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# Distance domination and amplifier placement problems

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**Abstract**

We consider the optimisation problem defined on a connected undirected graph with given root vertex and a parameter $\bar{s}$, in which we seek a spanning tree with the smallest number of special (amplifying) vertices such that each vertex in the tree is preceded on its unique path from the root vertex by an amplifying vertex no more than $\bar{s}$ edges distant. This problem, which we call the amplified spanning tree (AST) problem, is motivated by a practical problem in local access television cable networks. We show that even restricted to planar graphs with no vertex degree exceeding four, AST is NP-complete for every fixed $\bar{s} \geq 1$. Making use of a connection with distance domination problems, we show that two related problems, including total distance domination, are also NP-complete and we derive an approximability upper bound for AST.

# 1 Introduction

In this paper we consider the combinatorial optimisation problem known as the amplified spanning tree (AST) problem, which was introduced in [10] under the name STPA ("spanning tree problem with amplification"). The study of AST was originally motivated by a problem in the design of local access coaxial copper cable networks for the broadcast of cable television signal. A problem with a similar flavour was studied in [8].

We shall extend the existing computational complexity results for AST and for two related problems, and use the new results to derive an approximation ratio for AST.

In this introduction we give the notation to be used throughout the paper, and review the results obtained in [10].

## 1.1 Notation and problem definition

As explained in [10], AST requires the construction of a spanning tree on a finite simple connected undirected graph. Throughout the paper, unless otherwise noted, $G$ shall denote such a graph with vertex set $V$ and edge set $E$, and a distinguished vertex $v_0 \in V$ that is determined by physical conditions. We refer to $v_0$ as the *root vertex*. In terms of the original motivating cable network problem, the spanning tree serves to carry a signal from the root vertex to all other vertices in $V$.

**Definition 1** *[10] Given a graph $G$, let $T = (V, E_T)$ be a spanning tree on $G$ and $P \subseteq V \setminus \{v_0\}$ be a vertex subset; then $(T, P)$ is called an* amplified tree *on $G$. We call $P$ an* amplification set*, and the members of $P$* amplifying vertices*; we call $P \cup \{v_0\}$ a* supply set*.*

Given an amplified tree $(T, P)$, for each vertex $v \in V$ the integer $s(v)$ represents the strength of the signal entering $v$. The signal strengths are dependent on an integer $\bar{s}$ called the *attenuation parameter* that is determined by physical conditions. In the following definition, for each vertex $v \in V \setminus \{v_0\}$ we denote by $\mu(v)$ the *parent vertex* of $v$, which immediately precedes $v$ on the unique path in $T$ from $v_0$ to $v$.

**Definition 2** *[10] Let $(T, P)$ be an amplified tree on a graph $G$. For each vertex $v \in V$, the* supply strength *$s(v)$ is calculated as follows: set $s(v_0) = \bar{s}$, and recursively calculate $s(v)$ for all $v \in V \setminus \{v_0\}$ using the formula*

$$s(v) = \begin{cases} \bar{s} - 1, & \text{if } \mu(v) \in P \cup \{v_0\}; \\ s(\mu(v)) - 1, & \text{otherwise.} \end{cases}$$

*If $s(v) \geq 0$ for all $v \in V \setminus \{v_0\}$, we say that $(T, P)$ is $\bar{s}$-*feasible *and that $P \cup \{v_0\}$ is an $\bar{s}$-*feasible supply set*, and we define the* cost *of $(T, P)$ as the cardinality $|P|$ of $P$.*

Where the dependence on $\bar{s}$ is implicit, we shall use the term *feasible* synonymously with "$\bar{s}$-feasible".

Observe that for a vertex $v \in V \setminus \{v_0\}$ and a feasible amplified tree $(T, P)$, $s(v) = 0$ implies either that $v \in P$ or that $v$ is a leaf of $T$.

In terms of the cable network application, a signal of high strength is continuously supplied by some means to the root vertex $v_0$ of the network, or equivalently the signal leaving $v_0$ is generated there without cost. The signal leaves the root vertex having the given maximum strength $\bar{s}$, and the signal attenuates as it travels along edges of the tree $T$. In our model, all edges impose an identical attenuation of one unit of signal strength per edge. Therefore, with no amplifying vertices, the signal can travel at most $\bar{s}$ edges away from the root vertex before vanishing. Yet we require that a signal of non-negative strength reach all vertices in the graph; hence at any vertex there is the possibility of signal amplification, which sets the signal leaving that vertex to the maximum strength $\bar{s}$. Signal amplification incurs a uniform cost at each vertex where it occurs, i.e., for each vertex in $P$. (We can interpret the maximum signal strength at the root vertex as having been achieved by zero-cost amplification at $v_0$.)

Generally for a given rooted graph $G$, the number of trees, and hence the number of feasible amplified trees, is exponential in $|V|$ (see [11], for example). As discussed in [10], the optimisation problem variant of AST seeks a feasible amplified tree of minimal cost, under the assumption that signal amplification has the same cost at every amplifying vertex; thus the sought-after feasible amplified tree $(T, P)$ is that for which $|P|$ is minimal.

Throughout the paper we shall be concerned with the decision problem variant of AST, which is defined as follows.

**Definition 3** *[10] Given a graph $G$ and a positive integer $K \leq |V| - 1$, we say that $I = (G, K)$ is an* instance *of the AST problem, and we ask whether there exists on $G$ a feasible amplified tree $(T, P)$ with cost no greater than $K$.*

For a particular value $\bar{s}$ of the attenuation parameter, we may say that $(G, K)$ is an instance of the AST($\bar{s}$) problem. Where the dependence on $\bar{s}$ is implicit, as is usually the case, we shall refer to the problem simply as AST.

As previously noted, [10] called the above problem the spanning tree problem with amplification (STPA). We prefer to call it the amplified spanning tree (AST) problem to fit the established pattern for names of tree problems such as minimum spanning tree, capacitated spanning tree, Steiner tree, shortest total path length tree, degree constrained spanning tree, maximum leaf spanning tree, and bounded diameter spanning tree (see [3]).

To clarify the nature of the AST problem, we repeat an example from [10]: consider the graph $G$ as shown in Figure 1, with $V = \{1, 2, 3, 4, 5, 6\}$, $E = \{a, b, c, d, e, f, g, h\}$, and $v_0 = 1$. Let $\bar{s} = 2$.

Consider the spanning tree $T$ with edge set $E_T = \{a, c, e, f, h\}$. Without any amplifying vertices, we have $s_1 = 2,\ s_2 = 1,\ s_3 = 0,\ s_4 = -1,\ s_5 = -2$, and $s_6 = -3$. We deduce that $(T, \emptyset)$ is not a feasible amplified tree. However $(T, \{2, 4\})$ is a feasible amplified tree: defining $\mathbf{s} = (s_1, s_2, s_3, s_4, s_5, s_6)$, we have $\mathbf{s} = (2, 1, 1, 0, 1, 0)$ so that

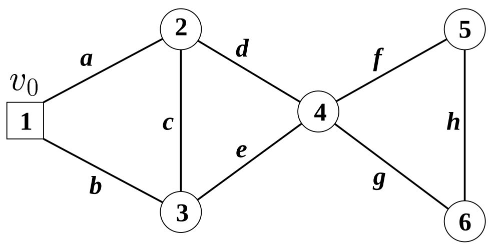


Figure 1: A small graph $G$ with $v_0 = 1$ and $\bar{s} = 2$.

$\mathbf{s} \geq \mathbf{0}$. An alternative feasible amplified tree is $(T, \{3,4\})$: in this case we have $\mathbf{s} = (2,1,0,1,1,0)$. Each of these feasible amplified trees $(T, P)$ has a cost of two, which in each case is the cardinality of the set $P$ of amplifying vertices. However, neither of these amplified trees is optimal: consider the spanning tree $T'$ with edge set $\{a,b,d,f,g\}$ and the amplification set $P = \{4\}$. We calculate $\mathbf{s} = (2,1,1,0,1,1)$ and deduce that $(T', \{4\})$ is feasible. In fact $(T', \{2\})$ is also feasible. Hence there exist feasible amplified trees of cost one. Yet for no amplification set $P$ of cardinality one is $(T, P)$ feasible with the original spanning tree $T$; thus $\{2,4\}$ and $\{3,4\}$ are amplification sets $P$ of minimal cardinality satisfying the condition that $(T, P)$ is feasible.

For a fixed spanning tree $T$, we say that an amplification set $P$ for which $(T, P)$ is feasible, and the cardinality of which is minimal over all such amplification sets, is *minimal for* $T$, and that the cardinality of a minimal amplification set is the *minimal amplifier cost of* $T$. A *minimal-cost spanning tree* is a spanning tree with the least minimal amplifier cost. The simple example above demonstrates that the minimal amplifier cost is quite sensitive to the choice of spanning tree $T$.

### 1.2 Existing results

The authors of [10] considered the optimisation problem variant of AST. They presented several integer programming formulations and a simple heuristic, outlined the relationship of AST to distance domination problems in graphs, and gave some initial complexity results. In their heuristic analysis they employed the result of [9] that given a spanning tree $T$, it is easy to construct an amplification set $P$ that is minimal for $T$ and hence obtain the minimal amplifier cost of $T$. This construction is achieved using a dynamic programming algorithm with running time bounded by a polynomial in the size of the graph.

Thus the AST problem is easily solved if the structure of a minimal-cost spanning tree is known. Finding such a spanning tree, however, is not straightforward. The heuristic approach in [10] randomly generates a large number of different trees of shortest paths from $v_0$, and selects the cheapest as computed by the dynamic programme. Experimentally this seemed to yield reasonable solutions; however the heuristic can give arbitrarily bad solutions on certain examples.

In an effort to find minimal solutions for several test instances of AST, the solution yielded by the randomised heuristic was used as an upper bound in conjunction with each of two mixed-integer programming (MIP) formulations. Even the better MIP formulation was shown to give poor lower bounds and feasible solutions.

The difficulty of solving AST in general was shown in [10] to be a consequence of its relationship to distance domination problems. For surveys in this context, see [5] and [6].

**Definition 4** *The* distance $d_G(u, v)$ *between two vertices* $u$ *and* $v$ *in a graph* $G$ *is the length of a shortest path in* $G$ *between them. A vertex* $v$ *is said to* $k$-dominate *precisely those vertices within distance* $k$ *of* $v$. *A subset* $D$ *of the vertices of a graph* $G$ *is said to* $k$*-dominate* $G$ *if every vertex in* $G$ *is* $k$*-dominated by some vertex in* $D$*, and in that case* $D$ *is called a* $k$-dominating set *of* $G$. *A* 1*-dominating set is also called simply a* dominating set*. A* connected dominating set *is a dominating set that induces a connected subgraph of* $G$.

The term "$k$-domination" is borrowed from [6, p.321], but the reader is warned that in [5, p.184] the same terminology is used in another context (that of multiple domination).

We note that all decision problems in the current paper are in NP (are 'non-deterministic polynomial'), as for each of them a candidate solution can be checked in polynomial time. In [10] it was shown that AST(1) is equivalent to the connected dominating set (CDS) problem, which is NP-complete even on planar graphs with no vertex degree greater than 4 (see [3]), and that therefore AST(1) is NP-complete on such graphs. This equivalence arises from the observation that if $\bar{s} = 1$ then a feasible supply set $P \cup \{v_0\}$ induces a connected subgraph of $G$ and is a dominating set.

**Lemma 1** *[10] AST*(1) *is NP-complete, even on planar graphs with no vertex degree exceeding 4.*

The case $\bar{s} > 1$ was then considered. We introduce a useful term, so as to rephrase Lemma 2 from [10] in accordance with the current notation.

**Definition 5** *For a positive integer* $k$*, the* $k$-th power graph $G^k$ *is a graph with the same vertices as* $G$ *and an edge between every pair of vertices that are within distance* $k$ *of each other in* $G$*. A set of vertices of* $G$ *is said to be* $k$-pervasive *if it induces a connected dominating set on* $G^k$.

Where the dependence on $k$ is implicit, the term *pervasive* is used synonymously with "$k$-pervasive". It is now quite straightforward to deduce the following result.

**Lemma 2** *[10] The supply set of a feasible amplified tree is pervasive.*

Thus every feasible supply set is a feasible solution to what we call the CDS($\bar{s}$) problem, which is that of finding a CDS in $G^{\bar{s}}$. Observe that CDS(1) is the ordinary CDS problem, which as noted is equivalent to AST(1). For $\bar{s} \geq 3$, however, it was

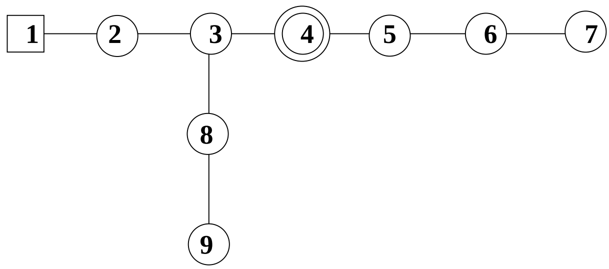


Figure 2: With $\bar{s} = 3$, {1,4} is a pervasive set that is not the supply set of a feasible amplified tree.

shown in [10] that a pervasive set $P \cup \{v_0\}$ is not necessarily a feasible supply set; Figure 2 repeats the example given there.

In the graph of Figure 2, let $\bar{s} = 3$. The set $\{1, 4\}$ is pervasive; however it is not a feasible supply set, since there is only one possible spanning tree $T$ and $(T, \{4\})$ is not a feasible amplified tree (as the supply strength $s(9)$ is negative).

### 1.3 New results

We begin by proving an equivalence that was stated without proof in [10]. First we require a definition.

**Definition 6** *Let* $(T, P)$ *be an amplified tree on the graph* $G$. *Consider* $v \in V \setminus \{v_0\}$ *and the unique path* $(v_0, v_1, v_2, \ldots, v_k = v)$ *in* $T$ *from* $v_0$ *to* $v$. *Let* $\bar{s}$ *be a positive integer. The* supply ancestors *of* $v$ *in* $T$ *are the members of the set* $A_{\bar{s}}(v) = \{v_{k-1}, v_{k-2}, \ldots, v_{k-\min\{k,\bar{s}\}}\} \cap (P \cup \{v_0\})$. *Also the* supply neighbours *of* $v$ *in* $T$ *are the members of the set* $N_{\bar{s}}(v) = \{s \in P \cup \{v_0\} : 0 < d_T(v, s) \leq \bar{s}\}$.

**Lemma 3** *The amplified tree* $(T, P)$ *on* $G$ *is feasible if and only if for all* $v \in V \setminus \{v_0\}$, $A_{\bar{s}}(v)$ *is non-empty.*

**Proof.** ($\Rightarrow$) Let $v \in V \setminus \{v_0\}$, let $(v_0, v_1, v_2, \ldots, v_k = v)$ be the unique path in $T$ from $v_0$ to $v$, and assume that $A_{\bar{s}}(v) = \emptyset$. By the definition of feasibility, $s(v_k) = s(v) \geq 0$. But also by the definition, $s(v_k) = s(\mu(v_k)) - 1 = s(v_{k-1}) - 1$ since $v_{k-1} \notin P \cup \{v_0\}$, and inductively $s(v_k) = s(v_{k-b}) - b$ for all $b \in \{1, 2, \ldots, \min\{k, \bar{s}\}\}$ since for all such $b$ we have $v_{k-b} \notin P \cup \{v_0\}$. Hence in particular, $s(v_{k-\min\{k,\bar{s}\}}) = s(v_k) + \min\{k, \bar{s}\} \geq \min\{k, \bar{s}\}$. But for all $w \in V \setminus \{v_0\}$ we have $s(w) < \bar{s}$, so either $\min\{k, \bar{s}\} = k$ or $v_{k-\min\{k,\bar{s}\}} = v_0$, and in either case $v_{k-\min\{k,\bar{s}\}} = v_0$, contradicting the original assumption.

($\Leftarrow$) Let $v \in V \setminus \{v_0\}$, and choose $v_{k-a}$ so that $a = \min\{b : v_{k-b} \in A_{\bar{s}}(v)\}$. If $a = 1$ then $\mu(v) \in P \cup \{v_0\}$, so $s(v) = \bar{s} - 1 \geq 0$. Otherwise, similar to above, inductively we have $s(v_k) = s(v_{k-b}) - b$ for all $b \in \{1, 2, \ldots, a-1\}$ since for all such $b$, $v_{k-b} \notin P \cup \{v_0\}$. But $s(v_{k-(a-1)}) = \bar{s} - 1$, and hence $s(v_k) = \bar{s} - 1 - (a-1) = \bar{s} - a$. But $a \leq \min\{k, \bar{s}\} \leq \bar{s}$, and we conclude that $s(v) \geq 0$. ■

**Corollary 1** *Let $(T, P)$ be an amplified tree on the graph $G$. If $(T, P)$ is $\bar{s}$-feasible then for all integers $t \geq \bar{s}$, $(T, P)$ is $t$-feasible.*

**Proof.** For all $v \in V \setminus \{v_0\}$, clearly $A_{\bar{s}}(v) \subseteq A_t(v)$. Hence if $A_{\bar{s}}(v)$ is non-empty then $A_t(v)$ is also. ■

The remainder of the paper proceeds as follows. In section 2 we show, via polynomial reduction from a known NP-complete problem (connected vertex cover), that AST is NP-complete for any fixed value of $\bar{s}$. We also consider the related question, which we call the amplified spanning tree with amplifiers known (ASTAK) problem, of whether for given amplifier locations $P$ in a graph $G$, a feasible amplified tree $(T, P)$ can be constructed. We show that ASTAK is NP-complete for all $\bar{s} \geq 3$. For $\bar{s} \leq 2$, ASTAK has the answer YES if and only if the supply set is pervasive, and for this case we give a polynomial algorithm to construct a feasible tree. Finally, we prove NP-completeness for the related problem of total distance domination, in which we seek a sufficiently small vertex subset $S$ such that each vertex in $G$ is $k$-dominated by some distinct vertex in $S$. Section 3 derives an approximability result for AST, which follows from its relationship to the CDS problem. These results highlight the close relationship between AST($k$) and $k$-domination problems.

## 2 Complexity Results

### 2.1 AST

It is already known that AST(1) is NP-complete (see [10], and section 1.2 above). Furthermore, as this follows by a transformation from the connected dominating set problem, the result remains valid even if AST(1) is restricted to regular graphs of degree 4 or planar graphs of maximum degree not exceeding 4 (see [3, p.339]).

We now prove that AST($\bar{s}$) is NP-complete for every fixed $\bar{s} \geq 2$. We transform from the connected vertex cover (CVC) problem, which is known to be NP-complete even on planar graphs with no vertex degree exceeding 4 (see [2]): this is a natural class of graphs to consider in the context of terrestrial cable networks. It is clear that AST is in NP: in polynomial time we can nondeterministically 'guess' the edges of a spanning tree $T$ of $G$ and a set $P$ of amplifying vertices, and then check whether $(T, P)$ is a feasible amplified tree with $|P|$ sufficiently low.

Let $I = (G, K)$ be an instance of CVC: we ask whether there is a subset $S \subseteq V$ with $|S| \leq K$ such that $S$ induces a connected subgraph of $G$ and for each edge $\{u, v\} \in E$ at least one of $u$ and $v$ belongs to $S$. If it satisfies these conditions, $S$ is called a *connected vertex cover* for $G$. We also restrict $G$ to be planar with no vertex degree exceeding 4.

Choose an edge $\{i, j\} \in E$; at least one of $i$ and $j$ must belong to any vertex cover $S$ for $G$. We polynomially transform to a set $\mathcal{C}$ of two instances $(G' = (V \cup V', E'), K - 1)$ of AST, containing one instance for each choice of root vertex $v_0$ from $\{i, j\}$. The graph $G'$ is obtained by subdividing each edge in $E$ into $\bar{s}$ edges, so that for each edge in $E$, $(\bar{s} - 1)$ new vertices are added to $G$. This gives $G' =$

$(V \cup V', E')$, where $V'$ is the set of $(\bar{s}-1)|E|$ new vertices and $E'$ is the set of $\bar{s}|E|$ edges replacing those in $E$. This transformation preserves both the planarity of the graph and the degrees of original vertices $V$; the new vertices in $V'$ are all of degree 2. Intuitively, a connected vertex cover in the original graph $G$ gives a feasible supply set of vertices in the transformed graph $G'$, as every vertex $v'$ in $G'$ is supplied by an amplifying vertex at one end of that edge of $G$ on which $v'$ lies.

We now show that the instance of CVC has the answer YES if and only if at least one of the two corresponding instances in $\mathcal{C}$ of AST has the answer YES. The following lemma is helpful.

**Lemma 4** *For a given CVC instance $(G, K)$, let $I = \big(G' = (V \cup V', E'), K-1\big)$ be an instance of AST that has been constructed as described above, with the root vertex $v_0 \in \{i, j\} \in E$. If $I$ has the answer YES, then the corresponding feasible amplified tree $(T, P')$ can be modified in polynomial time to give a feasible amplified tree $(T, P)$ such that $P \subseteq V \setminus \{v_0\}$ and $|P| \leq |P'|$.*

**Proof.** Consider the feasible amplified tree $(T, P')$ as hanging down from its root $v_0$. Observe that any vertex in $V$ has depth $c\bar{s}$ where $c$ is a non-negative integer, and that any vertex in $V'$, with depth $d$, has a unique descendant in $V$ of depth $\lceil d/\bar{s} \rceil \bar{s}$. We see that any vertex in $P' \cap V'$ of depth less than $\bar{s}$ can be replaced by its unique descendant farther down the tree of depth exactly $\bar{s}$, and that this operation does not affect the feasibility of the tree nor does it increase the cardinality of the set of amplifying vertices. Now assuming that all vertices in $P'$ of depth no more than $k\bar{s}$ are in $V$, we see that the same operation can be carried out to ensure that the vertices in $P'$ to a depth of $(k+1)\bar{s}$ are all in $V$. The entire process takes $O(|E|)$ time. ■

Thus if either instance of AST from the set $\mathcal{C}$ has the answer YES, then the set $P'$ of amplifying vertices of the feasible amplified tree $(T, P')$ is easily modified to give a new set $P$, so that the new amplifying vertices are all members of the vertex set $V$ of the original graph $G$. The vertices in $V'$ that are internal to an edge $e$ in $E$ can now be supplied in $G'$ only from an amplifying vertex that lies at one of the original endpoints of $e$, and hence each edge in $E$ is covered in $G$ by a vertex from the supply set $P \cup \{v_0\}$. Thus the supply set forms a vertex cover of $G$. Furthermore, as noted in Section 1, the supply set $P \cup \{v_0\}$ is connected in the $\bar{s}$-th power graph of $G'$; but the power graph of $G'$, restricted to $V$, is just $G$. Thus $P \cup \{v_0\}$ forms a connected vertex cover of $G$ with $|P \cup \{v_0\}| \leq K$, and the instance $I$ of CVC has the answer YES.

Conversely, if the instance $I$ of CVC has the answer YES, then there is a connected vertex cover $S \subseteq V$ of $G$ with $|S| \leq K$. At least one of $i$ and $j$ must belong to $S$, and we denote this vertex $v_0$. We then select the instance $I_{v_0} = \big(G', K-1\big) \in \mathcal{C}$ of AST, with root vertex $v_0$ and $G' = (V \cup V', E')$, and choose a set $P' = S \setminus \{v_0\}$ of amplifying vertices. Because $S = P' \cup \{v_0\}$ is connected, we may construct a spanning tree $T$ on the subgraph of $G$ induced by $S$, and then construct the corresponding tree $T'$ in $G'$. Given the special structure of $G'$, and as $V \cup V'$ is within distance $\bar{s}$ of $S$, it is clearly possible to augment the tree $T'$ with additional edges from $E'$ so that $T'$ becomes a spanning tree on $G'$ and $\big(T', S \setminus \{v_0\}\big)$ is a feasible amplified tree. Therefore the instance $I_{v_0} \in \mathcal{C}$ of AST has the answer YES.

Thus we see that AST is NP-complete for all $\bar{s} \geq 2$. We deduce, using the aforementioned result from [10], the desired theorem.

**Theorem 1** *AST is NP-complete for all $\bar{s} \geq 1$, even on planar graphs with no vertex degree exceeding 4.*

## 2.2 ASTAK

We now turn to the proof that the planar ASTAK (amplified spanning tree with amplifiers known) problem is NP-complete.

**Definition 7** *Given a graph $G$ with root vertex $v_0$, and a subset $P \subseteq V \setminus \{v_0\}$, the instance $\big(G, P\big)$ of ASTAK asks whether there exists a feasible amplified tree $(T, P)$ on $G$.*

In other words, for a given amplification set $P$ on the graph $G$ we seek a spanning tree $T$ that yields a feasible amplified tree $(T, P)$ on $G$. We restrict the problem to the case where $G$ is planar.

We shall prove that if $\bar{s} \geq 3$, then planar ASTAK is NP-complete. To show this we transform from the planar 3SAT problem, P3SAT ('planar satisfiability'), which is known to be NP-complete (see [7]).

An instance of 3SAT is a set $U = \{v_1, \ldots, v_n\}$ of truth variables and a Boolean formula over $U$ in conjunctive normal form with exactly three literals per clause. For an instance $B$ of the 3SAT problem, we ask whether there is a truth assignment of the given variables that makes $B$ true (in other words, we ask whether $B$ is 'satisfiable'). We write $B$ as a set of clauses $\{c_1, \ldots, c_m\}$, where each clause is a subset of three literals from the union of the sets $V = \{v_1, \ldots, v_n\}$ and $\overline{V} = \{\overline{v_1}, \ldots, \overline{v_n}\}$. We choose to write the clause $\{a_1, a_2, a_3\}$ in the form $(a_1 + a_2 + a_3)$. Lichtenstein [7] calls $G(B) = (N, A)$ the *graph of $B$*, where $N = \{c_i \mid 1 \leq i \leq m\} \cup \{v_j \mid 1 \leq j \leq n\}$ and $A = A_1 \cup A_2$ with

$$A_1 = \big\{\{c_i, v_j\} \;\big|\; v_j \in c_i \text{ or } \overline{v_j} \in c_i\big\}, \quad A_2 = \big\{\{v_j, v_{j+1}\} \;\big|\; 1 \leq j < n\big\} \cup \big\{\{v_n, v_1\}\big\}.$$

Clearly $G(B)$ may be constructed in time that is polynomial in the size of $B$. P3SAT ('planar satisfiability') is the restriction of 3SAT to instances where $G(B)$ is planar.

Let $B$ be an instance of P3SAT. When $G(B)$ is embedded in the plane the edges in $A_2$ form a closed curve that divides the plane into two regions. The embedding is *bipolar* if at each vertex representing a variable $v_j$, the edges representing positive instances $v_j$ of the variable are separated by the closed curve from the edges representing negative instances $\overline{v_j}$ of the variable. We say a graph is bipolar if it has a bipolar embedding in the plane. It is shown in [7] that planar satisfiability is NP-complete even when $G(B)$ is bipolar. We impose this condition on $B$, and shall assume that a bipolar embedding $G(B)$ is given as part of the instance. By means of illustration, let $B = c_1 \cdot c_2 \cdot c_3$, where $c_1 = (a_1 + \overline{a_2} + \overline{a_3})$, $c_2 = (\overline{a_3} + a_4 + a_4)$, and $c_3 = (\overline{a_1} + a_3 + \overline{a_4})$. The (bipolar) graph of $B$ is shown in Figure 3.

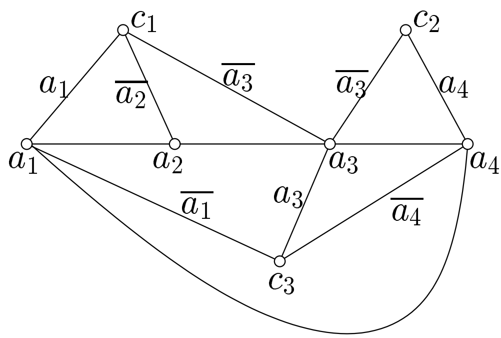


Figure 3: The graph $G(B)$ for the bipolar instance $B = (a_1 + \overline{a_2} + \overline{a_3}) \cdot (\overline{a_3} + a_4 + a_4) \cdot (\overline{a_1} + a_3 + \overline{a_4})$ of P3SAT.

We now give a polynomial-time algorithm that converts a bipolar embedding $G(B)$, for a formula $B$ in P3SAT, into an instance $I = (G, P)$ of ASTAK such that

(i) $G$ is planar;

(ii) $I$ has the answer YES if and only if $B$ has the answer YES.

The graph $G$ is constructed from $G(B)$ in polynomial time by adding the edge $\{v_0, v_1\}$, where $v_0$ is the root vertex, and replacing each edge $\{c_i, v_j\}$ by a path of $\bar{s} - 1$ edges (represented in Figures 4 and 5 by a dashed line). The edge $\{v_n, v_1\}$ is removed. In addition, for each of the variable vertices $\{v_1, \ldots, v_n\}$ a local replacement, a 4-cycle, is made in the manner demonstrated in Figure 4. This yields the graph $G$, as shown in Figure 5 for our previous example of $B$ (Figure 3). The square on the left of Figure 5 denotes the root vertex $v_0$, and double circles indicate amplifying vertices (the elements of $P$). Note that the supply set $P \cup \{v_0\}$ is a pervasive set, which as noted in Section 1 is necessary (but not sufficient, if $\bar{s} \geq 3$) for ASTAK to have the answer YES.

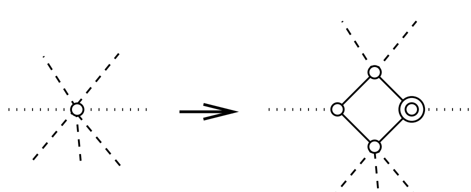

Figure 4: The local replacement of a variable vertex $v_j$ in $G(B)$. In the 4-cycle that is the local replacement, the edges representing positive instances $v_j$ are either all at the top or all at the bottom of the cycle, and the edges representing $\overline{v_j}$ are at the other side.

Assuming that $\bar{s} \geq 3$, a signal passes from the root vertex on the left through each of the amplifying vertices in turn, having a choice of one out of two paths

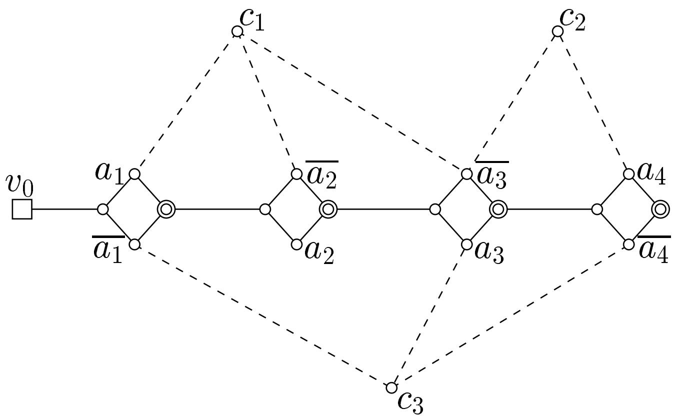


Figure 5: The instance of ASTAK corresponding to $B$ in Figure 3. A dashed line represents a path of $\bar{s}-1$ edges.

to reach each amplifying vertex from the preceding one. These choices correspond to a truth assignment for the literals. If the signal reaches an amplifying vertex through a vertex marked with a literal $x$, then the signal strength at $x$ is $\bar{s}-2$ (corresponding to FALSE), and the amplifying vertex can then supply a signal of strength $\bar{s}-1$ (corresponding to TRUE) to the negated literal vertex $\bar{x}$. Regardless of the assignment chosen, the signal has sufficient strength to supply every internal vertex on every path of $\bar{s}-1$ edges that lies between a clause vertex and a vertex corresponding to a literal. However, in order to supply the clause vertex itself, the signal must come from a vertex corresponding to a literal that is in the clause and has been assigned TRUE according to the scheme above. Thus it is clear that the instance $I$ can be answered in the affirmative if and only if $B$ can be satisfied. The desired result follows.

**Theorem 2** *Planar ASTAK is NP-complete for all* $\bar{s} \geq 3$.

Theorem 2 is in some sense the best result possible, as the following shall show.

**Theorem 3** *Suppose that* $\bar{s} \leq 2$*. There exists a deterministic polynomial time algorithm to solve ASTAK*($\bar{s}$)*. Let* $I = \big(G=(V,E), v_0, P\big)$ *be an instance of ASTAK*($\bar{s}$) *and* $S = P \cup \{v_0\}$ *the supply set. The answer to* $I$ *is YES if and only if* $S$ *is pervasive in* $G$.

**Proof.** We first give an algorithm to solve an instance of ASTAK(2) which involves growing a tree $T$ in two phases. At each point $U$ will denote the vertex set of $T$. Initially $T$ is a star consisting of $v_0$ and its neighbours in $V \setminus P$. At each step of phase 1 we identify an amplifying vertex $v_a \in P \setminus U$ that is adjacent to some vertex $v'_a \in U$. We add to $T$ a star centred on $v_a$ with edges connecting $v_a$ to $v'_a$ and to every neighbour of $v_a$ that is not in $U \cup P$. Phase 1 continues until no appropriate

$v_a$ can be found. We observe that if $S$ is connected in $G^2$, then at this stage it must be spanned by $T$ and every vertex that is either in $S$ or has a neighbour in $S$ must be included in $U$. Note that candidates for $v_a$ can easily be found by searching the neighbours of vertices in $U$ to find an element of $P$. There is no need to check the neighbourhood of a given vertex more than once, so phase 1 can easily be completed in time polynomial in $|V|$.

Let $U_1$ be a snapshot of $U$ at the end of phase 1. At the start of phase 2, for each vertex $v \in V \setminus U_1$ we identify a vertex $v' \in U_1$ that is adjacent to $v$. This is possible if $S$ forms a connected dominating set in $G^2$, because then every vertex is within distance 2 of some element of $S$ whose neighbours are all in $U_1$. If a $v'$ cannot be found for each $v$ then the algorithm returns FAIL. Otherwise, phase 2 is completed by adding the edge between $v$ and its corresponding $v'$ for each $v \in V \setminus U_1$. Note that the algorithm either constructs a spanning tree or returns FAIL. We claim that whenever it constructs a spanning tree that tree is $\bar{s}$-feasible, and that it returns FAIL only if no $\bar{s}$-feasible spanning tree exists.

To verify this claim first note that at every stage of phase 1 the signal has non-negative strength at all vertices in $U$ and has unit strength at vertices in $U \setminus S$. It then follows that all vertices added to $T$ in phase 2 have signal strength 0 and thus the final tree is feasible. It remains to show that the algorithm does not return FAIL if a feasible tree exists.

Suppose that a feasible spanning tree, $T_f$, exists and consider the state of our tree $T$ at the conclusion of phase 1. Suppose that $P \not\subset U_1$ and pick $v \in P \setminus U_1$ to be minimal with respect to distance from $v_0$ in $T_f$. The parent $v'$ of $v$ in $T_f$ cannot be in $U_1$ since otherwise $v$ would have been added to $U$ during phase 1. Therefore by the choice of $v$, $v'$ cannot be in $P$. Since $\bar{s} = 2$ the parent $v''$ of $v'$ in $T_f$ must be in $P$ and hence (by choice of $v$) in $U_1$. But this is a contradiction because it means $v'$ is in $U_1$ after all. Hence $P \subset U_1$. Also if $u \notin P$ is adjacent to $u' \in P$ then $u$ has been added to $U$ no later than the step in which $u'$ was added. Now consider $v \in V \setminus U_1$ and its parent $v'$ in $T_f$. Since $v \notin U_1$ it is not adjacent to an amplifying vertex and therefore by Lemma 3, $v'$ must be so adjacent or its parent must be $v_0$. Hence $v' \in U_1$ and $v$ has a vertex to which it can be joined in phase 2. Thus the algorithm does not return FAIL if $T_f$ exists.

The case $\bar{s} = 1$ is a simplification of the algorithm given above. Phase 1 consists of finding a spanning tree for the (connected) graph induced by $S$, while phase 2 remains as described.

The necessity of $S$ being connected in $G^{\bar{s}}$ for ASTAK($\bar{s}$) to yield a positive answer is true more generally for any $\bar{s} \geq 1$ and has been noted in Lemma 2. The sufficiency of this condition for $\bar{s} \leq 2$ follows from the observation that the algorithm outlined above finds a solution in this case. ∎

The description of a candidate tree for AST($\bar{s}$) is enough to determine its optimal set of amplifying vertices in polynomial time via a dynamic programme ([10]). However, Theorem 2 demonstrates that, unless P = NP, even construction of an $\bar{s}$-pervasive supply set of vertices does not in itself provide in reasonable time a feasible amplified tree for AST($\bar{s}$). This is because a polynomial-time algorithm for constructing a tree $T$ to connect any given $\bar{s}$-pervasive supply set would either succeed, where possible, in completing a feasible tree or would fail to do so within the polynomial-size bound on running time, and either way would answer the question posed by ASTAK($\bar{s}$). If P $\neq$ NP, therefore, any polynomial-time heuristic approach to AST must build at least a partial tree, or must deploy a supply set with more structure than merely the property of being $\bar{s}$-pervasive.

### 2.3 Total distance domination

In the ($\bar{s} = 1$) total domination problem, we consider a graph $G = (V, E)$ and a positive integer $K$ and ask whether there is a subset $S \subseteq V$ with $|S| \leq K$ such that for each $v \in V$ there is an edge $\{v, s\} \in E$ with $s \in S$. We could restate this by saying that we seek a small subset $S \subseteq V$ that is a dominating set and such that the subgraph induced by $S$ has no isolated vertices.

We shall consider a more general version known as the TOTAL DISTANCE DOMINATION problem.

**Definition 8** *[6] An instance $(G, K)$ of TOTAL DISTANCE DOMINATION asks whether, for a given positive integer $\bar{s}$, there is a subset $S \subseteq V$ with $|S| \leq K$ such that every vertex $v$ in $V$ is within distance $\bar{s}$ of some vertex of $S$ other than itself.*

Note that a pervasive set is necessarily a total distance dominating set except in the trivial case when it contains a single element.

Henning's chapter in [6, p.332] states that TOTAL DISTANCE DOMINATION "appears to be a computationally difficult problem". No formal justification of this statement is made, although references are given for NP-completeness proofs of related problems such as DISTANCE DOMINATION. In this section then, we plug what appears to be a gap in the literature.

We prove that planar TOTAL DISTANCE DOMINATION is NP-complete for any given $\bar{s} \geq 1$. To show this we transform, as in section 2.2 above, from P3SAT. We give a polynomial-time algorithm that converts a bipolar embedding $G(B)$ for an $n$-variable formula $B$ in P3SAT into an instance $I = \big(G, 2n\big)$ of TOTAL DISTANCE DOMINATION such that

(i) $G$ is planar;

(ii) $I$ has the answer YES if and only if $B$ has the answer YES.

The graph $G$ is constructed from $G(B)$ in polynomial time by firstly deleting the edges $\{v_j, v_{j+1}\}$ for each $j$ and the edge $\{v_n, v_1\}$. In addition we make a local replacement of each variable vertex $v_j$, in the manner demonstrated in Figure 6. The variable vertex $v$ is replaced by adjacent vertices labelled with the literals $v$ and $\bar{v}$. Where $G(B)$ had an edge from $v$ to a clause vertex $c_i$ we replace this edge with a path of $\bar{s}$ edges from $c_i$ to either $v$ or $\bar{v}$, depending on which literal is contained in $c_i$ in the formula $B$. For each variable $v$ we also add two extra vertices $v^*$ and $v^{**}$. Finally, we add three paths of $\bar{s}$ edges, one from $v$ to $v^*$, a second from $\bar{v}$ to $v^*$, and a third from $v^*$ to $v^{**}$. The resulting graph is still planar.

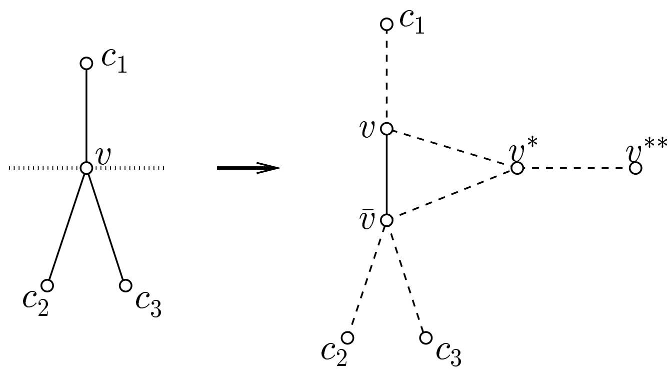


Figure 6: Local replacement for a variable vertex in the reduction from P3SAT to TOTAL DISTANCE DOMINATION. Dashed lines represent paths of $\bar{s}$ edges.

Now suppose that there exists a total distance dominating set $S \subseteq V$ such that $|S| \leq 2n$. For a given variable $v$, let $H_v$ be a subgraph of $G$ containing the vertices $v$, $\bar{v}$, $v^*$, $v^{**}$, and the paths connecting them (see Figure 6). We argue that $H_v$ must contain at least two elements of the total distance dominating set $S$. There must be a vertex $w \in S$ that $\bar{s}$-dominates $v^{**}$, and it can lie only on the path from $v^*$ to $v^{**}$. There must also be a vertex $w' \in S \setminus \{w\}$ that $\bar{s}$-dominates $w$, and $w'$ like $w$ must lie in $H_v$.

In fact if $S$ is to have no more than $2n$ elements then each subgraph $H_v$ must contain exactly two elements of $S$, and vertices such as clause vertices that are not in a subgraph $H_v$ cannot be in $S$. Suppose that $v$ appears in clause $c$ and $\overline{v}$ occurs in clause $c'$. (The case where one or both of the literals does not occur in any clause is an easy adaptation of the following.) Observe that one of the literal-labelled vertices in $H_v$ must be a member of $S$, so as to ensure that $S$ $\bar{s}$-dominates all the internal vertices in the paths that link $c$ to $v$ and $c'$ to $\overline{v}$. However a choice (corresponding to a truth assignment) must be made as to whether the positive literal vertex or the negative literal vertex is in $S$: it cannot be that both $v$ and $\overline{v}$ are in $S$, because the vertex $v^*$ must also be in $S$ in order for $S$ to $\bar{s}$-dominate $H_v$. Each clause vertex must be $\bar{s}$-dominated by a member of $S$ that corresponds to one of the literals in the clause.

Thus it is clear that $I$ can be answered in the affirmative if and only if $B$ can be satisfied, giving the following result:

**Theorem 4** *For any given $\bar{s}$, planar TOTAL DISTANCE DOMINATION is NP-complete.*

## 3 Approximation of AST

Consider a graph $G$ and the CDS($\bar{s}$) problem on $G$, which as noted in Section 1 is the connected dominating set problem on the $\bar{s}$-th power graph $G^{\bar{s}}$, or equivalently the problem of finding an $\bar{s}$-pervasive set $S$ on $G$.

**Theorem 5** *Given a graph $G$ and root vertex $v_0$, let $S$ be an $\bar{s}$-pervasive set such that $v_0 \in S$. We can construct from $S$ a feasible solution to the $AST(\bar{s})$ instance $(G, v_0)$, and this solution contains at most $\log_2(\bar{s}+1)$ amplifying vertices for each member of $S \setminus \{v_0\}$.*

By way of proof, we shall outline the method for constructing a feasible amplified tree. Assume the existence of a suitable feasible solution $S$ to CDS($\bar{s}$), which means $S$ is $\bar{s}$-pervasive in $G$. The first thing we do is construct a spanning tree $T$ of $G$ in which $S$ is still $\bar{s}$-pervasive. To show that this is possible, we need the following lemma.

**Lemma 5** *Let $S$ be an $\bar{s}$-pervasive set on $G$. There exists a spanning tree $T$ of $G$ such that $S$ is $\bar{s}$-pervasive on $T$.*

**Proof.** Give each vertex of $G$ a score that is its distance from $S$ (possibly 0), and give each edge a weight that is the sum of the scores of its endpoints. Now use Kruskal's minimum spanning tree algorithm (see, for example, [1]) to construct $T$, a minimum-weight spanning tree on $G$. The distance from any vertex to $S$ must be the same in $T$ as it is in $G$, as can easily be checked by induction on the distance of a vertex from $S$. During the construction of the minimal spanning tree, after all possible edges of weight $k$ and less have been added, these edges induce a forest in which the set $S$ restricted to each component tree is a $(k+1)$-pervasive set in that tree. So if $S$ is not $\bar{s}$-pervasive in $T$ there must be two vertices $s_1, s_2 \in S$ such that the path from $s_1$ to $s_2$ in $T$ includes no other vertex of $S$, but includes an edge $e$ with weight at least $\bar{s}$. However, since $S$ is pervasive in $G$ there must be a path in $G$ from $s_1$ to $s_2$ in which every edge has weight less than $\bar{s}$. Some edge $e'$ on this path must connect the two components of $T \setminus \{e\}$, and replacing $e$ with $e'$ yields a spanning tree of weight lower than $T$. We conclude that in fact $S$ must be pervasive in $T$. ∎

Next we find an amplification set $P$ to complete our $\bar{s}$-feasible amplified tree $(T, P)$. To illustrate our approach consider the subgraph in Figure 7, for which $\bar{s} = 9$. This should be considered as part of a larger tree. Vertices $s_\mu$ and $s$ are members of $S$, with $s_\mu$ closer in $T$ to $v_0$ than $s$ is. Vertices such as $v_1$ and $v_2$ are $\bar{s}$-dominated by $s$ but not necessarily by $s_\mu$, and due to the direction of signal flow into $s$ those vertices cannot be supplied by $s$. They can be supplied, however, by adding to $P$ the vertices $q_1$ and $q_2$: the set $\{q_1, q_2\}$ defines a feasible set $Q_s$ of amplifying vertices for the subgraph of the spanning tree $T$ that is $\bar{s}$-dominated by $s$.

Note that the amplifying vertex $q_1$, two edges back from $s$ along the path towards $s_\mu$, supplies vertex $v_1$: $v_1$ is $\bar{s}$-dominated by $s$ by means of the edge adjacent to $s$, but signal cannot flow back towards $s_\mu$ along this edge, so the new amplifying vertex $q_1$ is added one edge farther back. Similarly, vertex $q_1$ cannot supply vertex $v_2$. Vertex $v_2$ is distance-dominated by $s$ by means of the three edges going back

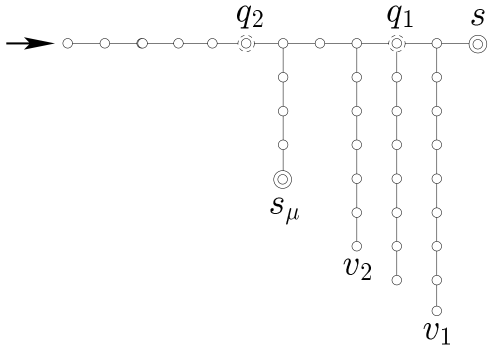


Figure 7: A subgraph of the spanning tree $T$, showing two members $s_\mu$ and $s$ of the pervasive set, with $\bar{s} = 9$. The arrow shows the direction of signal flow in $T$ from the root vertex to $s$. The broken double circles indicate additional vertices included in the amplification set in order to supply vertices, such as $v_1$ and $v_2$, that are $\bar{s}$-dominated but not supplied by $s$.

towards $s_\mu$, which means that we may place the second new amplifying vertex $q_2$ at most three more edges back: signal flowing forwards from $q_2$ supplies vertex $v_2$.

In short, we add a number of new amplifying vertices on the unique path $p_a$ in $T$ from $s$ to $v_0$. The first new amplifying vertex is located $a_1 = 2$ edges back from $s$ along $p_a$. Each successive new amplifying vertex is located $a_{n+1} = 2(a_n + 1)$ edges back from $s$. Solving this recurrence yields $a_n = 2^{n+1} - 2$, for $n \geq 1$. This gives the sequence $(a_n) = (2, 6, 14, 30, \ldots)$, but we need at most $k = \min\left\{i : a_i \geq \left\lceil \frac{1}{2}\bar{s} \right\rceil - 1\right\} = \lfloor \log_2(\bar{s} + 1) \rfloor - 1$ amplifying vertices (because for any positive integer $m$, $\left\lceil \log_2 \left\lceil \frac{m+1}{2} \right\rceil \right\rceil = \lfloor \log_2 m \rfloor$).

We present the details of the algorithm below:

**Algorithm 1** *Approximation algorithm for AST($\bar{s}$)*

*Generate $S \subseteq V$ such that $v_0 \in S$ and $S$ is a CDS($\bar{s}$) on $G$*
*Form a spanning tree $T$ on $G$ such that $S$ is a CDS($\bar{s}$) on $T$*
*Let $T^{\bar{s}}$ be the $\bar{s}$-th power graph of $T$*
*Let $T^{\bar{s}}(S)$ be the subgraph of $T^{\bar{s}}$ induced by $S$*
*Let $T_p$ be a tree of shortest paths from $v_0$ in $T^{\bar{s}}(S)$*
*Let $Q = \emptyset$ (the set of extra amplifying vertices added thus far)*
*Order $S$ by the lengths of paths in $T_p$ from $v_0$ to the elements of $S$.*
***for*** *each $s \in S$*
  *Let $p_a$ be the unique path in $T$ from $s$ to $v_0$*
  *Let $l = \min\{\bar{s}, \text{length}(p_a)\} - 1$*
  *Let $Q_s = \emptyset$; let $k = 1$*
  ***while*** *$k \leq \lfloor \log_2(l + 2) \rfloor - 1$*
    *Add to $Q_s$ the vertex $q_k$ on $p_a$ that is $2^{k+1} - 2$ edges distant from $s$*
    *Let $k = k + 1$*
  *Let $Q = Q \cup Q_s$*

***return*** $(T, S \cup Q)$

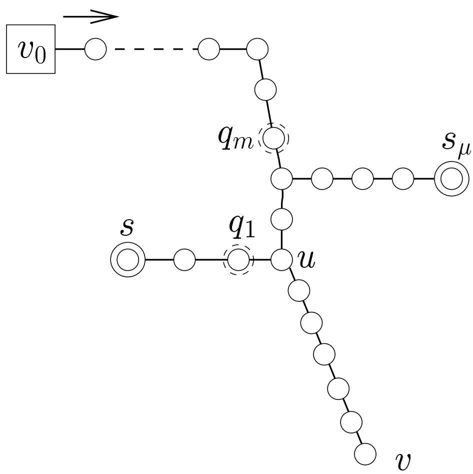


Figure 8: A subgraph of $T$, showing two members $s$ and $s_\mu$ of the pervasive set $S$, with $\bar{s} = 9$. The arrow from the root vertex $v_0$ shows the direction of signal flow in $T$ towards $v$. New amplifying vertices are placed at vertices $q_1$ and $q_m$.

**Lemma 6** *The amplified tree $(T, S \cup Q)$ returned by Algorithm 1 is $\bar{s}$-feasible.*

**Proof.** First consider a vertex $s_n \in S \backslash \{v_0\}$, and the unique path $(s_0 = v_0, s_1, s_2, \ldots, s_n)$ from $v_0$ to $s_n$ in $T_p$. We want to show that the depth of $s_{n-1}$ in $T$ is strictly less than that of $s_n$ in $T$. We note that the depth of $v_0$ in $T$ is strictly less than that of $s_1$ in $T$. Now by induction suppose that the depth of $s_{b-1}$ in $T$ is strictly less than that of $s_b$ in $T$, where $1 \leq b < n$, and assume by contradiction that the depth of $s_b$ in $T$ is not strictly less than that of $s_{b+1}$ in $T$. Then $s_{b+1}$ cannot be a descendant in $T$ of $s_b$ and therefore there is a unique closest common ancestor $a_b$ in $T$ of both $s_b$ and $s_{b+1}$, possibly satisfying $a_b = s_{b+1}$ but not $a_b = s_b$, such that $d_T(a_b, s_{b+1}) \leq d_T(a_b, s_b)$. Clearly there is also a unique closest common ancestor $a_{b-1}$ in $T$ of both $s_{b-1}$ and $s_b$, possibly satisfying $a_{b-1} = s_{b-1}$ but not $a_{b-1} = s_b$, such that $d_T(a_{b-1}, s_{b-1}) < d_T(a_{b-1}, s_b)$. Both $a_{b-1}$ and $a_b$ are ancestors in $T$ of $s_b$, and possibly $a_{b-1} = a_b$. Choosing $a^*$ to be the member of $\{a_{b-1}, a_b\}$ that is of lesser depth in $T$, we see that $a^*$ is the unique closest common ancestor in $T$ of $s_{b-1}$, $s_b$, and $s_{b+1}$.

Note that $d_T(s_{b-1}, s_{b+1}) = d_T(s_{b-1}, a_{b-1}) + d_T(a_{b-1}, a_b) + d_T(a_b, s_{b+1})$. Now either $a^* = a_{b-1}$ or $a^* = a_b$ (or possibly both). If $a^* = a_{b-1}$ then we have $d_T(s_{b-1}, s_{b+1}) = d_T(s_{b-1}, a_b) + d_T(a_b, s_{b+1}) \leq d_T(s_{b-1}, a_b) + d_T(a_b, s_b) = d_T(s_{b-1}, s_b) \leq \bar{s}$, and if $a^* = a_b$ then we have $d_T(s_{b-1}, s_{b+1}) < d_T(s_b, a_{b-1}) + d_T(a_{b-1}, a_b) + d_T(a_b, s_{b+1}) = d_T(s_b, s_{b+1}) \leq \bar{s}$. Thus in either case $s_{b-1}$ and $s_{b+1}$ are adjacent in $T^{\bar{s}}(S)$, which contradicts the assumption that $(v_0, s_1, s_2, \ldots, s_n)$ is a shortest path in $T^{\bar{s}}(S)$.

Let $v \in V \setminus \{v_0\}$, and let $p_0$ be the unique path in $T$ from $v$ to $v_0$. We must show that the set $\{w \in p_0 \cap (S \cup Q) : 0 < d_T(w,v) \leq \bar{s}\}$ is non-empty. Let $N_{\bar{s}}(v) = \{s \in S : 0 < d_T(v,s) \leq \bar{s}\}$ be the supply neighbours of $v$. Choose $s \in N_{\bar{s}}(v)$ such that the distance in $T_p$ from $s$ to $v_0$ is minimised. We can assume that $p_0 \cap N_{\bar{s}}(v)$ is empty, for otherwise we have finished. Thus, regarding $T$ as a tree rooted at $v_0$, we see that $v$ is not a descendant in $T$ of $s$.

Let $s_\mu$ be the parent in $T_p$ of $s$ ($s_\mu$ must exist; otherwise $s = v_0$, and then $v$ is a descendant of $s$). We have shown above that the depth of $s_\mu$ is $T$ is strictly less than that of $s$ in $T$. It cannot be that $s_\mu = v$, for otherwise the parent in $T_p$ of $s_\mu$ would be in $N_{\bar{s}}(v)$, contradicting the choice of $s$. We note that $v$ is not an ancestor in $T$ of $s$, or else $s_\mu$ or its parent in $T_p$ would be in $N_{\bar{s}}(v)$, again contradicting the choice of $s$. Hence there exists a unique closest common ancestor $u$ of both $v$ and $s$ in $T$ such that $u \neq v$ and $u \neq s$ (see Figure 8). Note that because $s_\mu$ is closer than $s$ is to $v_0$ in $T$, $s_\mu$ cannot be $u$ and cannot be a descendant of $u$, for otherwise $s_\mu$ would be closer to $v$ than $s$ is, and thus in $N_{\bar{s}}(v)$, again contradicting the choice of $s$.

Let $L$ be the length of the path in $T$ from $u$ to $s$, and let $q_m$ be the closest ancestor of $u$ (possibly $u$ itself) that is in $Q_s$. By the construction in Algorithm 1, $q_m$ is the $m$-th vertex to be added to $Q_s$ where $m$ satisfies $2^m - 2 < L \leq 2^{m+1} - 2$. Hence $m = \lceil \log_2(L+2) \rceil - 1$, so $q_m$ is at a distance of no more than $2^{\lceil \log_2(L+2) \rceil} - 2$ from $s$ along the path in $T$ from $s$ to $v_0$, and is an ancestor of $u$ and hence of $v$. The distance $d_T(q_m, v)$ in $T$ from $q_m$ to $v$ is equal to $d_T(q_m,u) + d_T(u,v) = d_T(q_m,s) - d_T(u,s) + d_T(u,v) \leq \left(2^{\lceil \log_2(L+2) \rceil} - 2\right) - L + (\bar{s} - L)$. Making use of the fact that $2^{\lceil \log_2 x \rceil} \leq 2(x-1)$ for any integer $x \geq 2$, we deduce that $d_T(q_m, v) \leq \bar{s}$. ■

In general, then, given an $\bar{s}$-pervasive set $S$ including $v_0$, we obtain a feasible amplifying set $P$ that contains $S \setminus \{v_0\}$ and satisfies

$$|P| \leq \big(|S| - 1\big)(1 + k) \leq \big(|S| - 1\big)\log_2(\bar{s} + 1).$$

Any solution to CDS($\bar{s}$) remains $\bar{s}$-pervasive if the vertex $v_0$ is added to it. Therefore if we define $\mathrm{OPT}_X(I)$ to be an optimal solution to the problem $X$ on an instance $I$, and choose $S = \{v_0\} \cup \mathrm{OPT}_{\mathrm{CDS}(\bar{s})}(G)$, then we see that

$$\big|\mathrm{OPT}_{\mathrm{AST}(\bar{s})}(G, v_0)\big| \;\leq\; |P| \;\leq\; \big(|S| - 1\big)\log_2(\bar{s}+1) \;\leq\; \big|\mathrm{OPT}_{\mathrm{CDS}(\bar{s})}(G)\big|\log_2(\bar{s}+1).$$

Any approximation algorithm for CDS can be applied to $G^{\bar{s}}$ to yield an $\bar{s}$-pervasive set $S$ on $G$, with $v_0$ added to $S$ if necessary. In [4] two greedy algorithms are given for finding a CDS on a graph $G$, in which the approximation ratio $f$ depends on $\Delta$, the maximum degree of $G$. The maximum degree of $G^{\bar{s}}$ is bounded from above by $\Delta^{\bar{s}}$. Thus any CDS approximation algorithm with such a ratio $f(\cdot)$ yields, by the above procedure, an algorithm for AST($\bar{s}$) that generates a feasible amplification set $P$ satisfying

$$\begin{aligned} |P| &\leq \log_2(\bar{s}+1)(|S| - 1) \\ &\leq \log_2(\bar{s}+1)\Big(f(\Delta^{\bar{s}})\big|\mathrm{OPT}_{\mathrm{CDS}}(G^{\bar{s}})\big|\Big) \\ &\leq \log_2(\bar{s}+1) f(\Delta^{\bar{s}})\Big(\big|\mathrm{OPT}_{\mathrm{AST}(\bar{s})}(G, v_0)\big| + 1\Big), \end{aligned} \tag{1}$$

because we know that the supply set in a solution to AST($\bar{s}$) certainly remains a solution to CDS($\bar{s}$) if the vertex $v_0$ is added.

The two algorithms in [4] have ratios $f_1(\Delta) = 2\big(1+H(\Delta)\big)$ and $f_2(\Delta) = \log\Delta+3$, where $H(\Delta) \approx \log\Delta+0.7$ is the harmonic function. Thus by (1) we obtain for AST($\bar{s}$) an asymptotic approximation ratio of

$$\log_2(\bar{s}+1)\big(\log(\Delta^{\bar{s}})+3\big) = O(\bar{s}\log\Delta\log(\bar{s})).$$

For fixed $\bar{s}$, this reduces to $O(\log\Delta)$. Moreover, it is easily checked that this construction is completed in polynomial time, since it is based on standard polynomial-time operations such as finding minimum weight spanning trees and shortest paths.

**Corollary 2** *There is a polynomial-time algorithm that, given an instance $(G, v_0)$ of AST($\bar{s}$), finds a solution that is within a factor $O(\bar{s}\log\bar{s}\,\log\Delta)$ of the optimal solution, where $\Delta$ is the maximum degree of $G$.*

## 4 Conclusions

We have studied a problem known as AST which seeks the cheapest distribution of amplifying vertices in a broadcast network, given that in our model the signal can travel along at most $\bar{s}$ edges without amplification. We have shown that the decision problem variant of AST($\bar{s}$) is NP-complete for any fixed $\bar{s}$. We have also shown that the related problem ASTAK($\bar{s}$), in which a set of amplifying vertices is given and it is asked whether we can construct a feasible spanning tree that connects these vertices, is NP-complete for all $\bar{s} \geq 3$. For the case in which $\bar{s} \leq 2$, ASTAK($\bar{s}$) was shown to have the answer YES if and only if the supply set is a connected dominating set in the $\bar{s}$-th power graph $G^{\bar{s}}$, and we gave a polynomial algorithm to either construct a feasible tree or report that no such tree exists. We then showed that the related problem of total distance domination is also NP-complete. Finally, we reported an approximability result for AST($\bar{s}$) which follows from its close relationship to the connected dominating set problem.

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