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On the integer-magic spectra of tessellation graphs

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Abstract

Let A be an abelian group with non-identity elements A^*. A graph is A-magic if it has an edge-labeling by elements of A^* which induces a constant vertex labeling of the graph. In this paper we determine, for certain classes of triominoes and polyominoes, for which values of $k \geq 2$ the graphs are Z_k-magic.

1 Introduction

Let $G = (V, E)$ be a connected graph without multiple edges or loops. For any abelian group A (written additively), let $A^* = A - \{0\}$. A function $f : E(G) \to A^*$ is called a *labeling* of G. Any labeling induces a map $f^+ : V(G) \to A$, defined by $f^+(v) = \Sigma f(u, v)$ where $(u, v) \in E(G)$. If there exists a labeling f which induces a constant label c on $V(G)$, we say that f is an *A-magic labeling* and that G is an *A-magic* graph with *index* c. We denote by Z_k the group of integers (mod k). In this paper, we are interested in determining for which values of $k \geq 2$ a graph is Z_k-magic. The set $\{k : G \text{ is } Z_k\text{-magic}, k \geq 2\}$ is called the *integer-magic spectrum* of a graph G and is denoted by $\mathrm{IM}(G)$.

Z-magic graphs were considered by Stanley [19, 20], where he pointed out that the theory of magic labelings could be studied in the general context of linear homogeneous diophantine equations. They were also considered in [1, 16]. Doob [2, 3, 4] and others [9, 12, 13, 15] have studied A-magic graphs and Z_k-magic graphs were investigated in [6, 8, 10, 11, 14].

Within the mathematical literature, various definitions of magic graphs have been introduced. The original concept of an A-magic graph is due to J. Sedlacek [17, 18], who defined it to be a graph with real-valued edge labeling such that (i) distinct edges have distinct nonnegative labels, and (ii) the sum of the labels of the edges incident to a particular vertex is the same for all vertices. Previously, Kotzig and Rosa [7] had introduced yet another definition of a magic graph. Over the years, there has been great research interest in graph labeling problems. In fact, many different graph labelings have been introduced into the literature. They include edge-magicness, vertex-magicness, anti-magicness as well as countless others. The interested reader is directed to Wallis' [21] recent monograph on magic graphs and to Gallian's [5] excellent dynamic survey of graph labelings.

2 Tessellation graphs

A *tessellation* is a tiling of the plane, using polygons. If a tessellation consists of congruent polygons, it is a *regular tessellation.* Thus, there are only three regular tessellations, utilizing equilateral triangles, squares, or regular hexagons. A *tessellation graph* is a finite subgraph of a regular tessellation, consisting of a grid of congruent polygons where each polygon shares at least one common edge with another.

Definition. A region Ω in the plane is *n-connected* if the complement of Ω has exactly n components.

Definition. For $n \geq 2$, an *n-tessellation graph* is a graph which tessellates an n-connected region in the plane.

For example, a 1-tessellation graph tessellates a simply-connected, bounded region in the plane. The reader should note the following remarks.
Remarks. Let G be an n-tessellation graph, $n \geq 2$.
(i) In G, the boundaries of a hole and the outer boundary of G have no vertices in common.
(ii) If G is an n-tessellation graph with $n \geq 3$, then the boundaries of any two holes have no vertices in common.

In this paper, we examine the integer-magic spectra of tessellation graphs constructed from equilateral triangles and squares, respectively. Lee and Wang [14] have analyzed the integer-magic spectrum of the honeycomb graphs.

3 Triominoes

Consider a tessellation of the plane, using congruent equilateral triangles. Two triangles are *connected* if they share a common edge. Let T be a connected collection of triangles. Then, T is a connected planar graph, consisting of a grid of C_3's with each C_3 sharing at least one common edge with another. A connected collection of triangles is called a *triomino.* T is called an *n-triomino* if it is an n-tessellation graph.

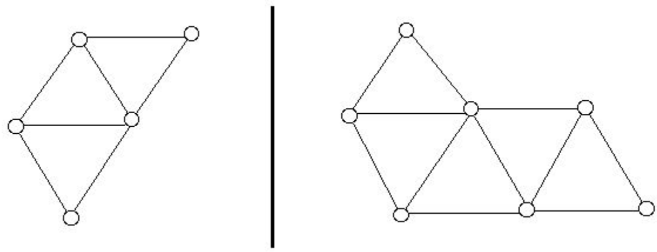

Figure 1. Two different triominoes.

Definition. A graph G is *outerplanar* if it can be embedded in the plane so that every vertex of G lies on the boundary of the exterior region.

We first analyze the integer-magic spectra of a few classes of 1-triominoes. The following two theorems which were proved in other papers will be needed. In their study of maximal outerplanar graphs, Lee, Ho and Low [8] showed the following result:

Theorem A. *Every outerplanar 1-triomino is Z_{2k}-magic, for all $k \geq 2$.*

While studying eulerian graphs, Low and Lee [15] showed the following theorem:

Theorem B. *Every eulerian graph is Z_{2k}-magic, for all $k \geq 1$.*

We begin with the following definition.

Definition. A *snake of length n* is a 1-triomino, formed by n equilateral triangles in the following way:

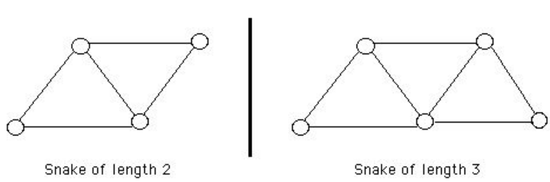

Figure 2.

Remarks.
(i) The only snake which is Z_2-magic is the one of length 1.
(ii) A snake of length 1 has integer-magic spectrum $N - \{1\}$.
(iii) In [8], it was shown that a snake of length 2 has integer-magic spectrum $2N - \{2\}$.

Theorem 3. *Let S be a snake of length n, $n \geq 3$. Then,* $\mathrm{IM}(S) = N - \{1, 2\}$.

Proof. Since the vertices of S are not of the same parity, S is not Z_2-magic. Also, since S is outerplanar, by Theorem A we have that $2N - \{2\} \subseteq \mathrm{IM}(S)$. In analyzing the rest of $\mathrm{IM}(S)$, we consider the possible cases.

CASE 1 Suppose n is odd. The following diagram gives a Z_{2k+1}-magic labeling, $k \geq 1$, for S:

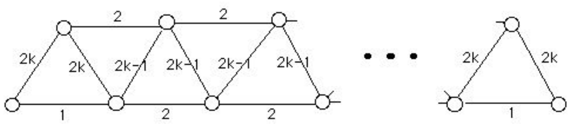

Figure 3.

CASE 2 Suppose n is even and $n \equiv 0 \pmod 4$. The following diagrams give Z_{2k+1}-magic labelings, $k \geq 1$, for S:

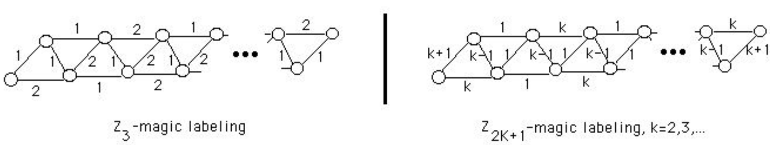

Figure 4.

CASE 3 Suppose n is even and $n \equiv 2 \pmod 4$. We first illustrate a Z_{2k+1}-magic labeling, $k \geq 1$, for a snake of length 6.

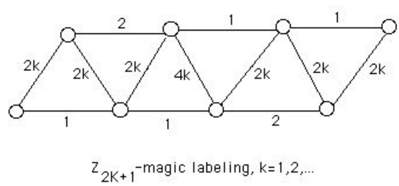

Figure 5.

Note that the induced vertex labeling is 0. For snakes of length 10, 14, 18, ..., we obtain Z_{2k+1}-magic labelings, $k \geq 1$, by attaching snakes of length 4 to this particular snake of length 6. The edges which are identified with each other have a new labeling, namely the sum of the original labelings. The induced vertex labeling is 0 and none of the edges are labeled 0. The following diagrams illustrate the Z_{2k+1}-magic labelings:

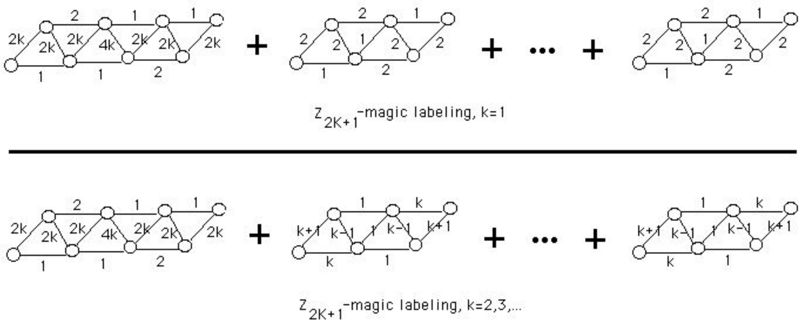

Figure 6.

□

Definition. A *pyramid of height* n is a 1-triomino, formed by layering n snakes in the following way:

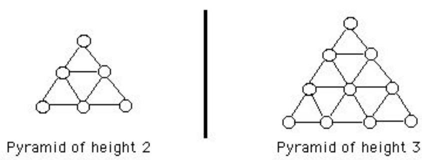

Figure 7.

The reader should note the following observations.

Observations. Let P be a pyramid of height n, $n \geq 1$.

(i) Every layer of P is a snake of odd length.

(ii) The base layer of P is a snake of length $2n - 1$.

(iii) P is an eulerian graph.

(iv) P has an odd number of edges if and only if $n \equiv 1$ or $2 \pmod 4$.

(v) If $n \equiv 1 \pmod 4$ and $n > 1$, then the top $n - 1$ layers of P form an eulerian pyramid having an even number of edges.

(vi) If $n \equiv 2 \pmod 4$ and $n > 2$, then the top $n - 2$ layers of P form an eulerian pyramid having an even number of edges.

(vii) If $n = 1$, then $\text{IM}(P) = N - \{1\}$.

Theorem 4. *Let P be a pyramid of height* 2. *Then,* $\text{IM}(P) = N - \{1, 3\}$.

Proof. Since P is eulerian, Theorem B implies that $2N \subseteq \text{IM}(P)$. It is straightforward to show via an indirect proof, that P is not Z_3-magic. For brevity, we will omit this particular detail. The following diagram illustrates a Z_{2k+1}-magic labeling, $k \geq 2$, for P:

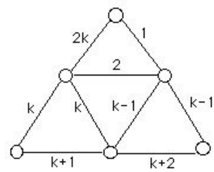

Figure 8.

□

Theorem 5. *Let P be a pyramid of height n, $n \geq 3$. If $n \equiv 0$ or $3 \pmod 4$, then* $\mathrm{IM}(P) = N - \{1\}$. *Otherwise,* $\{2, 4, 5, 6, ...\} \subseteq \mathrm{IM}(P)$.

Proof. Since P is eulerian, Theorem B implies that $2N \subseteq \mathrm{IM}(P)$. To analyze the rest of $\mathrm{IM}(P)$, we consider the possible cases.

CASE 1 Suppose $n \equiv 0$ or $3 \pmod 4$. Then, P is an eulerian graph with an even number of edges. Let $e_1 e_2 e_3 \cdots e_{2n}$ be an eulerian circuit, starting and ending at vertex v. The following labeling scheme will give an Z_{2k+1}-magic labeling of P, $k \geq 1$:

$$f(e_i) = \begin{cases} 1, & \text{if } i \text{ is odd.} \\ 2k, & \text{if } i \text{ is even.} \end{cases}$$

The induced vertex labeling is 0. Thus, $\mathrm{IM}(P) = N - \{1\}$.

CASE 2 Suppose $n \equiv 1 \pmod 4$. Note that P can be formed by attaching a snake of length $2n - 1$ to a pyramid of height $n - 1$. We obtain a Z_{2k+1}-magic labeling of P, $k \geq 2$, by labeling these two components individually and then performing the attachment. By observation (v), the top $n - 1$ layers of P form an eulerian graph with an even number of edges. Label this pyramid of height $n - 1$, using the labeling scheme described in CASE 1. Now, label the snake of length $2n - 1$, using the scheme described in the proof of Theorem 3. Both Z_{2k+1}-magic labelings, $k \geq 1$, have induced vertex labeling 0. The bottom edges of the pyramid of height $n - 1$ are labeled with either 1 or $2k$. The top edges of the snake of length $2n - 1$ are labeled with 2. By attaching the components together, each of the identified edges has a new labeling, namely the sum of the respective original ones. We now have a Z_{2k+1}-magic labeling of P, $k \geq 2$. Thus, $\{2, 4, 5, 6, ...\} \subseteq \mathrm{IM}(P)$.

CASE 3 Suppose $n \equiv 2 \pmod 4$. Note that P can be formed by attaching two snakes in succession to a pyramid of height $n - 2$. We obtain a Z_{2k+1}-magic labeling of P, $k \geq 2$, by labeling these three components individually and then performing the attachments. By observation (vi), the top $n - 2$ layers of P form an eulerian graph with an even number of edges. Label this pyramid of height $n - 2$, using the labeling scheme described in CASE 1. Now, label the snakes of lengths $2n - 3$ and $2n - 1$, using the scheme described in the proof of Theorem 3. As in CASE 2, attaching the snake of length $2n - 3$ to the pyramid of height $n - 2$ yields a Z_{2k+1}-magic labeling, $k \geq 2$, of a pyramid of height $n - 1$. In this labeling, note that the bottom edges of the pyramid of height $n - 1$ are labeled with 1 or 2. In similar fashion, we now attach the labeled snake of length $2n-1$ to obtain a Z_{2k+1}-magic labeling, $k \geq 2$, of P. Thus, $\{2, 4, 5, 6, ...\} \subseteq \mathrm{IM}(P)$.

□

Note that in general, the Z_3-magic case has not been proven to be inside the integer-magic spectrum, when $n \equiv 1$ or $2 \pmod 4$. However, we have the additional result for $n = 5$.

Theorem 6. *Let P be a pyramid of height 5. Then,* $\mathrm{IM}(P) = N - \{1\}$.

Proof. Since P is eulerian, Theorem B implies that $2N \subseteq \mathrm{IM}(P)$. The following diagram gives a Z_{2k+1}-magic labeling, $k \geq 1$, for P.

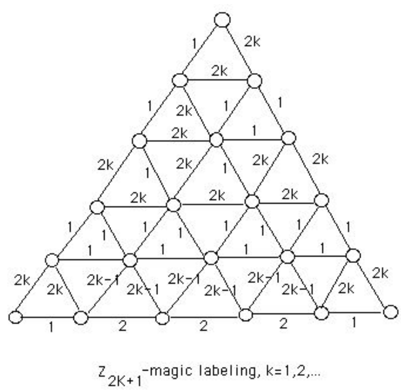

Figure 9.

□

Let us now focus on the integer-magic spectra of more general triominoes. We begin with the following definition, which will be used to construct a specific type of triomino.

Definition. A *diamond* is a triomino, consisting of two equilateral triangles.

Theorem 7. *Let T be a 1-triomino, constructed by tiling diamonds. Then for $k \geq 2$, T has a Z_{2k}-magic labeling with induced vertex labeling 0. Furthermore, this labeling labels the interior edges of T with k and the boundary edges of T with $1, k-1, k+1$, or $2k-1$.*

Proof. We induct on the number of diamonds n used to tile T.

(ANCHOR CASE) If $n = 1$, then T can be labeled in the following way:

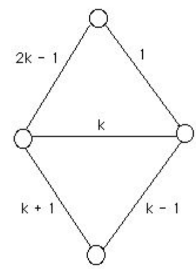

Figure 10. Fundamental labeling of a diamond.

This particular labeling of the diamond will be used throughout the proof. We call this the *fundamental labeling* of a diamond.

(INDUCTION HYPOTHESIS) Assume that any 1-triomino which can be tiled using $n-1$ diamonds has a Z_{2k}-magic labeling with induced vertex labeling 0, in which the interior edges are labeled k and the boundary edges are labeled $1, k-1, k+1$, or $2k-1$.

Now, let T be a 1-triomino which can be tiled with n diamonds. Thus, T can be constructed by adjoining a diamond D_n to a 1-triomino T_* arising from the tiling of $n-1$ diamonds. Since T_* is a 1-triomino, the adjoining of D_n to T_* is accomplished by identifying one, two, or three edges (and their respective vertices) of D_n and T_*. By the induction hypothesis, T_* has a Z_{2k}-magic labeling with induced vertex labeling 0, in which the interior edges are labeled k and the boundary edges are labeled $1, k-1, k+1$, or $2k-1$. For $k \geq 2$, we now construct a Z_{2k}-magic labeling of T with these properties.

In adjoining D_n to T_*, one of four possible cases can occur:

1. Identifying exactly one edge of D_n with a labeled edge of T_*.
2. Identifying two edges of D_n with two labeled edges of T_*, as shown in Fig. (ii).
3. Identifying two edges of D_n with two labeled edges of T_*, as shown in Fig. (iii).
4. Identifying three edges of D_n with three labeled edges of T_*, as shown in Fig. (iv).

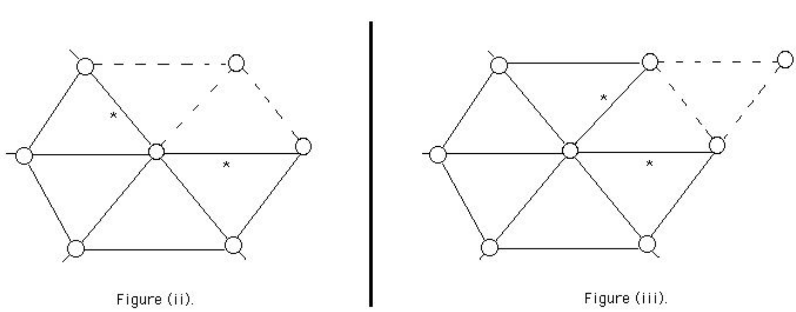

Figure 11.

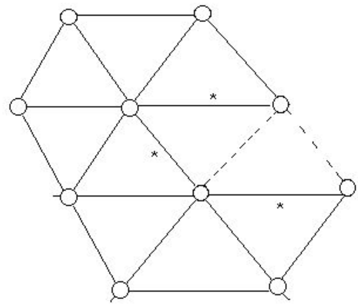

Figure (iv).

Figure 12.

We will label T according to the following scheme:

CASE (i) Let e_1^* be the labeled edge of T_* which is to be identified with an edge of D_n. If $e_1^* = 1$, adjoin D_n to T_* by identifying the labeled edge $k-1$ from the fundamental labeling of D_n to T_* and adding the two labels. This gives a Z_{2k}-magic labeling of T with induced vertex labeling 0, in which the interior edges are labeled k and the boundary edges are labeled $1, k-1, k+1$, or $2k-1$. In the cases where $e_1^* = k-1$, $e_1^* = k+1$ or $e_1^* = 2k-1$, a Z_{2k}-magic labeling of T with the desired properties can be obtained in a similar manner. An example is illustrated in Figure 13.

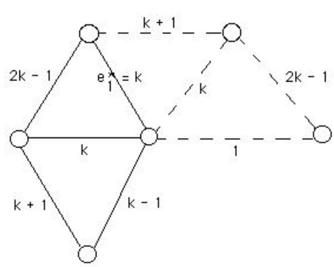

Figure 13.

CASE (ii) Let e_1^* and e_2^* be the two labeled edges of T_* which are to be identified with two edges of D_n. Then, we have the following labeled configuration:

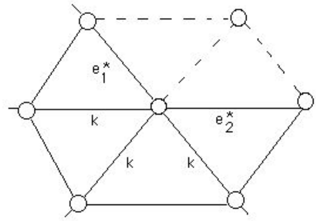

Figure 14.

Thus, $k+k+k+e_1^*+e_2^*=0 \pmod{2k}$. If $e_1^*=1$, then $e_2^*=k-1$. If $e_1^*=k-1$, then $e_2^*=1$. If $e_1^*=2k-1$, then $e_2^*=k+1$. If $e_1^*=k+1$, then $e_2^*=2k-1$. Thus for $k \geq 2$, we can use the fundamental labeling of D_n and adjoin D_n to T_* to give a Z_{2k}-magic labeling of T. Furthermore, this labeling has an induced vertex labeling of 0, the interior edges are labeled k and the boundary edges are labeled $1, k-1, k+1$, or $2k-1$. An example is shown in Figure 15.

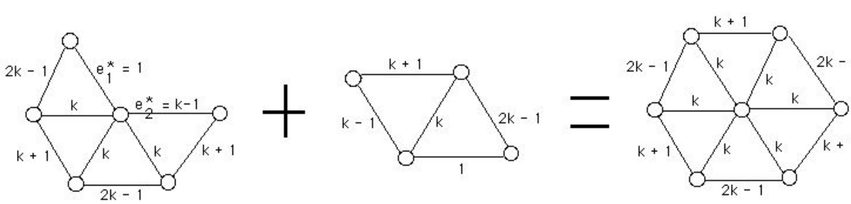

Figure 15.

CASE (iii) Let e_1^* and e_2^* be the two labeled edges of T_* which are to be identified with two edges of D_n. Then, we have the following labeled configuration:

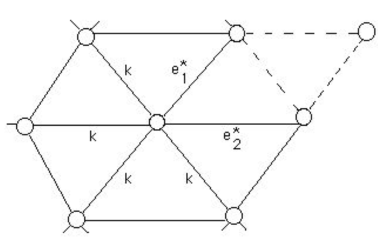

Figure 16.

Thus, $k+k+k+k+e_1^*+e_2^*=0 \pmod{2k}$. If $e_1^*=1$, then $e_2^*=2k-1$. If $e_1^*=2k-1$, then $e_2^*=1$. If $e_1^*=k+1$, then $e_2^*=k-1$. If $e_1^*=k-1$, then $e_2^*=k+1$. Thus for $k \geq 2$, we can use the fundamental labeling of D_n and adjoin D_n to T_* to give a

Z_{2k}-magic labeling of T. Furthermore, this labeling has an induced vertex labeling of 0, the interior edges are labeled k and the boundary edges are labeled $1, k-1, k+1$, or $2k-1$. An example is shown in Figure 17.

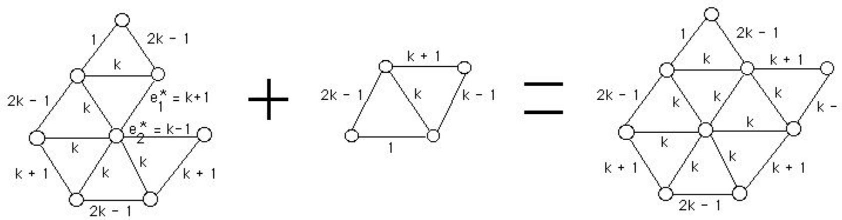

Figure 17.

CASE (iv) Let e_1^*, e_2^* and e_3^* be the three labeled edges of T_* which are to be identified with three edges of D_n. Then, we have the following labeled configuration:

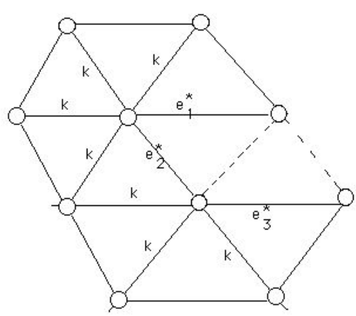

Figure 18.

Thus, $k+k+k+k+e_1^*+e_2^* = 0 \pmod{2k}$ and $k+k+k+e_2^*+e_3^* = 0 \pmod{2k}$. If $e_1^* = 1$, then $e_2^* = 2k-1$ and $e_3^* = k+1$. If $e_1^* = 2k-1$, then $e_2^* = 1$ and $e_3^* = k-1$. If $e_1^* = k+1$, then $e_2^* = k-1$ and $e_3^* = 1$. If $e_1^* = k-1$, then $e_2^* = k+1$ and $e_3^* = 2k-1$. Thus for $k \geq 2$, we can use the fundamental labeling of D_n and adjoin D_n to T_* to give a Z_{2k}-magic labeling of T. Furthermore, this labeling has an induced vertex labeling of 0, the interior edges are labeled k and the boundary edges are labeled $1, k-1, k+1$, or $2k-1$. An example is shown in Figure 19.

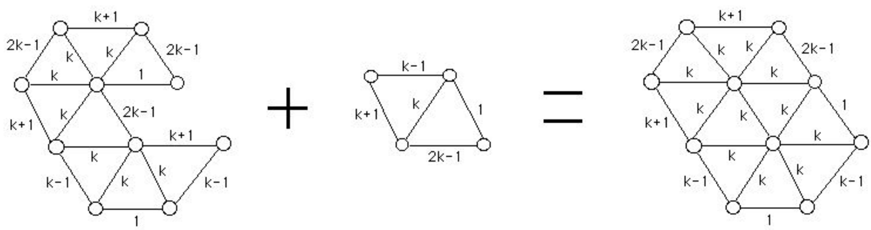

Figure 19.

This finishes the induction proof. □

Corollary 1. *Let T be a 1-triomino, constructed by tiling diamonds. If every vertex of T has the same parity, then $2N \subseteq \mathrm{IM}(T)$. Otherwise, $2N - \{2\} \subseteq \mathrm{IM}(T)$.*

One should note that the integer-magic spectrum of 1-triominoes constructed by tiling diamonds can sometimes contain more than just $2N - \{2\}$. For example, the integer-magic spectrum of $K_1 + P_5$ is $N - \{1, 2, 3\}$.

Theorem 8. *Let T be an n-triomino, $n \geq 2$, constructed by tiling diamonds. Furthermore, suppose that each hole in T can be tessellated with diamonds. If every vertex of T has the same parity, then $2N \subseteq \mathrm{IM}(T)$. Otherwise, $2N - \{2\} \subseteq \mathrm{IM}(T)$.*

Proof. Clearly, if every vertex of T has the same parity, then T is Z_2-magic. Let $k \geq 2$. We wish to find a Z_{2k}-magic labeling of T. Now, suppose that $T_1, T_2, ..., T_{n-1}$ denote the holes in T. For each hole T_j, there is an associated 1-triomino T_j^*, obtained by tessellating T_j with diamonds. Also, let T^* be the 1-triomino obtained from T, by filling in the holes T_j with diamonds. Thus, $V(T) = V(T^*) - \cup_j\{\text{interior vertices of } T_j^*\}$ and $E(T) = E(T^*) - \cup_j\{\text{interior edges of } T_j^*\}$.

From Theorem 7, there are Z_{2k}-magic labelings of T^* and each of the T_j^* in which the internal edges of T^* and T_j^* are labeled k, with the vertices having an induced labeling 0.

Now, overlay the labeled T_j^* onto the labeled T^*, by adding the values of the respective edges. This gives a new labeling of T^*. In particular, this overlaying process causes the internal edges of T_j^* to be labeled $k + k$, which is congruent to 0 (mod $2k$). Also, the boundary edges of the T_j^* are now labeled with $(2k - 1) + k$, $1 + k$, $(k + 1) + k$, or $(k - 1) + k$. Note that none of these are congruent to 0 (mod $2k$), $k \geq 2$.

In this new labeling of T^*, delete all edges which are labeled 0 as well as any disconnected vertices. The resulting labeled graph will yield a Z_{2k}-magic labeling of T, for $k \geq 2$. This labeling has magic index 0. □

Figure 20 illustrates the overlaying of labeled T_1^* onto labeled T^*, for a 2-triomino T. For the sake of clarity, only the relevant edges have been labeled.

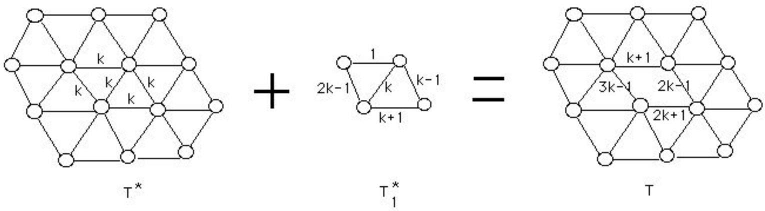

Figure 20. A Z_{2k}-magic labeling, obtained by overlaying process.

4 Polyominoes

A *cell* is the boundary of a unit square ($\cong C_4$) in the xy-plane, where the vertices of the square are at lattice points. Two cells are *connected* if they share a common edge. Let S be a connected collection of connected cells. Thus, S can be viewed as a connected planar graph, consisting of a grid of C_4's with each C_4 sharing at least one common edge with another. A connected collection of connected cells is called a *polyomino.*

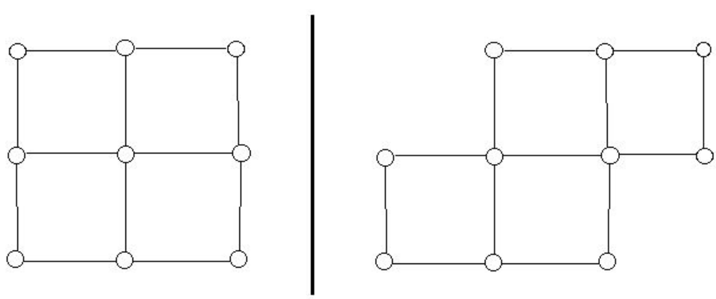

Figure 21. Two different four–cell polyominoes.

In this section, we examine the integer-magic spectrum of polyominoes.

Lemma 1. *If $c \in 2N$ and $2k \nmid (\frac{c}{2}+k)$, then $2k \nmid (\frac{c}{2}-k)$.*

Theorem 9. *For every outerplanar 1–polyomino P, there exists a Z_{2k}-magic labeling having index c, where $c \in 2N$, $2k \nmid \frac{c}{2}$, and $2k \nmid (\frac{c}{2}+k)$.*

Proof. In P, let C_0 be a cell which has a vertex of degree two and label all of its edges with $\frac{c}{2}$. This gives a Z_{2k}-magic labeling of C_0, of index c.

For each cell C_j adjacent to C_0, let e_j^* denote the common edge between C_j and C_0. Now, perform the following labeling scheme on each C_j:

CASE 1 Suppose e_j^* is labeled $\frac{c}{2}$. In this case, label the unlabeled edges of C_j and re-label e_j^* as described in Figure A.

CASE 2 Suppose e_j^* is labeled $\frac{c}{2}+k$. In this case, label the unlabeled edges of C_j and re-label e_j^* as described in Figure B.

CASE 3 Suppose e_j^* is labeled $\frac{c}{2}-k$. In this case, label the unlabeled edges of C_j and re-label e_j^* as described in Figure C.

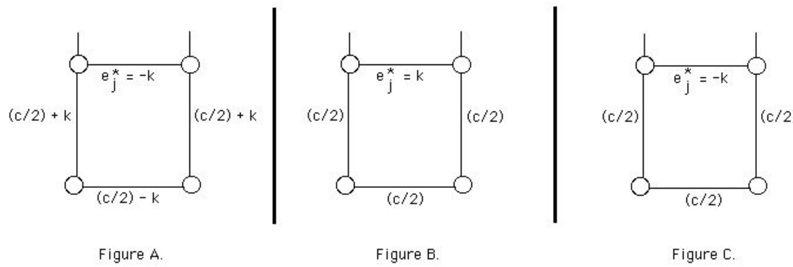

Figure 22.

Notice that in all of these cases, none of the edges have been labeled 0. This follows from the hypothesis and Lemma 1. Furthermore, the subgraph (comprised of C_0 and the adjacent cells C_j) has an induced vertex labeling of c.

Since P is an outerplanar 1-polyomino, we can continue to perform this labeling scheme on cells adjacent to labeled cells C_j, etc..., until P has been completely labeled. The final labeling will be a Z_{2k}-magic labeling with index c. □

Theorem 10. *Let P be an n-polyomino, $n \geq 1$. If every vertex of P has the same parity, then* $\mathrm{IM}(P) = N - \{1\}$*. Otherwise,* $\mathrm{IM}(P) = N - \{1, 2\}$*.*

Proof. Clearly, if every vertex of P has the same parity, then P is Z_2-magic. In P, let C_0 be a cell which has a vertex of degree two. Label the edges of C_0 as found in Figure 23. This gives a Z_k-magic labeling ($k \geq 3$) for C_0.

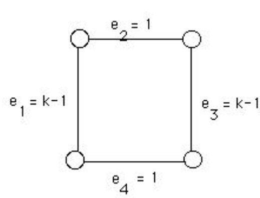

Figure 23.

Now, perform the following labeling scheme for all cells C_j adjacent to C_0:

A. Label all the respective unlabeled edges e_i in the same way as described in Figure 23.
B. Replace each previously labeled edge e_i (having label $f(e_i)$) in C_j with the new label $2 \cdot f(e_i)$.
Observe that labeling the adjacent cells C_j in this way preserves the induced vertex labeling of 0 on the vertices. Since $f(e_i) = 1$ or $k-1$, we have that $2 \cdot f(e_i) \neq 0 \pmod{k}$, for $k \geq 3$.

Continue this labeling scheme on cells adjacent to cells C_j, etc..., until P has been completely labeled. The final labeling will be a Z_k–magic labeling ($k \geq 3$) with an induced vertex labeling of 0. ☐

Figure 24 gives a few examples which illustrate Theorem 10.

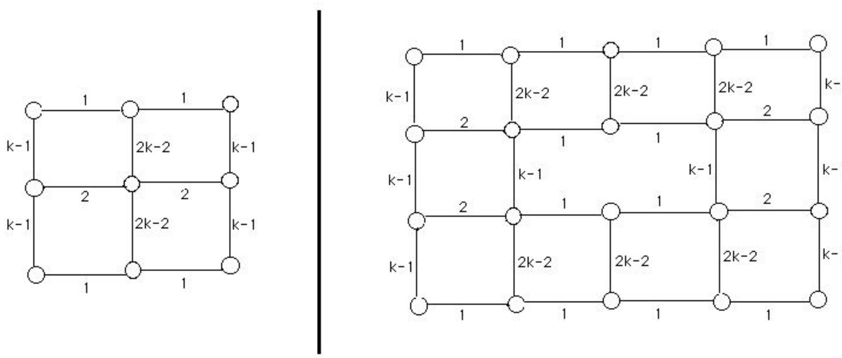

Figure 24. Two examples of a Z_k-magic labeling, $k \geq 3$.

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