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Partial $S(k-1,k,v)$'s inducing P_k-decompositions of K_v

Robert E. Jamison

Department of Mathematics
Clemson University
Clemson, SC 29634–0975
U.S.A.
rejam@clemson.edu

Gaetano Quattrocchi*

Dipartimento di Matematica e Informatica
Università di Catania
viale A. Doria, 6 95125 Catania
ITALY
quattrocchi@dmi.unict.it

Abstract

Let P_k be the path with k vertices and $k-1$ edges, $k \geq 4$. For every integer v, $v \equiv 0$ or $1 \pmod{2(k-1)}$ if k is odd or $v \equiv 0$ or $1 \pmod{k-1}$ if k is even, we produce a P_k-design $(V, \mathcal{B})$ of order v such that no two blocks have $k-1$ or k common vertices, i.e. $(V, \mathcal{B})$ is a partial $S(k-1, k, v)$.

1 Preliminaries

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple undirected graph and let P_k be the path with k vertices and $k-1$ edges, $k \geq 3$. A P_k-decomposition $(V(\Gamma), \mathcal{B})$ of Γ is an edge-disjoint decomposition $\mathcal{B}$ of Γ into copies of P_k, called blocks. Usually $\mathcal{B}$ is called the block set of the P_k-decomposition.

A P_k-design of order v is a P_k-decomposition of K_v, the complete undirected graph on v vertices. Tarsi [4] proved that the necessary conditions for the existence of a P_k-design of order v, $v \geq k$ (if $v > 1$) and $v(v-1) \equiv 0 \pmod{2(k-1)}$, are also sufficient.

We say that a P_k-decomposition $(V(\Gamma), \mathcal{B})$ is *good* if it induces a partial $S(k-1, k, v)$ where $v = |V(\Gamma)|$, i.e. if every $(k-1)$-element subset of $V(\Gamma)$ is contained in the vertex-set of at most one block of $\mathcal{B}$.

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A good P_k-design of order v will be denoted by $GP(v,k,1)$.

It is trivial to see that a $GP(v,3,1)$ cannot exist, while, for $k \geq 4$, a good P_k-design can exist as shown by the following example. Let $\mathcal{B} = \{[i, 1+i, 3+i, 6+i] \mid i \in \mathbb{Z}_7\}$, then $(\mathbb{Z}_7, \mathcal{B})$ is a $GP(7,4,1)$. On the other hand the P_4-design $(\mathbb{Z}_7, \mathcal{C})$, where $\mathcal{C} = \{[i, 1+i, 6+i, 2+i] \mid i \in \mathbb{Z}_7\}$, is not good. A simple counting argument shows that a $GP(6,4,1)$ and a $GP(k,k,1)$, for every even $k \geq 4$, cannot exist.

The aim of the present paper is to construct a $GP(v,k,1)$ for every pair of integers v and k such that

- $v \equiv 0$ or $1 \pmod 4$, $v \geq 7$ if $k = 4$;
- $v \equiv 0$ or $1 \pmod{2(k-1)}$, $v \geq 2(k-1)$ if k is odd, $k \geq 5$;
- $v \equiv 0$ or $1 \pmod{k-1}$, $v \geq 2(k-1)$ if k is even, $k \geq 6$.

These results will be proved applying recursive constructions to starting P_k-designs. We construct these designs by difference methods [1, 2, 3]. So we give only the base blocks (writing short blocks in *slanted*). Let G be an additive abelian group of order n. We identify the vertex set of our designs with one of the following sets: G, $G^+ = G \cup \{\infty\}$ (∞ being a symbol not in G), $G \times \{1,2\}$. If $B_i = [a_1^i, a_2^i, \ldots, a_k^i]$, $i = 1, 2, \ldots, m$, are the base blocks then $\mathcal{B} = \cup_{i=1}^m dev_G(B_i)$, where $dev_G(B_i) = \{[a_1^i + g, a_2^i + g, \ldots, a_k^i + g] \mid g \in G\}$ (the block-orbit under the action of G) and composition law is given by

- the composition law of G, if $V = G$;
- the composition law of G extended by the rule $\infty + g = g + \infty = \infty$, if $V = G^+$;
- $(x,i) + g = (x+g, i)$, $i = 1, 2$, if $V = G \times \{1,2\}$.

Example 1. Let $V = \mathbb{Z}_{10}$. Let $B_1 = [0,1,4,2]$ and $B_2 =$ *[0,4,9,5]* be the base blocks (B_2 is *short*). The block-orbits under the action of $\mathbb{Z}_{10}$ are $dev_{\mathbb{Z}_{10}} B_1 = \{[i, 1+i, 4+i, 2+i] \mid i \in \mathbb{Z}_{10}\}$, $dev_{\mathbb{Z}_{10}} B_2 = \{[i, 4+i, 9+i, 5+i] \mid i \in \mathbb{Z}_{10}\}$. Then $(V, \mathcal{B})$ is a cyclic P_4-design, where $\mathcal{B} = dev_{\mathbb{Z}_{10}} B_1 \cup dev_{\mathbb{Z}_{10}} B_2$ (note that $|dev_{\mathbb{Z}_{10}} B_1| = 10$ and $|dev_{\mathbb{Z}_{10}} B_2| = 5$).

Example 2. Let $V = \{\infty\} \cup \mathbb{Z}_8$. Let $B_1 = [\infty, 0, 1, 3]$ and $B_2 =$ *[0,3,7,4]* be the base blocks (B_2 is *short*). The block-orbits under the action of $\mathbb{Z}_8$ are $dev_{\mathbb{Z}_8} B_1 = \{[\infty, i, 1+i, 3+i] \mid i \in \mathbb{Z}_8\}$, $dev_{\mathbb{Z}_8} B_2 = \{[i, 3+i, 7+i, 4+i] \mid i \in \mathbb{Z}_8\}$. Then $(V, \mathcal{B})$ is a 1-rotational P_4-design, where $\mathcal{B} = dev_{\mathbb{Z}_8} B_1 \cup dev_{\mathbb{Z}_8} B_2$ (note that $|dev_{\mathbb{Z}_8} B_1| = 8$ and $|dev_{\mathbb{Z}_8} B_2| = 4$).

Example 3. Let $V = \mathbb{Z}_6 \times \{1,2\}$. Let $B_1 = [(0,1),(0,2),(1,1),(2,2)]$ and $B_2 = [(0,1),(3,2),(1,1),(5,2)]$ be the base blocks. The block-orbits under the action of $\mathbb{Z}_6$ (leaving the second coordinate unchanged in the cyclic development) are $dev_{\mathbb{Z}_6} B_1 = \{[(i,1),(i,2),(1+i,1),(2+i,2)] \mid i \in \mathbb{Z}_6\}$, $dev_{\mathbb{Z}_6} B_2 = \{[(i,1),(3+i,2),(1+i,1),(5+i,2)] \mid i \in \mathbb{Z}_6\}$. Then $(V, \mathcal{B})$, $\mathcal{B} = dev_{\mathbb{Z}_6} B_1 \cup dev_{\mathbb{Z}_6} B_2$, is a P_4-decomposition of the complete bipartite graph K_{V_1,V_2}, $V_i = V \times \{i\}$ for $i = 1, 2$.

Given a set X, a *multiset* on X is a list $L=\{x_1,x_2,\ldots,x_n\}$ of elements from X where repetitions are allowed. Formally, a multiset L on X is a map $\mu_L : X \to \mathbb{N}$ where $\mu_L(x)$ is the *multiplicity* of x. Let $(V,\mathcal{B})$, where $V=G$ or $V=G^+$, be a P_k-design constructed by difference methods under the action of a group G and let B, C be two distinct base blocks. We denote by

- $\mu_B(g)$ the multiplicity of $g \in G$ in the multiset $\Delta(B)=\{b-c \mid b,c \in B\cap G, b\neq c\}$;
- $\mu_{B,C}(g)$ the multiplicity of $g\in G$ in the multiset $\Delta(B,C)=\{b-c \mid b\in B\cap G, c\in C\cap G\}$.

Put $V_i=G\times\{i\}$, $i=1,2$. Let $(V_1\cup V_2,\mathcal{B})$ be a P_k-decomposition of the complete bipartite graph K_{V_1,V_2} by difference methods under the action of G (see Example 3), and let B, C be two distinct base blocks. We denote by

- $\mu^i_B(g)$, $i=1,2$, the multiplicity of $g\in G$ in the multiset $\Delta^i(B)=\{b-c \mid (b,i),(c,i)\in B, b\neq c\}$;
- $\mu^i_{B,C}(g)$ the multiplicity of $g\in G$ in the multiset $\Delta^i(B,C)=\{b-c \mid (b,i)\in B,(c,i)\in C\}$.

The proofs of the following two lemmas are an easy consequence of the difference methods and the fact that there is at most one base block containing ∞. We suppose that $G=\mathbb{Z}_n$. Moreover if n is even and B is a short base block, then the nonzero element $g\in\mathbb{Z}_n$ such that $B+g=B$ is given by $g=\frac{n}{2}$.

Lemma 1 *Let $(V,\mathcal{B})$ be a P_k-design constructed by difference methods under the action of $\mathbb{Z}_n$. $(V,\mathcal{B})$ is a $GP(v,k,1)$ if and only if the following conditions are satisfied:*

1. *For every base block B one of the following conditions holds:*
   - *If B is not short and $\infty\notin B$, then $\mu_B(g)\leq k-2$ for every $g\in\mathbb{Z}_n\setminus\{0\}$.*
   - *If B is not short and $\infty\in B$, then $\mu_B(g)\leq k-3$ for every $g\in\mathbb{Z}_n\setminus\{0\}$.*
   - *If B is short, then $\mu_B(g)\leq k-2$ for every $g\in\mathbb{Z}_n\setminus\{0,\frac{n}{2}\}$.*
2. *For every pair of distinct base blocks B and C, then $\mu_{B,C}(g)\leq k-2$ for every $g\in G$.*

Let $(V,\mathcal{B})$ be the P_4-design of Example 1. It is $\Delta(B_1)=\{1,1,2,2,3,4,6,7,8,8,9,9\}$, $\Delta(B_2)=\{1,1,4,4,5,5,5,5,6,6,9,9\}$ and $\Delta(B_1,B_2)=\{0,0,1,1,2,2,3,4,5,5,6,6,7,7,8,9\}$. By Lemma 1, $(V,\mathcal{B})$ is good. Also the P_4-design of Example 2 is good.

Lemma 2 *Let $V_i=G\times\{i\}$, $i=1,2$. Let $(V_1\cup V_2,\mathcal{B})$ be a P_k-decomposition of the complete bipartite graph K_{V_1,V_2} by difference methods under the action of $\mathbb{Z}_n$ (see Example 3). $(V_1\cup V_2,\mathcal{B})$ is good if and only if the following conditions are satisfied:*

1. *For every base block* B, $\mu_B^1(g) + \mu_B^2(g) \leq k-2$.
2. *For every pair of distinct base blocks* B *and* C, $\mu_{B,C}^1(g) + \mu_{B,C}^2(g) \leq k-2$.

By Lemma 2, the P_4-decomposition of Example 3 is good.

2 $\mathbf{GP(v,4,1)}$

In this section we study the existence of $GP(v,4,1)$'s. We treat separately this case because there is not any $GP(6,4,1)$.

Lemma 3 *There is a* $GP(v,4,1)$, $v = 7,9,10,12,13,15,16,18$.

Proof A $GP(v,4,1)$ for $v = 7,9,10$, is given in Section 1. The remaining path designs are shown below. We leave to the reader to prove that the following base blocks satisfy the conditions of Lemma 1 or Lemma 2.

$GP(12,4,1)$. Base blocks: $[\infty,0,4,9]$, $[0,1,3,6]$.

$GP(13,4,1)$. Let $V_i = \mathbb{Z}_6 \times \{i\}$ and let $[\{\infty\} \cup V_i, \mathcal{B}_i]$ be a $GP(7,4,1)$, $i=1,2$. Let $(V_1 \cup V_2, \mathcal{B}_3)$ be the good P_4-decomposition of K_{V_1,V_2} of Example 3. Then $(\{\infty\} \cup V_1 \cup V_2, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a $GP(13,4,1)$.

$GP(15,4,1)$. Base blocks: $[\infty,0,4,13]$, $[0,1,4,2]$, *[0,6,13,7]*.

$GP(16,4,1)$. Base blocks: $[0,1,4,2]$, $[0,4,10,5]$, *[0,7,15,8]*.

$GP(18,4,1)$. Let $V_i = \mathbb{Z}_9 \times \{i\}$, $i=1,2$. The base blocks $[(0,1),(0,2),(1,1),(2,2)]$, $[(0,1),(3,2),(1,1),(5,2)]$ and $[(0,1),(6,2),(1,1),(8,2)]$ generate a good decomposition $(V_1 \cup V_2, \mathcal{B})$ of K_{V_1,V_2} into P_4's under the action of $\mathbb{Z}_9$. Let $(V_i, \mathcal{C}_i)$, $i=1,2$, be a $GP(9,4,1)$. Then $(V_1 \cup V_2, \mathcal{B} \cup \mathcal{C}_1 \cup \mathcal{C}_2)$ is a $GP(18,4,1)$. □

Lemma 4 *If there is a* $GP(v,4,1)$ *then there is a* $GP(v+12,4,1)$.

Proof Let $(W,\mathcal{B}_1)$ be a $GP(12,4,1)$ with $W = \mathbb{Z}_{12}$. Suppose $v=2t$. Let $(V,\mathcal{B}_2)$ be a $GP(v,4,1)$ with $V = \{x_0,x_1,\ldots,x_{2t-1}\}$. Put $\mathcal{B}_3 = \{[j,x_{2i},4+j,x_{2i+1}],\ [j,x_{2i+3},8+j,x_{2i}] \mid i=0,1,\ldots,t-1,\ j=0,1,2,3\}$ (the suffices are $\pmod{2t}$). It is easy to see that $\mathcal{B}_3$ is a good P_4-decomposition of $K_{V,W}$. Then $(V \cup W, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a $GP(v+12,4,1)$.

Let $v = 2t+1$. Put $V_1 = \{x_0,x_1,\ldots,x_{2t-3}\}$ and $V_2 = \{y_0,y_1,y_2\}$. Let $\mathcal{B}_3$ be the block set of a good P_4-decomposition of $K_{V_1,W}$ (since $|V_1| \equiv 0 \pmod 2$, we can construct this decomposition as in the previous case). It is easy to check that $\mathcal{B}_4 = \{[y_0,i,y_1,i+2],\ [y_2,6+i,y_1,4+i],\ [y_1,8+i,y_2,10+i],\ [y_0,4+i,y_2,i],\ [y_2,2+i,y_0,8+i],\ [y_1,10+i,y_0,6+i] \mid i=0,1\}$ is a good P_4-decomposition of $K_{V_2,W}$. Then $(V \cup W, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4)$ is a $GP(v+12,4,1)$. □

Theorem 1 *For every* $v \equiv 0$ or $1 \pmod 3$, $v \geq 7$, *there is a* $GP(v,4,1)$.

Proof Apply Lemmas 3 and 4. □

3 $\mathbf{GP(v,k,1),\ k \geq 5}$

Lemma 5 *There exists a* $GP(4t+1, 2t+1, 1)$*,* $t \geq 2$*.*

Proof Put $\alpha_i = 2t+4-i$, $\beta_i = 5+i$, $i = 0, 1, \ldots, t-2$. Then

$$B = [0, 1, 3, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_{t-2}, \beta_{t-2}]$$

is the base block of a P_{2t+1}-design $(\mathbb{Z}_{4t+1}, \mathcal{B})$. It is easy to verify that $\mu_B(g) \leq 2t-1$ for every $g \in \mathbb{Z}_{4t+1}$. By Lemma 1, $(\mathbb{Z}_{4t+1}, \mathcal{B})$ is good. □

Lemma 6 *There exists a* $GP(4t+2, 2t+2, 1)$*,* $t \geq 2$*.*

Proof Let

$$B = [\infty, 0, 1, 3, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_{t-2}, \beta_{t-2}]$$

where α_i and β_i are defined as in Lemma 5. It follows $\mu_B(g) \leq 2t-1$ for every $g \in \mathbb{Z}_{4t+1}$. By Lemma 1, B is the base block of a $GP(4t+2, 2t+2, 1)$ on vertex set $\{\infty\} \cup \mathbb{Z}_{4t+1}$. □

Lemma 7 *There exists a* $GP(4t-1, 2t, 1)$*,* $t \geq 3$*.*

Proof Put $\alpha_i = 2t+2-i$, $\beta_i = 4+i$, $i = 0, 1, \ldots, t-3$. Then

$$B = \begin{cases} [0, 1, 3, 8, 5, 9] & \text{if} \quad t = 3 \\ [0, 1, 3, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_{t-3}, \beta_{t-3}, t+4] & \text{if} \quad t \geq 4 \end{cases}.$$

is the base block of a P_{2t}-design $(\mathbb{Z}_{4t-1}, \mathcal{B})$. It is easy to verify that $\mu_B(g) \leq 2t-2$ for every $g \in \mathbb{Z}_{4t-1}$. By Lemma 1, $(\mathbb{Z}_{4t-1}, \mathcal{B})$ is good. □

Lemma 8 *There exists a* $GP(4t, 2t+1, 1)$*,* $t \geq 2$*.*

Proof Let

$$B = [\infty, 0, 1, 3, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_{t-3}, \beta_{t-3}, t+4]$$

where α_i and β_i defined as in Lemma 7. It follows $\mu_B(g) \leq 2t-2$ for every $g \in \mathbb{Z}_{4t-1}$. By Lemma 1, B is the base block of a $GP(4t, 2t+1, 1)$ on vertex set $\{\infty\} \cup \mathbb{Z}_{4t-1}$. □

Lemma 9 *There exists a* $GP(6t-3, 2t, 1)$*,* $t \geq 3$*.*

Proof Let $(V, \mathcal{B})$ be the P_{2t}-design with vertex set $V = \{\infty\} \cup \mathbb{Z}_{6t-4}$ and block set obtained as devolopment of the following base blocks:

- $B_1 = [\infty, 0, 1, 3, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_{t-3}, \beta_{t-3}]$, where $\alpha_i = 4t+1-i$, $\beta_i = 2t+4+i$, $i = 0, 1, \ldots, t-3$;

- $B_2 = [0, \alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \alpha_{t-2}, \beta_{t-2}, 3t-2]$, where $\alpha_i = 2t-1-i$, $\beta_i = 4t-1+i$, $i = 0, 1, \ldots, t-2$ (B_2 is short).

B_1 and B_2 satisfy the hypothesis of Lemma 1. Then $(V, \mathcal{B})$ is good. □

Lemma 10 *There exists a $GP(6t-2, 2t, 1)$, $t \geq 3$.*

Proof Let $(V, \mathcal{B})$ be the P_{2t}-design with vertex set $V = \mathbb{Z}_{6t-2}$ and block set obtained as development of the following base blocks:

- $B_1 = [0, 1, 3, \alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_{t-3}, \beta_{t-3}, 3t+4]$, where $\alpha_i = 4t+2-i$, $\beta_i = 2t+4+i$, $i = 0, 1, \dots, t-3$;
- $B_2 = [0, \alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_{t-2}, \beta_{t-2}, 3t-1]$, where $\alpha_i = 2t-i$, $\beta_i = 4t+1+i$, $i = 0, 1, \dots, t-2$ (B_2 is short).

B_1 and B_2 satisfy the hypothesis of Lemma 1. Then $(V, \mathcal{B})$ is good. □

Theorem 2 *If there is a $GP(v, 2t+1, 1)$, $v \geq 4t$ and $t \geq 2$, then there is a $GP(v+4t, 2t+1, 1)$.*

Proof Let v be even, $v \geq 4t$, $t \geq 2$. Put $V = V_1 \cup V_2$ where $V_i = \mathbb{Z}_{\frac{v}{2}} \times \{i\}$, $i = 1, 2$. Let $X = \{x_0, x_1, \dots, x_{4t-1}\}$. By assumption there is a $GP(v, 2t+1, 1)$ $(V, \mathcal{B}_1)$. By Lemma 8 there a $GP(4t, 2t+1, 1)$ $(X, \mathcal{B}_2)$. So it suffices to produce a good P_{2t+1}-decomposition $(V \cup X, \mathcal{B}_3)$ of $K_{V,X}$. Define the block set $\mathcal{B}_3$ as follows:

- Let $t = 2$. For every $i \in \mathbb{Z}_{\frac{v}{2}}$ and $j = 0, 1, 2, 3$, $[(i,1), x_{jt}, (i,2), x_{jt+1}, (i-1, 1)] \in \mathcal{B}_3$.
- Let t be even, $t \geq 4$. For every $i \in \mathbb{Z}_{\frac{v}{2}}$ and $j = 0, 1, 2, 3$, $[(i,1), x_{jt}, (i,2), x_{jt+1}, (1+i, 1), x_{jt+2}, (1+i, 2), x_{jt+3}, \dots$ $\dots, (\frac{t-2}{2}+i, 1), x_{jt+t-2}, (\frac{t-2}{2}+i, 2), x_{jt+t-1}, (\frac{t}{2}+i, 1)] \in \mathcal{B}_3$.
- Let t be odd, $t \geq 3$. For every $i \in \mathbb{Z}_{\frac{v}{2}}$ and $j = 0, 1, 2, 3$, $[(i,1), x_{jt}, (i,2), x_{jt+1}, (1+i, 1), x_{jt+2}, (1+i, 2), x_{jt+3}, \dots, (\frac{t-3}{2}+i, 1),$ $x_{jt+t-3}, (\frac{t-3}{2}+i, 2), x_{jt+t-2}, (\frac{t-1}{2}+i, 1), x_{jt+t-1}, \frac{t-1}{2}+i, 2)] \in \mathcal{B}_3$.

Let v be odd, $v \geq 4t+1$, $t \geq 2$. Let $V = V_1 \cup V_2$, where $V_1 = \{0, 1, \dots, 2t-2\}$ and $V_2 = \{2t-1, 2t, \dots, v-1\}$. Let $X = X_1 \cup X_2$, where $X_j = \{\infty_j\} \cup \{x_0^j, x_1^j, \dots, x_{2t-2}^j\}$, $j = 1, 2$. It suffices to produce a good P_{2t+1}-decomposition of $K_{V,X}$ or, equivalently, a good P_{2t+1}-decomposition of K_{V_1,X_j} and $K_{V_2,X}$. Being $|V_2|$ even, a good P_{2t+1}-decomposition of $K_{V_2,X}$ can be produced as in the previous case. The required decomposition of K_{V_1,X_j} is $[x_i^j, i, x_{1+i}^j, 2t-2+i, x_{2+i}^j, 2t-3+i, \dots, x_{t-1+i}^j, t+i, \infty_j]$, $j = 1, 2$, $i \in \mathbb{Z}_{2t-1}$. □

Theorem 3 *If there is a $GP(v, 2t, 1)$, $v \geq 4t-2$ and $t \geq 3$, then there is a $GP(v+4t-2, 2t, 1)$.*

Proof Let v be even, $v \geq 4t-2$, $t \geq 3$. Put $V = V_1 \cup V_2$ where $V_i = \mathbb{Z}_{\frac{v}{2}} \times \{i\}$, $i = 1, 2$. Let $X = \{\infty_0, \infty_1\} \cup X_1 \cup X_2$, where $X_1 = \{x_0, x_1, \dots, x_{2t-3}\}$, $X_2 = \{y_0, y_1, \dots, y_{2t-3}\}$. By assumption, there is a $GP(v, 2t, 1)$ $(V, \mathcal{B}_1)$. By Lemma 6, there is a $GP(4t-2, 2t, 1)$ $(X, \mathcal{B}_2)$. So it suffices to produce a good P_{2t}-decomposition $(V \cup X, \mathcal{B}_3)$ of $K_{V,X}$. Define the block set $\mathcal{B}_3$ as follows:

- Let t be even. For every $i \in \mathbb{Z}_{\frac{v}{2}}$ and $j=0,1$,
  $[(i,1),x_{j(t-1)},(i,2),x_{1+j(t-1)},(1+i,1),x_{2+j(t-1)},(1+i,2),x_{3+j(t-1)},\ldots$
  $\ldots,(\frac{t-2}{2}+i,1),x_{t-2+j(t-1)},(\frac{t-2}{2}+i,2),\infty_j] \in \mathcal{B}_3$, and
  $[(0+i,2),y_{j(t-1)},(i,1),y_{1+j(t-1)},(1+i,2),y_{2+j(t-1)},(1+i,1),$
  $y_{3+j(t-1)},\ldots,(\frac{t-2}{2}+i,2),y_{t-2+j(t-1)},(\frac{t-2}{2}+i,1),\infty_j] \in \mathcal{B}_3$.

- Let t be odd. For every $i \in \mathbb{Z}_{\frac{v}{2}}$ and $j=0,1$,
  $[(i,1),x_{j(t-1)},(i,2),x_{1+j(t-1)},(1+i,1),x_{2+j(t-1)},(1+i,2),x_{3+j(t-1)},\ldots$
  $\ldots,(\frac{t-3}{2}+i,1),x_{t-1+j(t-1)},(\frac{t-3}{2}+i,2),x_{t-2+j(t-1)},(\frac{t-1}{2}+i,1),\infty_j] \in \mathcal{B}_3$, and
  $[(i,2),y_{j(t-1)},(i,1),y_{1+j(t-1)},(1+i,2),y_{2+j(t-1)},(1+i,1),y_{3+j(t-1)},\ldots$
  $\ldots,(\frac{t-3}{2}+i,2),y_{t-1+j(t-1)},(\frac{t-3}{2}+i,1),y_{t-2+j(t-1)},(\frac{t-1}{2}+i,2),\infty_j] \in \mathcal{B}_3$.

Let v be odd, $v \geq 4t-1$, $t \geq 3$. Let $V = V_1 \cup V_2$, where $V_1 = \{0,1,\ldots,2t-2\}$ and $V_2 = \{2t-1,2t,\ldots,v-1\}$. Let $X = X_1 \cup X_2$, $X_j = \{x_0^j, x_1^j, \ldots, x_{2t-2}^j\}$, $j=1,2$. It suffices to produce a good P_{2t}-decomposition of $K_{V,X}$ or, equivalently, a good P_{2t}-decomposition of K_{V_1,X_j} and $K_{V_2,X}$. Being $|V_2|$ even , a good P_{2t}-decomposition of $K_{V_2,X}$ can be produced as in the previous case. The required decomposition of K_{V_1,X_j} is
$[i,x_i^j,2t-2+i,x_{1+i}^j,2t-3+i,x_{2+i}^j,\ldots,2t-t+i,x_{t-1+i}^j]$, $i \in \mathbb{Z}_{2t-1}$. □

Theorem 4 *Let $k \geq 5$. Let v be an integer such that $v \geq 2(k-1)$ and*

- $v \equiv 0$ or $1 \pmod{2(k-1)}$ *if k is odd;*
- $v \equiv 0$ or $1 \pmod{k-1}$ *if k is even.*

Then there is a $GP(v,k,1)$.

Proof Apply Lemmas 5, 6, 7, 8, 9, 10 and Theorems 2 and 3. □

4 Open Questions

- The problem studied in this paper can be generalized as follows: for every integer t, $3 \leq t \leq k-1$, determine the spectrum of the integers v such that there is a P_k-design $(V,\mathcal{B})$ of order v having the property that no two blocks have h, $t \leq h \leq k$ common vertices, i.e. $(V,\mathcal{B})$ is a partial $S(t,k,v)$.

- For every admissible $v \equiv 4 \pmod 6$, $v \geq 10$, determine a $GP(v,4,1)$ embeddable into a Steiner quadruple system of order v as shown in the following example.

  Example 4. Let $(V,\mathcal{B}_1)$ be the $GP(10,4,1)$ with
  $V=\{a,b,c,d,e,f,g,h,l,m\}$, $\mathcal{B}_1 = \{emlf, lhfg, mdhb, amfd, cmga,$
  $fcgd, lead, gbde, hafb, heca, cbef, glab, mblc, ldch, mhge\}$.
  Let $\mathcal{B}_2 = \{abcd, aefg, behl, cfhm, dglm, ahlm, bfgm, cegl, defh, ecdm,$
  $fbdl, gbch, hadg, lacf, mabe\}$. Then $(V,\mathcal{B}_1 \cup \mathcal{B}_2)$ is a Steiner quadruple system of order 10 (considering the blocks of $\mathcal{B}_1 \cup \mathcal{B}_2$ as K_4's).

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