AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **36** (2006), Pages 13–26

Square contractions of graphs

Richard Hammack*

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, Virginia 23284–2014
U.S.A.
rhammack@vcu.edu

Abstract

The problem of contracting an arbitrary graph to a square is known to be NP-complete. This paper proves the problem is tractable for the class of graphs whose complements have radius unequal to 2. As an application, the results are used to aid in computing the cyclicity of graphs.

1 Introduction

A simple graph $G = (V(G), E(G))$ is *contractible* to a graph $H = (V(H), E(H))$ if there is a partition $\{V_y | y \in V(H)\}$ of $V(G)$ for which the subgraph of G induced by V_y is connected for each $y \in V(H)$, and an edge of G joins V_y to V_z if and only if $yz \in E(H)$.

This article examines the problem of contracting an arbitrary graph to C_4, the cycle on 4 vertices, also called the *square*. Brouwer and Veldman [4] state that this problem is NP-complete, and they provide an outline of a proof. We show the problem is tractable for a large class of graphs, namely those whose complements have radii unequal to 2.

The article is organized as follows. Section 2 contains some simple results relating the C_4-contractibility of a graph to the radius of its complement. Section 3 presents an algorithm that contracts an arbitrary graph to a square, and the algorithm is proved to be polynomial for graphs G with $\text{rad}(\overline{G}) \notin \{2, 3\}$, and exponential otherwise. Section 4 addresses the case of graphs G with $\text{rad}(\overline{G}) \geq 3$, and presents a second polynomial algorithm for finding C_4 contractions of such graphs. Finally, Section 5 applies these ideas to the problem of computing the cyclicity of a graph. The author is indebted to the referee for suggesting many improvements, and especially for suggesting a strategy that simplified the algorithm for the case $\text{rad}(\overline{G}) \geq 3$.

The remainder of this section recalls some standard definitions. Given a graph G, its *complement*, $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and with $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$. If X and Y are disjoint subsets of $V(G)$, then $E_G(X, Y) =$

* Supported by a grant from the Walter Williams Craigie Teaching Endowment

$\{xy \in E(G) | x \in X, y \in Y\}$. The *neighborhood* of a vertex $x \in V(G)$ is the set $N(x) = \{y \in V(G) | xy \in E(G)\}$. The distance $d_G(x, y)$ between two vertices x and y of G is the length of the shortest path from x to y in G, or ∞ if no such path exists. The *radius* of G is $\text{rad}(G) = \min_{x \in V(G)}\{\max_{y \in V(G)}\{d_G(x, y)\}\}$. If $X \subseteq V(G)$, then the subgraph of G induced on X is denoted $G[X]$.

We regard the vertex set of the cycle C_k as being the elements $\{0, 1, 2, \cdots, k-1\}$ of the cyclic group $\mathbb{Z}_k$, with $E(C_k) = \{ij | i - j = \pm 1\}$. A contraction of G to C_k is denoted by a k-tuple $(V_0, V_1, \cdots, V_{k-1})$ of pairwise disjoint nonempty subsets of $V(G)$, where $V(G) = \bigcup_{i \in \mathbb{Z}_k} V_i$, and each $G[V_i]$ is connected, and for all distinct $i, j \in \mathbb{Z}_k$, $E_G(V_i, V_j) \neq \emptyset$ if and only if $i - j = \pm 1$.

2 A Complement Criterion for Contraction

This section presents two results linking contractibility of a graph to C_k to the radius of its complement. We first show that G can be contracted to a square only if $\text{rad}(\overline{G}) \in \{2, 3, \infty\}$. This will allow our contraction algorithm to reject any graph G for which $\text{rad}(\overline{G}) \notin \{2, 3, \infty\}$.

Lemma 1: If a graph G can be contracted to a square, then $\text{rad}(\overline{G}) \in \{2, 3, \infty\}$.

Proof. Suppose (V_0, V_1, V_2, V_3) is a contraction of G to C_4. Since $E_G(V_0, V_2) = \emptyset = E_G(V_1, V_3)$, it follows that $\overline{G}$ has as subgraphs the disjoint complete bipartite graphs with partite sets V_0, V_2, and V_1, V_3, respectively. If $\overline{G}$ is disconnected, then $\text{rad}(\overline{G}) = \infty$. Otherwise there must be some edge xy of $\overline{G}$ joining these two complete bipartite graphs, and this means one of x or y is in some V_i and the other is in V_{i+1}. Without loss of generality, we may say $x \in V_0$, and $y \in V_1$. Now let z be any vertex of $V(\overline{G})$. Figures 1A–1D show that whether z is in V_0, V_1, V_2 or V_3, there is always a path in $\overline{G}$ from x to z having length at most 3. (Note that the edges in Figure 2 are edges in $\overline{G}$, not G.) Thus $d_{\overline{G}}(x, z) \leq 3$ for all $z \in V(\overline{G})$, so $\text{rad}(\overline{G}) \leq 3$.

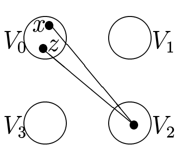

Figure 1A

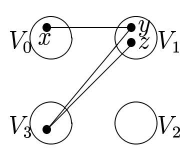

Figure 1B

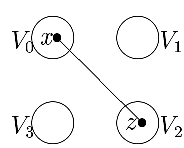

Figure 1C

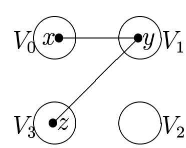

Figure 1D

On the other hand, it is impossible for $\text{rad}(\overline{G}) = 0$ or $\text{rad}(\overline{G}) = 1$, for then G is either trivial or disconnected, and cannot be contracted to a square. Thus $\text{rad}(\overline{G}) \in \{2, 3, \infty\}$. ■

To see that there are graphs G, contractible to a square, with $\text{rad}(\overline{G}) = 2, 3, \infty$, consider the Cartesian product graphs $G = K_n \times K_2$. For $n = 2$, $\text{rad}(\overline{G}) = \infty$, and for $n > 2$, $\text{rad}(\overline{G}) = 3$. Also, if $G = C_4 \times K_2$, then $\text{rad}(\overline{G}) = 2$.

Lemma 2: If G can be contracted to C_k, with $k > 4$, then $\text{rad}(\overline{G}) = 2$.

Proof. Suppose $(V_0, V_1, V_2, \cdots, V_{k-1})$ is a contraction of G to C_k. Let x and y be arbitrary vertices of G. We first show that $d_{\overline{G}}(x, y) \leq 2$, which implies $\mathrm{rad}(\overline{G}) \leq 2$. Say $x \in V_i$. If $y \in V_i \cup V_{i-1}$, choose $z \in V_{i+2}$ and note xzy is a path of length 2 in $\overline{G}$, so $d_{\overline{G}}(x, y) \leq 2$. Likewise, if $y \in V_{i+1}$, choose $z \in V_{i+3}$ and note xzy is a path of length 2 in $\overline{G}$. If $y \notin V_{i-1} \cup V_i \cup V_{i+1}$, then $xy \in E(\overline{G})$, and $d_{\overline{G}}(x, y) = 1$. Thus $\mathrm{rad}(\overline{G}) \leq 2$. Also, since G can be contracted to C_k, it has no isolated vertices, so for any $x \in V(G)$ there is a $y \in V(G)$ with $d_{\overline{G}}(x, y) > 1$, hence $\mathrm{rad}(\overline{G}) \geq 2$. ∎

3 First Contraction Algorithm

This section introduces a simple algorithm that contracts any graph G to a square, or reports if such a contraction is not possible. Although its complexity is polynomial provided $\mathrm{rad}(\overline{G}) \notin \{2, 3\}$, it is not sufficiently sophisticated to efficiently handle the case $\mathrm{rad}(\overline{G}) = 3$. However, it has the advantage of simplicity, and it introduces some ideas employed in our second, more complex, algorithm. It also illuminates a structural property possessed by graphs with $\mathrm{rad}(\overline{G}) = 2$ that makes finding their C_4 contractions problematic.

The algorithm searches for a contraction of a graph to C_4 by examining the graph's 2-colorings. A *2-coloring* of a graph is an assignment of two colors (say black and white) to its vertices. A *proper* 2-coloring is one with the property that no two adjacent vertices have the same color. We say that a 2-coloring of G *induces a contraction of* G *to* C_4 if there is a contraction (V_0, V_1, V_2, V_3) of G to C_4 where $V_0 \cup V_2$ are the black vertices of the 2-coloring and $V_1 \cup V_3$ are the white vertices. Figure 1 is an example of a 2-coloring which induces a contraction of a graph to C_4. Obviously, any graph that is contractible to C_4 has a 2-coloring that induces that contraction.

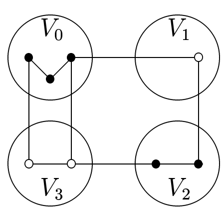

Figure 2

Now comes a construction that plays a key role in this article. If G is a graph, then $\widetilde{G}$ is the spanning subgraph of G whose edge set is $\{xy \in E(G) |\ d_{\overline{G}}(x, y) \geq 3\}$. Notice that if the radius of $\overline{G}$ is at least 3 then no component of $\widetilde{G}$ is trivial, but if $\overline{G}$ has a radius 2, then every vertex in the center of $\overline{G}$ is a trivial component of $\widetilde{G}$.

Examples 1: If G is the Cartesian product $K_2 \times K_n$, with $n > 2$, then $\widetilde{G}$ has n components, namely the n edges $(a, x)(b, x)$ over K_n. Also, $\widetilde{K_n} = K_n$, and $\widetilde{C_5} = \overline{K_5}$, and $\widetilde{C_4} = C_4$.

The following remarks are used extensively in this article.

Remark 1: If $xy \in E(G)$, then $xy \in E(\widetilde{G})$ if and only if $N(x) \cup N(y) = V(G)$. This is true because $N(x) \cup N(y) = V(G) \iff \overline{G}$ has no path of form $xzy \iff$

$d_{\overline{G}}(x,y) > 2 \iff xy \in E(\widetilde{G})$. (Because of this, an edge of $\widetilde{G}$ is middle edge of a *spanning double star* of G [3].)

Remark 2: The importance of $\widetilde{G}$ is this: Given any contraction of G to C_4, each edge xy of $\widetilde{G}$ must necessarily join some V_i to V_{i+1}. The reason is that $xy \in E(\widetilde{G}) \implies d_{\overline{G}}(x,y) > 2 \implies x, y \notin V_i \cup V_{i+2}$ for all $i \in \mathbb{Z}_4 \implies xy \in E_G(V_i, V_{i+1})$ for some $i \in \mathbb{Z}_4$. It follows that if some 2-coloring of G induces a contraction of G to C_4, then this 2-coloring must be a *proper* 2-coloring of the subgraph $\widetilde{G}$. The next result uses this idea.

Lemma 3: A graph G is contractible to C_4 if and only if $\widetilde{G}$ is bipartite and has a proper two-coloring that induces a contraction of G to C_4.

Proof. The necessity clear. Conversely, suppose G is contractible to C_4, and let (V_0, V_1, V_2, V_3) be such a contraction. Color the vertices $V_0 \cup V_2$ black and color the vertices $V_1 \cup V_3$ white. By Remark 2, the vertices of any edge of G have opposite colors. Thus $\widetilde{G}$ is properly 2-colored, and is therefore bipartite. By construction, this 2-coloring induces a contraction of G to C_4. ■

Lemma 3 implies that we can find a C_4 contraction of G by constructing $\widetilde{G}$, confirming that it is bipartite, and then examining all its proper 2-colorings and selecting any one that induces a C_4 contraction. (Or conclude that no such contraction exists, if none is found.) Suppose $\widetilde{G}$ is bipartite and its components are $\widetilde{G}_1, \widetilde{G}_2, \cdots, \widetilde{G}_c$, and for each $1 \le i \le c$ we select an $x_i \in V(\widetilde{G}_i)$. Then any proper 2-coloring of $\widetilde{G}$ is determined by an assignment of a color black or white to each x_i. Thus $\widetilde{G}$ has potentially 2^c proper 2-colorings to be examined. Algorithm 1 works by examining these two-colorings.

Algorithm 1:
Input: A simple graph G
Output: A contraction (V_0, V_1, V_2, V_3), of G to C_4 if such is possible, or $\emptyset$ otherwise

1. Compute $\text{rad}(\overline{G})$.
2. If $\text{rad}(\overline{G}) \notin \{2, 3, \infty\}$, then **return**($\emptyset$) and **stop**. *(By Lemma 1, no contraction is possible.)*
3. Construct $\widetilde{G}$.
4. Find the components of $\widetilde{G}$ and list them as $\widetilde{G}_1, \widetilde{G}_2, \cdots, \widetilde{G}_c$.
5. Check that $\widetilde{G}$ is bipartite. If it is not, then **return**($\emptyset$) and **stop**. *(By Lemma 3, G can't be contracted to C_4.)*
6. For each $1 \le i \le c$, choose a vertex $x_i \in V(\widetilde{G}_i)$.
7. For each c-digit binary number $b_1 b_2 b_3 \cdots b_c$, do the following:

   7.1 For each $1 \le i \le c$, give $\widetilde{G}_i$ the proper 2-coloring in which x_i is white if $b_i = 1$ or black if $b_i = 0$. *(If $\widetilde{G}_i$ is trivial, only one color is needed.)*

   7.2 Let the subgraph of G induced on black vertices have components $C_0, C_2, \cdots, C_{2b}$

7.3 Let the subgraph of G induced on white vertices have components $C_1, C_3, \cdots, C_{2w+1}$. *Now we check if this coloring induces a contraction of G to C_4 by checking that the subgraphs induced on black and white vertices, respectively, each have 2 components, and some edge of G joins any two components of different colors.*

7.4 If $b = w = 1$, and $E_G(C_i, C_j) \neq \emptyset$ whenever i and j have opposite parity, Then **return**$(V(C_0), V(C_1), V(C_2), V(C_3))$, and **stop**. *(This is a C_4 contraction.)*

8. **Return**$(\emptyset)$, and **stop**. *(If this step is reached, all proper 2-colorings of $\widetilde{G}$ have been exhausted, and none induced a contraction.)* ■

Proposition 1: Suppose G is a graph with n vertices, and for which $\widetilde{G}$ has c components. If $\text{rad}(\overline{G}) \notin \{2, 3\}$, then the complexity of Algorithm 1 is $O(n^4)$. Otherwise, if $\text{rad}(\overline{G}) \in \{2, 3\}$, the complexity is $O(n^4 + 2^c n^2)$.

Proof. Computing $\text{rad}(\overline{G})$ in Line 1 is $O(n^4)$, using Algorithm 8.8 of [8]. If, in Line 2, $\text{rad}(\overline{G}) \notin \{2, 3, \infty\}$, the algorithm terminates with total complexity $\mathcal{O}(n^4)$. Thus, for the remainder of the proof, assume that $\text{rad}(\overline{G}) \in \{2, 3, \infty\}$. In Line 3, $\widetilde{G}$ may be constructed by setting $V(\widetilde{G}) = V(G)$ and $E(\widetilde{G}) = \{xy \in E(G) \mid N(x) \cup N(y) = V(G)\}$. (See Remark 1.) Since in forming $E(\widetilde{G})$, there are $O(n^2)$ edges $xy \in E(G)$ to test, and forming $N(x)$ and $N(y)$ is $O(n)$, it follows that the complexity of Line 3 is $O(n^3)$. Line 4 is $O(n^2)$ by Algorithm 8.3 of [8]. In Line 5, testing for bipartiteness can be done with a standard $O(n^2)$ depth-first search. If the algorithm terminates in Line 5, the total complexity is $\mathcal{O}(n^4)$. The complexity of Line 6 is $\mathcal{O}(n)$, and lines 1 through 6 have a total complexity of $\mathcal{O}(n^4)$.

Step 7 executes at most 2^c iterations. In Line 7.1, the 2-coloring can be attained by an $O(n^2)$ depth-first traversal of each component of $\widetilde{G}$, alternatively coloring vertices black or white, starting with x_i. Steps 7.2 and 7.3 are each $O(n^2)$, by Algorithm 8.3 of [8]. Line 7.4 is $O(n^2)$. Therefore the total complexity of Step 7 is thus $O(2^c n^2)$.

Thus, if $\text{rad}(\overline{G}) \in \{2, 3, \infty\}$ and lines 1–7 are executed, their net complexity is $O(n^4 + 2^c n^2)$.

However, if $\text{rad}(\overline{G}) = \infty$, we claim that $c = 1$, making the complexity $O(n^4)$. Choose $x, y \in V(G)$ for which $d_{\overline{G}}(x, y) = \infty$, so $xy \in E(\widetilde{G})$. Suppose for the sake of contradiction that there is a vertex w that is in a component of $\widetilde{G}$ that does not contain the edge xy. Then, since w is adjacent to neither x nor y in $\widetilde{G}$, it follows by definition of $\widetilde{G}$ that $d_{\overline{G}}(x, w) < 3$ and $d_{\overline{G}}(w, y) < 3$. This means $\overline{G}$ has a path from x to y — routed through w — of length no greater than 4, contradicting the fact that $d_{\overline{G}}(x, y) = \infty$. ■

4 Graphs with Complement Radius 3 or Greater

This section treats the problem of contracting G to C_4 in the case $\text{rad}(\overline{G}) \geq 3$. According to Proposition 1, Algorithm 1 is not guaranteed to do this in polynomial time. The problem is that when $\mathrm{rad}(\overline{G}) = 3$, the subgraph $\widetilde{G}$ can have as many as $O(n)$ components, thus $O(2^n)$ proper 2-colorings, potentially forcing the algorithm to execute exponentially many steps.[1]

To overcome this problem, we create a special algorithm for the case $\mathrm{rad}(\overline{G}) \geq 3$. The basic idea is the same as that of Algorithm 1, namely to examine 2-colorings of $\widetilde{G}$. However, some additional structure is introduced to accelerate the coloring of vertices. The algorithm operates by examining 4-tuples (v_0, v_1, v_2, v_3) of vertices of G, and searching for a C_4 contraction (V_0, V_1, V_2, V_3) of G with $v_i \in V_i$ for $i \in \mathbb{Z}_4$. In order to efficiently extend the 4-tuple to a contraction, it will be necessary that each component of $\widetilde{G}$ be nontrivial, and this is guaranteed by the condition $\mathrm{rad}(\overline{G}) \geq 3$.

Algorithm 2:
Input: A connected simple graph G with n vertices and with $\mathrm{rad}(\overline{G}) \geq 3$
Output: A contraction (V_0, V_1, V_2, V_3), of G to C_4 if such is possible, or $\emptyset$ otherwise

1. Construct $\widetilde{G}$.
In what follows, suppose there is a contraction (V_0, V_1, V_2, V_3), of G to C_4, with V_0 and V_2 colored black and V_1 and V_3 colored white. This assumption forces certain vertices to be in the sets V_i, as follows.

2. Select an edge $v_0v_1 \in E(\widetilde{G})$.
Such an edge exists because because rad$(\overline{G}) \geq 3$, so all components of $\widetilde{G}$ are nontrivial. By Remark 1, one endpoint of this edge is in some V_i and the other is in V_{i+1}. Without loss of generality, it may be assumed that $v_0 \in V_0$ (and is colored black) and $v_1 \in V_1$ (and is colored white). Now notice G must have some edge v_2v_3 with $v_2 \in V_2$ and $v_3 \in V_3$. Moreover, since $v_0v_1 \in E(\widetilde{G})$, Remark 1 implies that $\{v_2, v_3\} \subseteq N(v_0) \cup N(v_1)$. As $E_G(V_0, V_2) = \emptyset = E_G(V_1, V_3)$, it then follows that $v_0v_3, v_1v_2 \in E(G)$, and $v_0v_1v_2v_3v_0$ is a square in G. (See Figure 3.) The next step searches for a contraction by examining all such squares.

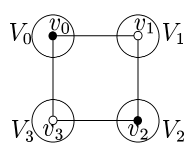

Figure 3

3. For each $v_2v_3 \in E(G)$, with $v_2 \in N(v_1) - N(v_0)$ and $v_3 \in N(v_0) - N(v_1)$, do the following:

3.1 For $i \in \mathbb{Z}_4$, set $V_i := \{v_i\}$.
This initializes the sets V_i as containing vertices that are thus far known to

[1]For an example of a graph for which $\widetilde{G}$ has $O(n)$ components, recall Example 1, in which $G = K_n \times K_2$, with $n > 2$, and $\widetilde{G}$ has n components. Perhaps this is not a particularly interesting example, since all but two proper 2-colorings of $\widetilde{G}$ will induce a contraction of G to C_4. For a less trivial example, consider the graph H obtained from G as follows. Let $\mathcal{E}$ be any set of pairwise nonadjacent edges of K_n, and for each $xy \in \mathcal{E}$, replace each pair of edges $(x,0)(y,0)$, $(x,1)(y,1)$ in G with $(x,0)(y,1)$, $(x,1)(y,0)$. Then $\widetilde{H}$ consists of the $|\mathcal{E}|$ squares $(x,0)(y,1)(y,0)(x,1)(x,0)$.

be in them. Assuming the current V_i can be extended to a C_4 contraction, other vertices must be appended to these sets as follows.

3.2 Repeat the following step until no further changes in the V_i are produced.

3.2.1 Search for an $x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3)$ that, for some $i \in \mathbb{Z}_4$, is adjacent to vertices of each of V_{i-1}, V_i and V_{i+1}, but is not adjacent to any vertex of V_{i+2};
If such a vertex x is found, put $V_i := V_i \cup \{x\}$.
Reason: Inserting x in any other V_j would force $E_G(V_0, V_2) \neq \emptyset$ or $E_G(V_1, V_3) \neq \emptyset$, which is forbidden if (V_0, V_1, V_2, V_3) is to be extended to a C_4 contraction. The complexity of step 3.2 is $\mathcal{O}(n^3)$ because 3.2.1 is executed no more than $n-4$ times, and each iteration involves examining no more than $n-4$ vertices and their their $\mathcal{O}(n)$ adjacencies.

3.3 Search for $x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3)$ that is adjacent to vertices of V_i, for all $i \in \mathbb{Z}_4$. If such an x is found, **exit** this iteration of Step 3; Else continue in Step 3.
Reason: If such a vertex x is found, it cannot be inserted in any V_j without forcing $E_G(V_0, V_2) \neq \emptyset$ or $E_G(V_1, V_3) \neq \emptyset$, so a new choice of the edge v_2v_3 must be made. The complexity of step 3.3 is $\mathcal{O}(n^2)$ because it examines no more than $n-4$ vertices and their their $\mathcal{O}(n)$ adjacencies.

If the iteration of Loop 3 was not exited in Line 3.3, it is now the case that the 4-tuple (V_0, V_1, V_2, V_3) formed by the current V_i's is a C_4 contraction of the subgraph $G[V_0 \cup V_1 \cup V_2 \cup V_3]$ of G. However, there may be vertices of G that have not yet been inserted in any V_i. By steps 3.2 and 3.3, any such $x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3)$ is adjacent to fewer than three of the sets V_i. We claim that such an x is adjacent to exactly two of the V_i: Consider an edge $xy \in E(\widetilde{G})$, which exists, since $\widetilde{G}$ has no trivial components. If x were adjacent to none of the V_i, then Remark 1 would make y adjacent to each element of $\{v_0, v_1, v_2, v_3\}$, an impossibility since that would have terminated the current iteration of Loop 3 in Line 3.3. Thus x is adjacent to V_k, for some $k \in \mathbb{Z}_4$. If x is also adjacent to one of the vertices $\{v_{k+1}, v_{k+2}, v_{k+3}\}$, then certainly it's adjacent to two of the V_i; If it's adjacent to none of them, then y is adjacent to all of them, hence $y \in V_{k+2}$ (by Step 3.2), making x adjacent to both V_k and V_{k+2}. Thus any x that has not yet been inserted into any of the sets $\{V_0, V_1, V_2, V_3\}$ is adjacent to exactly two of these sets.

We claim further that such an x cannot be adjacent to both V_0 and V_1, nor can it be adjacent to both V_2 and V_3. Suppose x is adjacent to both V_0 and V_1, and let y be as in the previous paragraph, so y is adjacent to v_2 and v_3. Since $v_0v_1 \in E(\widetilde{G})$, Remark 1 implies that y is also adjacent to either v_0 or v_1. Hence y has been inserted into either V_2 or V_3 (in Step 3.2), a contradiction, for now x is adjacent to three of the V_i. If x were adjacent to both V_2 and V_3, then since $v_0v_1 \in E(\widetilde{G})$, one of v_0 or v_1 would be adjacent to x, making x adjacent to three of the V_i.

Now it is clear that any uninserted x is adjacent to both elements of one of the pairs $\{V_0, V_2\}$, $\{V_0, V_3\}$, $\{V_1, V_2\}$, $\{V_1, V_3\}$, so these uninserted vertices can be partitioned into sets $\mathcal{L}$, $\mathcal{R}$, $\mathcal{B}$ and $\mathcal{W}$, as follows. (See Figure 4.)

3.4 $\mathcal{L} := \{x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3) \mid N(x) \cap V_0 \neq \emptyset \neq N(x) \cap V_3\}$
This is the set of "left" vertices, adjacent to V_0 and V_3 but to neither V_1 nor V_2. If (V_0, V_1, V_2, V_3) is to be extended into a C_4 contraction of G, then any vertex of $\mathcal{L}$ must be inserted either in V_0 or V_3 to prevent a diagonal. Since V_0 and V_3 are of different colors, no vertex in $\mathcal{L}$ can yet be assigned a color. Execution of this line is $\mathcal{O}(n^2)$ because forming the set $\mathcal{L}$ involves examining no more than $n-4$ vertices and their adjacencies.

3.5 $\mathcal{R} := \{x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3) \mid N(x) \cap V_1 \neq \emptyset \neq N(x) \cap V_2\}$
This is the set of "right" vertices, adjacent to V_1 and V_2 but to neither V_0 nor V_3. If (V_0, V_1, V_2, V_3) is to be extended into a C_4 contraction of G, then any vertex of $\mathcal{R}$ must be inserted either in V_1 or V_2 to prevent a diagonal. Since V_1 and V_2 are of different colors, no vertex in $\mathcal{R}$ can yet be assigned a color. Execution of this line is $\mathcal{O}(n^2)$.

3.6 $\mathcal{B} := \{x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3) \mid N(x) \cap V_1 \neq \emptyset \neq N(x) \cap V_3\}$
This is the set of "black" vertices adjacent to V_1 and V_3 but to neither V_0 nor V_2. If (V_0, V_1, V_2, V_3) is to be extended into a C_4 contraction of G, then any vertex of $\mathcal{B}$ must be inserted in one of the black sets V_0 or V_2 to prevent a diagonal, and hence these vertices must be colored black. This line is $\mathcal{O}(n^2)$.

3.7 $\mathcal{W} := \{x \in V(G) - (V_0 \cup V_1 \cup V_2 \cup V_3) \mid N(x) \cap V_0 \neq \emptyset \neq N(x) \cap V_2\}$
This is the set of "white" vertices adjacent to V_0 and V_2, but to neither V_1 nor V_3. If (V_0, V_1, V_2, V_3) is to be extended into a C_4 contraction of G, then any vertex of $\mathcal{W}$ must be inserted in one of the white sets V_1 or V_3 to prevent a diagonal, and hence these vertices must be colored white. This line is $\mathcal{O}(n^2)$.

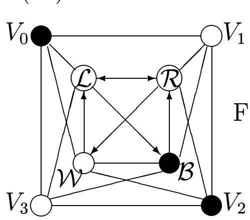

Figure 4

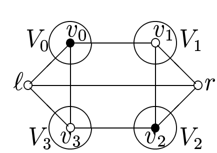

Figure 5

3.8 If $\mathcal{B} \cup \mathcal{W} = \emptyset$, set $V_0 := V_0 \cup \mathcal{L}$ and $V_1 := V_1 \cup \mathcal{R}$;
Return(V_0, V_1, V_2, V_3) and **stop**.
Reason: This is a C_4 contraction of G. This step is $\mathcal{O}(n)$.

3.9 If $\mathcal{L} \cup \mathcal{R} = \emptyset$, then **exit** this iteration of Step 3.
Reason: If this line is reached, then $\mathcal{B} \cup \mathcal{W} \neq \emptyset$, for otherwise the algorithm would have terminated in the previous step. If (V_0, V_1, V_2, V_3) were extended to a C_4 contraction, then any vertex of $\mathcal{B}$ would be connected

to v_0 or v_2 by a path in G whose vertices were colored black. Similarly, any vertex of $\mathcal{W}$ would be connected to v_1 or v_3 by a path in G whose vertices were colored white. Such paths would necessarily contain vertices from $\mathcal{L} \cup \mathcal{R}$ (see figure 4), but no such vertices exist. Thus (V_0, V_1, V_2, V_3) cannot be extended to a C_4 contraction, so a new choice of edge v_2v_3 must be made.

At this point, $\mathcal{B} \cup \mathcal{W} \neq \emptyset \neq \mathcal{L} \cup \mathcal{R}$. The following observation is needed in the remainder of the algorithm. Observe that if $\ell r \in E(\widetilde{G})$, and $\ell \in \mathcal{L}$, then $r \in \mathcal{R}$. The reason is that, as ℓ is adjacent to neither v_1 nor v_2, Remark 1 implies that r is adjacent to both v_1 and v_2. (See Figure 5.) Now r can't be in V_1 or V_2 because then ℓ would be adjacent to three of the V_i. Also, r is in neither V_0 or V_3 because that would force $E_G(V_0, V_2) \neq \emptyset$ or $E_G(V_1, V_3) \neq \emptyset$. Thus, r has not been inserted into any V_i and it is is adjacent to V_1 and V_2, so $r \in \mathcal{R}$. Similarly, if $\ell r \in E(\widetilde{G})$ and $r \in \mathcal{R}$, then $\ell \in \mathcal{L}$.

In particular, by the previous paragraph, $\ell \in \mathcal{L}$ implies $\ell v_0, \ell v_3 \in E(G)$, and $r \in \mathcal{R}$ implies $rv_1, rv_2 \in E(G)$.

Next, a certain digraph D is constructed, with $V(D) = \mathcal{L} \cup \mathcal{R} \cup \mathcal{B} \cup \mathcal{W}$. The meaning of an arc (x, y) of D (which we interpret as an arrow from x to y) is as follows. If (V_0, V_1, V_2, V_3) is extended to a C_4 contraction of G, then $(x, y) \in E(D)$ means that that if x has been inserted up (i.e. either in V_0 or V_1), then y must be inserted up also; and that if y has been inserted down (i.e. either in V_2 or V_3), then x must be inserted down too. The arcs of D are created in the following steps.

3.10 For any two vertices $b, b' \in \mathcal{B}$ with $bb' \in E(G)$, add arcs (b, b') and (b', b) to D.
Reason: If one of b and b' were inserted up, and the other down, then they would be in the black sets V_0 and V_2. Edge bb' would force $E_G(V_0, V_2) \neq \emptyset$. So b and b' must be inserted both up, or both down, and that is encoded by the arcs (b, b') and (b', b). This step can be done by considering each $b \in \mathcal{B}$ and adding arc (b, b') for every $b' \in N(b) \cap \mathcal{B}$, hence is of complexity $\mathcal{O}(n^2)$.

3.11 For any pair $w, w' \in \mathcal{W}$ with $ww' \in E(G)$, add arcs (w, w') and (w', w) to D.
Reason: If one of w and w' were inserted up, and the other down, then they would be in the white sets V_1 and V_3. Edge ww' would force $E_G(V_1, V_3) \neq \emptyset$. As above, the complexity of this step is $\mathcal{O}(n^2)$.

3.12 For any pair $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ with $\ell r \in E(G)$, add arcs (ℓ, r) and (r, ℓ) to D.
Reason: If one of ℓ and r were inserted up, and the other down, then they would be in V_0 and V_2 or V_1 and V_3. Either way, edge ℓr of G would be a diagonal. The complexity of this step is $\mathcal{O}(n^2)$.

3.13 For any two vertices $\ell \in \mathcal{L}$ and $b \in \mathcal{B}$ with $\ell b \in E(G)$, add arc (ℓ, b) to D.

Reason: Suppose ℓ is inserted up, so it is in V_0. If b were inserted down, it would be in V_2, forcing $E_G(V_0, V_2) \neq \emptyset$, so b must be inserted up. This is $\mathcal{O}(n^2)$.

3.14 For any two vertices $\ell \in \mathcal{L}$ and $w \in \mathcal{W}$ with $\ell w \in E(G)$, add arc (w, ℓ) to D.

Reason: Suppose w is inserted up, so it is in V_1. If ℓ were inserted down, it would be in V_3, forcing $E_G(V_1, V_3) \neq \emptyset$, so ℓ must be inserted up. This is $\mathcal{O}(n^2)$.

3.15 For any two vertices $r \in \mathcal{R}$ and $b \in \mathcal{B}$ with $rb \in E(G)$, add arc (b, r) to D.

Reason: Suppose b is inserted up, so it is in V_0. If r were inserted down, it would be in V_2, forcing $E_G(V_0, V_2) \neq \emptyset$, so r must be inserted up. This is $\mathcal{O}(n^2)$.

3.16 For any two vertices $r \in \mathcal{R}$ and $w \in \mathcal{W}$ with $rw \in E(G)$, add arc (r, w) to D.

Reason: Suppose r is inserted up, so it is in V_1. If w were inserted down, it would be in V_3, forcing $E_G(V_1, V_3) \neq \emptyset$, so w must be inserted up. This is $\mathcal{O}(n^2)$.

3.17 Enlarge D by replacing it with its transitive closure.

Justification: If $(x, y), (y, z) \in E(D)$, then if x is inserted up, so is y, hence also z. Therefore it is meaningful to add the arc (x, z) to D, etc. Forming the transitive closure is $\mathcal{O}(n^4)$, as it can be attained by no more than n iterations of the following procedure: Examine each $y \in V(D)$, and for each pair $(x, y), (y, z) \in E(D)$, add to D the arc (x, z).

If there is a pair of vertices in $\mathcal{L}$ that is not joined by an arc of D, then a C_4 contraction exists and is obtained in the following step.

3.18 Search for a pair $\ell_0, \ell_3 \in \mathcal{L}$ with neither arc (ℓ_0, ℓ_3) nor (ℓ_3, ℓ_0) in $E(D)$; If such a pair is found, a C_4 contraction of G is obtained as follows.

3.18.1 $V_0 := V_0 \cup \{\ell_0\}$. *(Insert ℓ_0 up.)*

3.18.2 $V_3 := V_3 \cup \{\ell_3\}$. *(Insert ℓ_3 down.)*

Next, vertices $x \in V(D)$ are forced to be inserted up or down, according to whether $(\ell_0, x) \in E(D)$ or $(x, \ell_3) \in E(D)$.

3.18.3 $V_0 := V_0 \cup \{x \in \mathcal{B} \cup \mathcal{L} \mid (\ell_0, x) \in E(D)\}$. *(Such x must be inserted up, in V_0.)*

3.18.4 $V_1 := V_1 \cup \{x \in \mathcal{W} \cup \mathcal{R} \mid (\ell_0, x) \in E(D)\}$. *(Such x must be inserted up, in V_1.)*

3.18.5 $V_2 := V_2 \cup \{x \in \mathcal{B} \cup \mathcal{R} \mid (x, \ell_3) \in E(D)\}$. *(Such x must be inserted down, in V_2.)*

3.18.6 $V_3 := V_3 \cup \{x \in \mathcal{W} \cup \mathcal{L} \mid (x, \ell_3) \in E(D)\}$. *(Such x must be inserted down, in V_3.)*

The extended sets V_i are still pairwise disjoint, because for any x, transitivity of D together with $(\ell_0, \ell_3) \notin E(D)$ means that (ℓ_0, x) and (x, ℓ_3) are not both in $E(D)$, so no $x \in V(D)$ is inserted both up and down.

Note that there are vertices $r_1, r_2 \in \mathcal{R}$ for which $\ell_0 r_1, \ell_3 r_2 \in E(\widetilde{G})$. Vertex r_1 is in V_1 (Step 3.18.4), and r_2 is in V_2 (Step 3.18.5).

Observe that every vertex of $\mathcal{B} \cup \mathcal{W}$ has now been inserted into some V_i, and is connected to v_i by a path in V_i: Suppose $b \in \mathcal{B}$. Now, one of $\ell_0 b$ or br_2 is in $E(G)$, for if neither is in $E(G)$, then (by Remark 1) $\ell_3 b, br_1 \in E(G)$, and by steps 3.13, 3.15, 3.12, D has a directed path $(\ell_3, b)(b, r_1)(r_1 \ell_0)$, and hence an arc (ℓ_3, ℓ_0), contrary to assumption. On one hand, if $\ell_0 b \in E(G)$, then $(\ell_0, b) \in E(D)$ by Step 3.13, and $b \in V_0$ (by Step 3.18.3), and b is connected to v_0 by the path $b\ell_0 v_0$. On the other hand, if $br_2 \in E(G)$, then by steps 3.15 and 3.12, D has directed path $(b, r_2)(r_2, \ell_3)$, so $(b, \ell_3) \in E(D)$, meaning $b \in V_2$ (Step 3.18.5) and is connected to v_2 by the path $br_2 v_2$. Symmetrically, if $w \in \mathcal{W}$, then w has been inserted in V_1 (or V_3) and is connected to v_1 (or v_3) by the path $wr_1 v_1$ in V_1 (or $wl_3 v_3$ in V_3).

Observe also that steps 3.18.1–3.18.6 have preserved the property $E_G(V_0, V_2) = \emptyset = E_G(V_1, V_3)$. For consider $x, y \in V(D)$, where x and y were inserted into V_0 and V_2, respectively, in lines 3.18.3 and 3.18.5. This means $(\ell_0, x), (y, \ell_3) \in E(D)$. Now, $x \in \mathcal{L} \cup \mathcal{B}$ and $y \in \mathcal{R} \cup \mathcal{B}$, and (by steps 3.10, 3.12, 3.13, 3.15), if $xy \in E(G)$ then $(x, y) \in E(D)$. But $xy \notin E(G)$, for otherwise D has the directed path $(\ell_0, x)(x, y)(y, \ell_3)$ contradicting $(\ell_0, \ell_3) \notin E(D)$. Thus $E_G(V_0, V_2) = \emptyset$. On the other hand, if x and y were inserted into V_1 and V_3, respectively, then again $(\ell_0, x), (y, \ell_3) \in E(D)$, but this time $x \in \mathcal{R} \cup \mathcal{W}$ and $y \in \mathcal{L} \cup \mathcal{W}$. Still, by steps 3.11, 3.12, 3.14, 3.16, if $xy \in E(G)$ then $(x, y) \in E(D)$, and we conclude $xy \notin E(G)$, for otherwise there is a directed path $(\ell_0, x)(x, y)(y, \ell_3)$, contrary to assumption.

There may still be vertices of $\mathcal{L} \cup \mathcal{R}$ that have not yet been inserted in any V_i. Let $\ell \in \mathcal{L}$ be such a vertex. Now, ℓ is not adjacent to any vertex in $x \in V_2$, for if it were, x would be a vertex of D that was inserted in V_2 (down), meaning $x \in \mathcal{B} \cup \mathcal{R}$ and $(x, \ell_3) \in E(D)$. But $x \in \mathcal{B} \cup \mathcal{R}$ means $(\ell, x) \in E(D)$, by steps 3.12 and 3.13, and D has a directed path $(\ell, x)(x, \ell_3)$, hence an arc (ℓ, ℓ_3), contrary to the assumption that ℓ has not yet been inserted. Since ℓ is not adjacent to V_2, it may be inserted in V_0 without creating a diagonal. Symmetrically, if $r \in \mathcal{R}$ has not yet been inserted in any V_i, then r is not adjacent to V_3, so r can be inserted into V_1.

3.18.7 $V_0 := V_0 \cup (\mathcal{L} - (V_0 \cup V_1 \cup V_2 \cup V_3))$.

3.18.8 $V_1 := V_1 \cup (\mathcal{R} - (V_0 \cup V_1 \cup V_2 \cup V_3))$.

3.18.9 **Return**(V_0, V_1, V_2, V_3) and **stop**. *(This is a C_4 contraction of G.)*

If a pair ℓ_0, ℓ_3 was not found in Step 3.18, then each pair of vertices in $\mathcal{L}$ is joined at least one arc of D. This property now allows for an efficient examination of all 2-colorings of G (with $V_0 \cup V_2$ white and $V_1 \cup V_3$ black) that could possibly produce a C_4 contraction.

3.19 Let L be the sub-digraph of D induced on the set $\mathcal{L}$. *By Step 3.18, each pair of vertices in L are joined by at least one arc. Forming L is $\mathcal{O}(n^2)$ because each of its $\mathcal{O}(n)$ vertices is incident with potentially $\mathcal{O}(n)$ arcs.*

3.20 Let $L_0,\ L_1,\ L_2,\ \cdots, L_m$ be the vertex sets of the strongly connected components of L.
This may be accomplished as follows. Select a vertex $x_0 \in V(L)$, and put $L_0 = \{x_0\} \cup \{x \in V(G) \mid (x_0, x), (x, x_0) \in E(L)\}$; then select $x_1 \in V(L) - L_0$, and put $L_1 = \{x_1\} \cup \{x \in V(G) \mid (x_1, x), (x, x_1) \in E(L)\}$, etc. The complexity of this step is $\mathcal{O}(n^2)$.

The significance of the strong components is that if one vertex of an L_i is inserted up (down) then every vertex of L_i must be inserted up (down).

3.21 Form a digraph $\widetilde{L}$ with $V(\widetilde{L}) = \{L_i \mid 0 \le i \le m\}$, and $(L_i, L_j) \in E(\widetilde{L})$ if and only if $(x_i, x_j) \in E(L)$ for some $x_i \in L_i$ and $x_j \in L_j$. *This step is $\mathcal{O}(n^2)$.*
The significance of $\widetilde{L}$ is that if all the vertices of L_i are inserted up, and $(L_i, L_j) \in E(\widetilde{L})$, then all the vertices of L_j must be inserted up also. Further, notice that $\widetilde{L}$ is the (unique) transitive tournament on $m+1$ vertices.

3.22 Reindex the vertices of $\widetilde{L}$ so the in-degree of each L_i is i. *This is $\mathcal{O}(n)$.*

Now it is the case that if the vertices in some L_k are inserted up, then the vertices in L_i are inserted up for each $k \le i \le m$. Thus if each L_i has been inserted up or down to produce a C_4 contraction of G, then it must be the case that there is some $0 \le k \le m$ for which $L_0, L_1, \cdots L_{k-1}$ are inserted down, and $L_k, L_{k+1}, \cdots L_m$ are inserted up. The following step examines each of the $m+2$ such up/down configurations. Each such configuration forces a coloring of all vertices of G, and the algorithm checks each coloring for an induced C_4 contraction.

3.23 For each $k \in \{0, 1, 2, \cdots, m+1\}$, do the following:

3.23.1 Set $\mathcal{L}_B := \bigcup_{k \le i \le m} L_i$. *These are the vertices of $\mathcal{L}$ that are colored black.*

3.23.2 Set $\mathcal{L}_W := \bigcup_{0 \le i < k} L_i$. *These are the vertices of $\mathcal{L}$ that are colored white.*
As noted previously, for any $r \in \mathcal{R}$ there is an $\ell \in \mathcal{L}$ with $\ell r \in E(\widetilde{G})$, and hence $(\ell, r) \in E(D)$, by construction of D. If $\ell \in \mathcal{L}_B$, then ℓ is inserted up, so r must be inserted up too, in V_1, so r is white; Otherwise, r must be adjacent to $\mathcal{L}_W$, so r must be inserted in V_2 and

colored black. Thus the following sets consist of elements of $\mathcal{R}$ that must be colored white or black, respectively.

3.23.3 Set $\mathcal{R}_W := \{r \in \mathcal{R} \mid N(r) \cap \mathcal{L}_B \neq \emptyset\}$.

3.23.4 Set $\mathcal{R}_B := \mathcal{R} - \mathcal{R}_W$.

Now every vertex of G has been assigned a color, and the next steps check if this 2-coloring induces a contraction.

3.23.5 Let $C_0, C_2, \cdots, C_{2b}$ be the components of the subgraph of G induced on the black vertices $V_0 \cup V_2 \cup \mathcal{B} \cup \mathcal{L}_B \cup \mathcal{R}_B$. *This is* $\mathcal{O}(n^2)$ *by Algorithm 8.3 of [8].*

3.23.6 Let $C_1, C_3, \cdots, C_{2w+1}$ be the components of the subgraph of G induced on the white vertices $V_1 \cup V_3 \cup \mathcal{W} \cup \mathcal{L}_W \cup \mathcal{R}_W$. *This is* $\mathcal{O}(n^2)$ *by Algorithm 8.3 of [8].*

3.23.7 If $b = w = 1$, and $E_G(C_i, C_j) \neq \emptyset$ whenever i and j have opposite parity,
Then **return**$(V(C_0), V(C_1), V(C_2), V(C_3))$, and **stop**. *(This is a* C_4 *contraction.)*

4. If this step is reached, all possibilities for a C_4 contraction have been examined, but no such contraction was found. Thus none exists. **Return**$(\emptyset)$, and **stop**. ■

Proposition 2: Algorithm 2 has complexity $O(n^6)$, where G has n vertices.

Proof. Step 1 is $\mathcal{O}(n^3)$, as noted in the proof of Proposition 1. Step 2 is trivial. Step 3 is the main loop, and it examines $\mathcal{O}(n^2)$ edges $v_2v_3 \in E(G)$. The complexities of each line in the body of Loop 3 are noted in the algorithm, and none of the lines 3.1–3.23 has a complexity greater than $\mathcal{O}(n^4)$. Thus, the total complexity of Step 3 is $\mathcal{O}(n^6)$. Step 4 is trivial. Thus the net complexity of the algorithm is $O(n^6)$. ■

The following theorem is the main complexity result of this paper.

Theorem 1: If G is a graph for which $\text{rad}(\overline{G}) \neq 2$, then it is decidable in polynomial time whether or not G can be contracted to a square.

Proof. Computing $\text{rad}(\overline{G})$ is $\mathcal{O}(n^2)$, as noted in the proof of Proposition 1. If $\text{rad}(\overline{G}) < 2$, then Lemma 1 says G can't be contracted to C_4. If $\text{rad}(\overline{G}) > 2$, then Algorithm 2 decides in $\mathcal{O}(n^6)$ time if G can be contracted to C_4. ■

5 Applications to Cyclicity

We now apply the results of the previous sections to the problem of computing the cyclicity of a graph. The *cyclicity*, $\eta(G)$, of a connected graph G is the largest integer k for which G is contractible to C_k. This graph invariant was introduced in [5] as an aid in the study of a related invariant called *circularity* (see [1, 2, 7]). In [6], formulas are given for cyclicity in several classes of graphs, and a polynomial algorithm for computing cyclicity of planar graphs is described.

A graph has cyclicity k if and only if it can be contracted to C_k but *cannot* be contracted to C_{k+1}. Thus, since the problem of contracting a graph to C_4 is NP-complete, it is NP-complete to determine if $\eta(G) = k$ for $k \geq 3$. However, the results of this paper allow for the tractable computation of the cyclicity of any graph whose complement radius is not 2.

Proposition 3: If G is connected and has a cycle and $3 < \text{rad}(\overline{G}) < \infty$, then $\eta(G) = 3$.

Proof. Note that any connected graph containing a cycle can be contracted to C_3. Thus $\eta(G) \geq 3$. Lemma 1 implies G can't be contracted to a square, so $\eta(G) \leq 3$. ■

Proposition 4: Suppose G is connected and has a cycle and $\text{rad}(\overline{G}) \neq 2$. Then $3 \leq \eta(G) \leq 4$. Moreover, it is decidable in polynomial time whether $\eta(G) = 4$ or $\eta(G) = 3$.

Proof. By Lemma 2, G can't be contracted to C_5, so $\eta(G) \leq 4$. Also, G has a cycle, so $3 \leq \eta(G)$. By Theorem 1, it is decidable in polynomial time whether G can be contracted to C_4, that is whether $\eta(G) = 3$ or $\eta(G) = 4$. ■

References

[1] H. Bell, E. Brown, R.F. Dickman and E.L. Green, Circularity of graphs and continua: combinatorics, *Houston J. Math.* **6** (1980), 455–469.

[2] H. Bell, E. Brown, R.F. Dickman and E.L. Green, Circularity of graphs and continua: topology, *Fund. Math.* **122** (1981), 103–110.

[3] G. Bloom, J.W. Kennedy and L.V. Quintas, A characterization of graphs of diameter two, *American Math. Monthly* (1988), 37–38.

[4] A.E. Brouwer and H.J. Veldman, Contractability and NP-completeness, *J. Graph Theory* **11** (1987), 71–79.

[5] D. Blum, *Circularity of Graphs*, PhD Thesis, Virginia Polytechnic Institute and State University, 1982.

[6] R. Hammack, Cyclicity of graphs, *J. Graph Theory* **32**(2) (1999), 160–170.

[7] R. Hammack, Circularity of planar graphs, *Houston J. Math.* **25**(2) (1999), 213–221.

[8] E. Reingold, J. Nievergelt and N. Deo, *Combinatorial Algorithms,* Prentice-Hall, Englewood Cliffs, NJ, 1977.

(Received 1 July 2004; revised 22 Dec 2005)