that each p_i is an unordered pair of points and each point is in precisely one pair p_i. The resulting probability space of pairings is denoted by $\mathcal{P}_{n,d}$. Form a d-regular pseudograph on n vertices by placing an edge between vertices i and j for each pair in P having one point in bucket i and one in bucket j. In order to prove that a property is a.a.s. true of a uniformly distributed random d-regular (simple) graph, it is enough to prove that it is a.a.s. true of the pseudograph corresponding to a random pairing (see Bollobás [1] and Wormald [10]).

As in [11], we redefine this model slightly by specifying that the pairs are chosen sequentially. The first point in a random pair may be selected using any rule

whatsoever, as long as the second point in that random pair is chosen u.a.r. from all the remaining free (unpaired) points. This preserves the uniform distribution of the final pairing.

When a pair has been determined in the sequential process, we say that it has been *exposed*. By exposing pairs in the order which an algorithm requests their existence, the generation of the random pairing may be combined with the algorithm (as in [4, 9, 11]). In this way, the algorithm such as the one in the previous section, which deletes edges, may be described in terms of operations incorporated into the pairing generation. The definition of the operations may be extended to do whatever other tasks the algorithm needs to carry out.

The algorithm proper acts upon the final (pseudo)graph of the generation process and the set of exposed pairs builds up this final graph during the course of the generation process which incorporates the algorithm. The order in which the edges are deleted corresponds to the order in which the pairs were exposed.

The setting of [12, Theorem 1] requires a number of definitions and may be described as follows. It concerns a class of processes applied to the random pairing. As described above, this may be defined in terms of the generation algorithm which exposes pairs. The beginning of the generation algorithm is the empty pairing G_0. The pairing G_{t+1} is obtained from G_t by applying an operation which may expose some of the pairs. The operation, op_t, which is applied to G_t must be one of some prespecified set of operations, Op_i, $i = 1, \dots, d$, where Op_i consists of selecting a bucket u of degree $d - i$ (corresponding to a vertex of degree i) in G_t u.a.r., and then applying some specified set of tasks involving some random choices, resulting in G_{t+1}; the *degree* of a bucket is the number of points it contains in exposed pairs. A subset T of $V(G) \cup E(G)$ is selected during the operations, with $T_0 = \emptyset$ initially, and $T = T_t$ for the pairing G_t.

For $1 \le i \le d$, let $Y_i = Y_i(t)$ denote the number of buckets of degree $d-i$ in G_t and let $Y_{d+1} = Y_{d+1}(t)$ denote cardinality of the set T_t. Put $\mathbf{Y}(\mathbf{t}) = (Y_1(t), \dots, Y_{d+1}(t))$. We refer the reader to [12, Theorem 1] for motivation for the following definitions and provide a little explanation below. Let $\mathbf{y}$ denote $(y_1(x), \dots, y_{d+1}(x))$. Given functions $f_{i,r}(x, \mathbf{y})$, define

$$\begin{aligned} \alpha_k(x, \mathbf{y}) &= f_{d-k-1,d-k}(x, \mathbf{y}), \\ \tau_k(x, \mathbf{y}) &= -f_{d-k-1,d-k-1}(x, \mathbf{y}), \end{aligned} \tag{1}$$

where

$$x = \frac{t}{n}, \qquad \mathbf{y}(x) = \frac{\mathbf{Y}(t)}{n}. \tag{2}$$

We will consider the equations

$$\frac{\mathrm{d}y_i}{\mathrm{d}x} = F(x, \mathbf{y}, i, k) \qquad \text{where} \tag{3}$$

$$F(x, \mathbf{y}, i, k) = \begin{cases} \frac{\tau_k}{\tau_k + \alpha_k} f_{i,d-k}(x, \mathbf{y}) + \frac{\alpha_k}{\tau_k + \alpha_k} f_{i,d-k-1}(x, \mathbf{y}) & (\text{if } k \le d-2) \\ f_{i,1}(x, \mathbf{y}) & (\text{if } k = d-1) \end{cases} \tag{4}$$

and work with the parameters of $f_{i,\ell}$ in the domain

$$\mathcal{D}_\epsilon = \{(x, \mathbf{y}) : 0 \le x \le d,\ 0 \le y_i \le d \text{ for } 1 \le i \le d+1,\ y_d \ge \epsilon\} \tag{5}$$

for some pre-chosen value of $\epsilon > 0$. The behaviour of the process will be described in terms of the function $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}(x) = (\tilde{y}_1(x), \ldots, \tilde{y}_{d+1}(x))$ defined as follows, with reference to an initial value $x = x_0 = t_0/n$ of interest:

$$\begin{aligned}&\tilde{y}_i(x_0) = Y_i(t_0)/n,\ i = 1, \ldots, d+1, \text{ and inductively for } k \ge 1, \tilde{\mathbf{y}} \text{ is the solution}\\ &\text{of (3) with initial conditions } \mathbf{y}(x_{k-1}) = \tilde{\mathbf{y}}(x_{k-1}), \text{ extending to all } x \in [x_{k-1}, x_k],\\ &\text{where } x_k \text{ is defined as the infimum of those } x > x_{k-1} \text{ for which at least one of}\\ &\text{the following holds: (i) } \tau_k \le 0 \text{ and } k < d-1; \text{ (ii) } \tau_k + \alpha_k \le \epsilon \text{ and } k < d-1;\\ &\text{(iii) } \tilde{y}_{d-k} \le 0; \text{ or (iv) the solution is outside } \mathcal{D}_\epsilon \text{ or ceases to exist.}\end{aligned} \tag{6}$$

The interval $[x_{k-1}, x_k]$ is called phase k. This inductive definition of $\tilde{\mathbf{y}}$ continues for phases $k = 1, 2, \ldots, m$, where

$$\begin{aligned}&m \text{ denotes the smallest } k \text{ for which either } k = d-1, \text{ or any of the termination}\\ &\text{conditions (ii), (iii) or (iv) for phase } k \text{ in (6) come into effect at } x_k.\end{aligned} \tag{7}$$

Wormald [12] proved that the phases all have nonempty interior provided

$$\begin{aligned}
\tau_k &> 0 \text{ at } (x_{k-1}, \tilde{\mathbf{y}}(x_{k-1})) \\
&\qquad (1 \le k \le \min\{d-2, m\}), \\
\tau_k + \alpha_k &> \epsilon \text{ at } (x_{k-1}, \tilde{\mathbf{y}}(x_{k-1})) \\
&\qquad (1 \le k \le \min\{d-2, m\}), \\
f_{d-1,d-1} &> 0 \text{ at } (x_0, \tilde{\mathbf{y}}(x_0)), \\
f'_{d-k,d-k}\tau_k + f_{d-k,d-k-1}f'_{d-k-1,d-k} &> 0 \text{ at } (x_{k-1}, \tilde{\mathbf{y}}(x_{k-1}))^+ \\
&\qquad (1 < k \le \min\{d-2, m\}), \\
f'_{d-k,d-k} &> 0 \text{ at } (x_{k-1}, \tilde{\mathbf{y}}(x_{k-1}))^- \quad (1 < k \le m), \\
f'_{1,1} &> 0 \text{ at } (x_{d-2}, \tilde{\mathbf{y}}(x_{d-2}))^+ \quad (\text{if } m = d-1),
\end{aligned} \tag{8}$$

with f' denoting $\frac{\mathrm{d}f(x,\tilde{\mathbf{y}}(x))}{\mathrm{d}x}$ and $(x, \tilde{\mathbf{y}}(x))^+$ and $(x, \tilde{\mathbf{y}}(x))^-$ referring to the right-hand and left-hand limits as functions of x.

We may now restate [12, Theorem 1] which we will use in the following section in connection with the total dominating set algorithm.

Theorem 2 ([12]) *Let $d \ge 3$. Assume that for some fixed $\epsilon > 0$ the operations Op_r satisfy*

$$\mathbf{E}\Big(Y_i(t+1) - Y_i(t) \mid G_t \wedge \{\mathrm{op}_t = \mathrm{Op}_r\}\Big) = f_{i,r}(t/n, Y_1/n, \ldots, Y_{d+1}/n) + o(1) \tag{9}$$

for some fixed functions $f_{i,r}(x, y_1(x), \ldots, y_{d+1}(x))$ and for $i = 1, \ldots, d+1$, $r = 1, \ldots, d$, with the convergence in $o(1)$ uniform over all t and G_t for which $Y_r(t) > 0$ and $Y_d(t) > \epsilon n$. Assume furthermore that

(i) there is an upper bound, depending only upon d, on the number of pairs exposed, and on the number of elements added to T (i.e. $|T_{t+1}| - |T_t|$), during any one operation;

(ii) the functions $f_{i,r}$ are rational functions of $x, y_1, \ldots, y_{d+1}$ with no pole in $\mathcal{D}_\epsilon$ defined in (5);

(iii) there exist positive constants C_1, C_2 and C_3 such that for $1 \le i < d$, everywhere on $\mathcal{D}_\epsilon$, $f_{i,r} \ge C_1 y_{i+1} - C_2 y_i$ when $r \ne i$, and $f_{i,r} \le C_3 y_{i+1}$ for all r.

Define $\tilde{\mathbf{y}}$ as in (6), set $x_0 = 0$, define m as in (7), and assume that (8) holds. Then there is a randomised algorithm on $\mathcal{P}_{n,d}$ for which a.a.s. there exists t such that $|T_t| = n\tilde{y}_{d+1}(x_m) + o(n)$ and $Y_i(t) = n\tilde{y}_i(x_m) + o(n)$ for $1 \le i \le d$. Also $\tilde{y}_i(x) \equiv 0$ for $x_{k-1} \le x \le x_k$, $1 \le i \le d-k-1$ $(1 \le k \le m)$.

Some of these definitions may be easily explained. The algorithm in Section 2 works by deleting edges; the edges deleted correspond to pairs exposed in the corresponding pairing generation algorithm as described above. In particular, a vertex of degree i in the original algorithm corresponds to a bucket of degree $d-i$ in the pairing version; we use *vertex degree* and *bucket degree* to distinguish these complementary measures. The algorithm gives higher priority to the buckets of highest degree (vertices of lowest degree). The phase is determined by the set of bucket degrees which are reasonably common (meaning, roughly, more than cn buckets have that degree for some $c > 0$). Phase k corresponds to a period in which the smallest such common vertex degree is $d-k$ (i.e. largest common bucket degree is k). At such a time, vertices of degree $d-k-1$, when created, will immediately be used up, by being chosen for u in the subsequent steps, until the minimum positive vertex degree returns to $d-k$. So phase k basically consists of a mixture of two operations: Op_{d-k} and Op_{d-k-1}. The functions α and τ represent respectively the expected net increase in Y_{k+1} in an Op_{d-k} and the expected net decrease in Y_{k+1} in an Op_{d-k-1}. From these quantities, one may estimate the proportions of these operations being performed at any stage. The randomised algorithm referred to in the theorem uses roughly the same mixture of operations. This in turn allows us calculate the expected changes in the variables, and the result is (4), which leads to the differential equation (3).

4 Proof of Theorem 1

We specify the tasks in Op_r, $1 \le r \le d$. Here Op_r must first select a random bucket, u, of degree $d-r$ and expose pairs in u. The set of randomised tasks consists of choosing a bucket v, exposing pairs in v, choosing a bucket w and exposing pairs in all buckets neighbouring v and w. For each bucket that attains degree d (that is not v, w or one of their neighbours), we add another bucket to T (along with v and w).

We may verify the hypotheses of Theorem 2. First we will show that, for functions $f_{i,r}$ defined below, (9) holds when $Y_d(t) > \epsilon n$ (for any $\epsilon > 0$). From here onwards in this description, *v-degree* refers to vertex degree, so v-degree i means bucket degree $d-i$. Let y_i denote Y_i/n and let

$$\chi_1 = (\mathrm{S}_1^{q_1})^r - (\mathrm{S}_1^{q_1-1})^r$$

where $\mathrm{S}_a^b = \sum_{j=a}^{b} \mathrm{P}_j$, $\mathrm{P}_j = \frac{jy_j}{s}$ and $s = \sum_{i=1}^{d} iy_i$. Then, when performing an instance of Op_r, the probability that the neighbours of u have maximum v-degree exactly q_1 is $\chi_1 + o(1)$ (see [4] for similar arguments).

Let

$$\chi_2 = (S_1^{q_2})^{q_1-1} - (S_1^{q_2-1})^{q_1-1}.$$

Then, when performing an instance of Op_r, assuming that the neighbours of u have maximum v-degree q_1, the probability that the neighbours of v (other than u) have maximum v-degree exactly q_2 is $\chi_2 + o(1)$.

Similarly, for an instance of Op_r, the probability that u has b_1 neighbours of v-degree q_1, given that the maximum v-degree amongst the neighbours of u is q_1, is $\beta_1 + o(1)$ where

$$\beta_1 = (\mathrm{P}_{q_1})^{b_1} \binom{r}{b_1} (S_1^{q_1-1})^{r-b_1}$$

and the probability that v has b_2 neighbours (other than u) of v-degree q_2, given that the maximum v-degree amongst the neighbours of u is q_1 and there are b_1 such neighbours, is $\beta_2 + o(1)$ where

$$\beta_2 = (\mathrm{P}_{q_2})^{b_2} \binom{q_1-1}{b_2} (S_1^{q_2-1})^{q_1-1-q_2}.$$

Also, the expected number of neighbours of u that have v-degree j_1, $1 \le j_1 \le q_1 - 1$, given that u has b_1 neighbours of v-degree q_1 and q_1 is the maximum v-degree of all neighbours of u, is $\gamma_1/\beta_1 + o(1)$ where

$$\gamma_1 = (\mathrm{P}_{q_1})^{b_1} \binom{r}{b_1} (S_1^{q_1-1})^{r-b_1-1} (r-b_1)\mathrm{P}_{j_1}.$$

The expected number of neighbours of v (other than u) that have v-degree j_2, $1 \le j_2 \le q_2 - 1$, given that u has b_1 neighbours of v-degree q_1, v has b_2 neighbours of v-degree q_2, q_1 is the maximum v-degree of all neighbours of u and q_2 is the maximum v-degree of all neighbours of v is $\gamma_2/\beta_2 + o(1)$ where

$$\gamma_2 = (\mathrm{P}_{q_2})^{b_2} \binom{q_1-1}{b_2} (S_1^{q_2-1})^{q_1-2-b_2} (q_1-1-b_2)\mathrm{P}_{j_2}.$$

The pairs incident with u are always exposed and the effect, on the expected change in Y_i, of changing the v-degree of u to 0 is just

$$-\delta_{i=r}$$

where, here and in the following, $\delta_R = 1$ if the statement R is true, 0 otherwise.

Similarly, the pairs incident with v and w are always exposed and the effect, on the expected change in Y_i, of changing the v-degree of v and w to 0 is just $-\delta_{i=q_1}$ and $-\delta_{i=q_2}$ respectively.

The effect, on the expected change in Y_i, of exposing pairs in all neighbours of w (other than v) is

$$(q_2 - 1)\mu_i$$

where

$$\mu_i = -\mathrm{P}_i + \rho_i \sum_{x=2}^{d}(x-1)\mathrm{P}_x + o(1) \qquad \text{and}$$

$$\rho_i = -\mathrm{P}_i + \mathrm{P}_{i+1}\delta_{i+1\le d} + o(1).$$

The effect, on the expected change in Y_i, of each of the neighbours of v of degree q_2 (other than w) having all their pairs exposed is

$$-\delta_{i=q_2} + (q_2-1)\rho_i.$$

The effect, on the expected change in Y_i, of each of the neighbours of v of degree j_2, $1 \le j_2 \le q_2 - 1$, having their pairs exposed is

$$-\delta_{i=j_2} + (j_2-1)\rho_i.$$

The effect, on the expected change in Y_i, of each of the neighbours of u of degree q_1 (other than v) each having their v-degree increased by 1 is

$$\delta_{i=q_1-1} - \delta_{i=q_1}.$$

The effect, on the expected change in Y_i, of each of the neighbours of u of degree j_1, $1 \le j_1 \le q_1 - 1$, each having their v-degree increased by 1 is

$$\delta_{i=j_1-1} - \delta_{i=j_1}.$$

So we have that (9) holds with

$$\begin{aligned} f_{i,r} = {} & -\delta_{i=r} + \sum_{q_1=1}^{d}\chi_1\left[-\delta_{i=q_1} + \sum_{b_1=1}^{r}\beta_1(b_1-1)(\delta_{i=q_1-1}-\delta_{i=q_1})\right] \\ & + \sum_{q_1=1}^{d}\chi_1\left[\sum_{j_1=1}^{q_1-1}\gamma_1(\delta_{i=j_1-1}-\delta_{i=j_1}) + \sum_{q_2=1}^{d}\chi_2\left[-\delta_{i=q_2}+(q_2-1)\mu_i\right]\right] \\ & + \sum_{q_1=1}^{d}\chi_1\sum_{q_2=1}^{d}\chi_2\left[\sum_{b_2=1}^{q_1-1}\beta_2(b_2-1)(-\delta_{i=q_2}+(q_2-1)\rho_i)\right] \\ & + \sum_{q_1=1}^{d}\chi_1\sum_{q_2=1}^{d}\chi_2\left[\sum_{j_2=1}^{q_2-1}\gamma_2(-\delta_{i=j_2}+(j_2-1)\rho_i)\right] \end{aligned} \tag{10}$$

It also follows that (9) also holds for $i = d+1$ with $f_{d+1,r}$ defined as $2 + f_{0,r} + o(1)$, since in each Op_r, two vertices are added to the total dominating set $\mathcal{T}$, the expected number of non-dominated buckets attaining degree d is $f_{0,r}$, as defined in (10), and the effect of not necessarily adding one vertex to $\mathcal{T}$ for each of these vertices is only $o(1)$.

Hypothesis (i) of Theorem 2 is immediate since in any operation only a bounded number of pairs are exposed and a bounded number of vertices are added to $\mathcal{T}$. The functions $f_{i,r}$ satisfy (ii) because from (10) their (possible) singularities satisfy $s = 0$, which lies outside $\mathcal{D}_\epsilon$ since in $\mathcal{D}_\epsilon$, $s \geq y_d \geq \epsilon$. Hypothesis (iii) follows from (10) again using $s \geq y_d \geq \epsilon$ and the boundedness of the functions y_i (which follows from the boundedness of $\mathcal{D}_\epsilon$). Thus, defining $\tilde{\mathbf{y}}$ as in (6) with $t_0 = 0$, $Y_d(0) = n$ and $Y_i(0) = 0$ for $i \neq d$, we may solve (3) numerically to find m, verifying (8) at the appropriate points of the computation.

It turns out that these hold for each d in Table 1, and that in each case $m = d-1$, for sufficiently small $\epsilon > 0$. For such ϵ, the value of $\tilde{y}_{d+1}(x_m)$ may be computed numerically (the result is shown as the constants $\mathcal{T}_u(d)$ in Table 1), and then by Theorem 2, this is the asymptotic value of the size of the total dominating set $\mathcal{T}$ (scaled by n) at the end of some randomised algorithm. So the conclusion is that a random d-regular graph a.a.s. has a total dominating set of size at most $n\tilde{y}_{d+1}(x_m) + o(n)$. Note also that (by the theorem) $\tilde{y}_i(x) \equiv 0$ in phase k for $1 \leq i \leq d-k-1$, and by the nature of the differential equation, $\tilde{y}_i(x)$ will be strictly positive for $i > d-k$. So by (6) and (7), the end of the process (for ϵ arbitrarily small) occurs in phase $d-1$ when $\tilde{y}_1$ becomes 0.

This completes the proof of Theorem 1.

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