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Bipartite graphs with even spanning trees

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Abstract

A tree is defined to be *even* if every pair of its leaves (= vertices of degree one) are an even distance apart. Teresa Haynes has asked if there is a polynomial-time algorithm to determine if a given bipartite graph G has an even spanning tree. We give such an algorithm here, as well as Hall-type necessary and sufficient conditions on G for the existence of such a tree.

1 Even trees and A-pairings

The vertices of degree one in a tree are called *leaves*. We declare a tree to be *even* if the distance between any two of its leaves is even. Teresa Haynes, in [3], queried about the computational status of the following problem: given a bipartite graph G, does it have an even spanning tree?

We will give a polynomial-time algorithm here, and also prove that a Hall-type condition is both necessary and sufficient for the existence of such a tree.

Note that trees, like all connected bipartite graphs, are uniquely two-colorable. And it's easy to check if a given tree is even: in the unique two-coloring, all leaves must

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receive the same color. In particular, an even spanning tree in a connected bipartite graph G with unique bipartition $\{A, B\}$ must either have all its leaves in A, or all its leaves in B. For definiteness, we will seek a spanning tree with all its leaves in B.

A spanning forest F in G is defined to be an A-*pairing* if each vertex in A is incident with exactly two edges of F. We omit the simple proof of the following lemma.

Lemma 1 *The connected bipartite graph G with bipartition $\{A, B\}$ has a spanning tree with all its leaves in B if and only if it has an A-pairing.* □

The following lemma gives necessary conditions for the existence of an A-pairing, which are similar to the well known Hall conditions for G to have a matching saturating A [2]. And, as we shall prove in the sequel, they are also sufficient!

Lemma 2 *If the bipartite graph G with bipartition $\{A, B\}$ has an A-pairing F, then for every non-empty $S \subseteq A$, $|S| < |N(S)|$. (Here $N(S)$ denotes the set of vertices of B having at least one neighbor in S.)*

Proof. Let F' be the sub-forest of F induced by $S \cup N(S)$. Then F' has $2|S|$ edges, and at most $|S| + |N(S)|$ vertices. But the number of edges of any forest is less than the number of its vertices, so

$$2|S| < |S| + |N(S)|.$$

□

As a consequence of Lemma 2, $|A| < |B|$ if G has an A-pairing. Thus our seemingly arbitrary decision to search for a spanning tree with all of its leaves in B, rather than A, was not so arbitrary. If the two color classes of G have the same cardinality, there is no even spanning tree; if not, then all the leaves of any such tree must be in the larger color class.

As a further consequence, we may assume each vertex $v \in A$ has degree at least two in G, since $1 = |\{v\}| < N(\{v\}) = d(v)$.

2 Matroids to the rescue

For everything you ever wanted to know about matroids (but were afraid to ask), see [4].

We will define two matroids on the set E of edges of G. Our goal in so doing is to characterize the A-pairings of G as precisely those sets of edges which are independent in both matroids. We can then invoke a well-known result – the Matroid Intersection Theorem – to provide an algorithmic answer to our existence question.

The matroid $M_1 = (E, \mathcal{I}_1)$ is the forest matroid of G. That is, a subset $I \subseteq E$ is declared to be independent in the matroid M_1, i.e. $I \in \mathcal{I}_1$, if and only if I is the set of edges of a forest in G.

The matroid $M_2 = (E, \mathcal{I}_2)$ is defined as follows: a subset $I \subseteq E$ is declared to be independent in the matroid M_2, i.e. $I \in \mathcal{I}_2$, if and only if every vertex of A is incident with at most two edges of I.

It is well known (and easy to prove) that M_1 and M_2 are indeed matroids. Equally transparent is the following lemma:

Lemma 3 *$I \subseteq E$ is the set of edges of an A-pairing if and only if $I \in \mathcal{I}_1 \cap \mathcal{I}_2$, and $2|A| \leq |I|$.*

We can now settle the computational issue. J. Edmonds [1] has given a polynomial-time algorithm for finding a largest cardinality set independent in each of two given matroids based on the same edge set, provided of course that independence in each matroid can be decided efficiently. This proviso certainly holds for our matroids M_1 and M_2, so we may apply this algorithm to find $I \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$ of largest cardinality in polynomial time. If $|I| = 2|A|$ (it can't be bigger!), then there is an A-pairing. If $|I| < 2|A|$, then clearly there isn't.

3 A max-min formula

Also from [1], we obtain the following important equation:

Theorem 1 *Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids on the same edge set E, with rank functions r_1 and r_2 respectively. Then*

$$\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min\{r_1(T) + r_2(E \setminus T) : T \subseteq E\}$$

We can now prove our main theorem:

Theorem 2 *Let G be a bipartite graph with bipartition (A, B). Then G has an A-pairing (which, for a connected G with $|A| \leq |B|$, is equivalent to the existence of an even spanning tree in G) if and only if,*

(1) *for all non-empty $S \subseteq A$, $|S| < |N(S)|$.*

Proof. We've settled all the easier bits in the statement of Theorem 2, and gathered our tools for the last bit. It comes to this: assuming (1), we need to show, for all $K \subseteq E$, that $2|A| \leq r_1(K) + r_2(E \setminus K)$, where r_1 and r_2 are the rank functions of the matroids M_1 and M_2 respectively defined in the previous section.

In aid of this, we define, for each $a \in A$, E_a to be the set of edges of G incident with a. Then

$$r_2(E \setminus K) = \sum_{a \in A} \min\{2, |E_a \setminus K|\}.$$

So we need to show, for all $K \subseteq E$, that

$$2|A| \leq r_1(K) + \sum_{a \in A} \min\{2, |E_a \setminus K|\}. \tag{*}$$

Before proving this for some specific $K \subseteq E$, we are free to adjust K, provided that we do not increase the right-hand side of (*). We will avail ourselves of two such adjustments.

Accordingly, let $a \in A$.

Case 1 $E_a \setminus K = \{e\}$ In this case, we may transfer e from $E \setminus K$ to K. This might increase $r_1(K)$, but if so, only by 1. But the sum representing $r_2(E \setminus K)$ definitely decreases by 1. So the net effect is that the right hand side of (*) either maintains its value, or drops by 1.

Case 2 $|E_a \setminus K| \geq 2$ In this case, we may transfer all of $E_a \cap K$ from K to $E \setminus K$.

This certainly won't increase $r_1(K)$, and the sum representing $r_2(E \setminus K)$ is unaffected.

Having made all these adjustments, we can now assume that for some $S \subseteq A$, $K = \bigcup_{a \in S} E_a$. Thus $r_2(E \setminus K) = 2|A| - 2|S|$, and further, $r_1(K) = |S| + |N(S)| - w(S)$, where $w(S)$ denotes the number of components of the subgraph of G induced by $S \cup N(S)$.

Plugging these formulae into (*) shows that G has an A-pairing if and only if

$$|S| + w(S) \leq |N(S)| \text{ for all } S \subseteq A. \tag{2}$$

This is beginning to resemble condition (1) in the statement of Theorem 2. In fact, (2) obviously implies (1), but it's the reverse implication we need to prove.

So assume (1) holds, and let $S \subseteq A$. Denote the components of the subgraph of G induced by $S \cup N(S)$ by C_i, $1 \leq i \leq w(S)$. Then by (1), $|S_i| + 1 \leq |N(S_i)|$, where $S_i = A \cap V(C_i)$. Summing these inequalities over all $1 \leq i \leq w(S)$ yields (2). □

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