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On 2-edge-robust r-identifying codes in the king grid

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Abstract

Assume that $G = (V, E)$ is a simple undirected graph, and C is a nonempty subset of V. For every $v \in V$, we denote $I_r(v) = \{u \in C \mid d_G(u, v) \leq r\}$, where $d_G(u, v)$ denotes the number of edges on any shortest path between u and v. If all the sets $I_r(v)$ for $v \in V$ are pairwise different, and none of them is the empty set, we say that C is an r-identifying code in G. If C is r-identifying in every graph G' that can be obtained by adding and deleting edges in such a way that the number of additions and deletions together is at most t, the code C is called t-edge-robust. Let K be the graph with vertex set $\mathbb{Z}^2$ in which two different vertices are adjacent if their Euclidean distance is at most $\sqrt{2}$. We study bounds on the possible densities of 2-edge-robust r-identifying codes in K.

1 Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set E. The distance between two vertices u and v of G is defined to be the number of edges on any shortest path from u to v, and is denoted by $d_G(u, v)$. Denote

$$B_r(v) = \{u \in V \mid d_G(u, v) \leq r\}.$$

If C is a code in G, i.e., a nonempty subset of V, we denote

$$I_r(v) = I_r(G, v) = C \cap B_r(v)$$

for all $v \in V$, and say that the code C is r-**identifying**, if the sets $I_r(v)$ for $v \in V$ are pairwise different, and none of them is the empty set.

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Identifying codes were introduced by Karpovsky, Chakrabarty and Levitin in [8]. They can be used in maintaining multiprocessor architectures in the following way. Assume that every vertex of G corresponds to a processor and every edge corresponds to a dedicated link between two processors. We ask some of the processors to check their r-neighbourhoods and report YES or NO according to whether they detected any problems or not. Assume that at most one of the processors is malfunctioning (it is easy to modify the definition for more general situations). Based on these YES/NO answers we wish to be able to tell which processor is malfunctioning or that all the processors are fine. Clearly, the requirement is that the vertices that correspond to the processors that were asked to perform the test form an r-identifying code.

Identifying codes have been studied in many graphs, see, e.g., [1], [2] and [3] for infinite grids and meshes, and, e.g., [7] for binary hypercubes.

In this paper we study the **(infinite) king grid** K, which has vertex set $\mathbb{Z}^2$ and in which two different vertices are adjacent if their Euclidean distance is at most $\sqrt{2}$. We denote by Q_n the set of vertices $(x, y) \in V$ with $|x| \leq n$ and $|y| \leq n$. The **density** of a code C is defined as

$$D = D(C) = \limsup_{n\to\infty} \frac{|C \cap Q_n|}{|Q_n|}.$$

It has been proved in [3] that for all $r > 1$ the smallest possible density of an r-identifying code in K is $\frac{1}{4r}$.

It is natural to study r-identifying codes that are strong enough so that they can be used for identification even in the presence of some errors in the test results. Such robust identifying codes have been studied, for instance, in [10], [6], [4] and [9]. The following definition is from [6].

Definition 1 *An r-identifying code $C \subseteq V$ is called* **t-edge-robust** *if C is r-identifying in every graph $G' = (V, E')$, where $E' = E \triangle F$, where $F \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ has size at most t. Here $E \triangle F$ denotes the symmetric difference $(E \setminus F) \cup (F \setminus E)$.*

In other words, whenever G' can be obtained from G by adding and deleting edges in such a way that the total number of additions and deletions together is at most t, then the code C should still be r-identifying in G'.

In what follows we always assume that $r \geq 1$.

The exact smallest possible density of a t-edge-robust r-identifying code in K has been determined in [9] and [5], except when $t = 2$ and $r > 1$. In particular, the smallest possible density of a 2-edge-robust 1-identifying code in K is $1/2$. In this paper we study this remaining case. In particular, we construct a sequence of 2-edge-robust r-identifying codes (C_r) with densities D_r such that $D_r \to \frac{3}{8}$ when $r \to \infty$. We also show that the density of every 2-edge-robust r-identifying code in K is at least $33/128$.

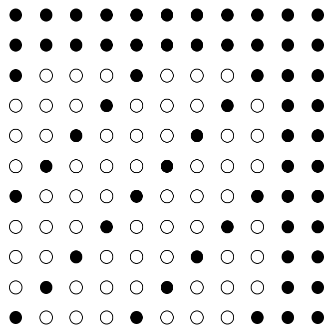

Figure 1: The tile used in Example 1 for $r = 6$.

2 Lower bounds

Assume that $r \geq 1$ and that C is a 2-edge-robust r-identifying code in K. Consider the following nine sets:

$$A_1(i,j) = \{(i,j),(i+1,j),(i,j+2r+1),(i+1,j+2r+1)\},$$

$$A_2(i,j) = \{(i,j),(i+1,j),(i,j+2r+1),(i+2r,j+2r+1)\},$$

$$A_3(i,j) = \{(i,j),(i+1,j),(i+2r-1,j+2r+1),(i+2r,j+2r+1)\},$$

$$A_4(i,j) = \{(i,j),(i+2r,j),(i,j+2r+1),(i+1,j+2r+1)\},$$

$$A_5(i,j) = \{(i,j),(i+2r,j),(i,j+2r+1),(i+2r,j+2r+1)\},$$

$$A_6(i,j) = \{(i,j),(i+2r,j),(i+2r-1,j+2r+1),(i+2r,j+2r+1)\},$$

$$A_7(i,j) = \{(i+2r-1,j),(i+2r,j),(i,j+2r+1),(i+1,j+2r+1)\},$$

$$A_8(i,j) = \{(i+2r-1,j),(i+2r,j),(i,j+2r+1),(i+2r,j+2r+1)\},$$

$$A_9(i,j) = \{(i+2r-1,j),(i+2r,j),(i+2r-1,j+2r+1),(i+2r,j+2r+1)\}.$$

Because C is a 2-edge-robust r-identifying code, each of them must contain at least one codeword. Indeed, they are the sets $B_r((i+r,j+r)) \triangle B_r((i+r,j+r+1))$ in the graphs obtained from K by adding an edge from $(i+r,j+r+1)$ to $(i+r+1,j+r-1)$, $(i+r,j+r-1)$ or $(i+r-1,j+r-1)$ and an edge from $(i+r,j+r)$ to $(i+r+1,j+r+2)$, $(i+r,j+r+2)$ or $(i+r-1,j+r+2)$.

The same is true for the nine sets $A_k^{\perp}(i,j)$ $(k = 1,2,\dots,9)$ obtained from the sets $A_k(i,j)$ by reflecting them in the line $y-j = x-i$ (the line with slope one that goes through (i,j)).

Theorem 1 *The density of a 2-edge-robust r-identifying code in the king grid is at least $(r+1)/(4r+2)$.*

Proof. Each of the $2r+1$ sets $A_1(i,j)$, $A_1(i+1,j)$, $\ldots$, $A_1(i+2r-1,j)$, and $A_5(i,j)$ contains at least one codeword of C, and each element in their union is contained in exactly two of the sets. Hence at least $r+1$ of the $4r+2$ elements in the union belong to the code. □

Example 1 If the only requirement for C were that each of the sets in the collections $\mathcal{A}(i,j)$ and $\mathcal{A}^{\perp}(i,j)$ must have at least one codeword, we could use a doubly periodic tiling with the tile in Figure 1 (illustrated in the case $r=6$) and periods $(0,2r-1)$ and $(2r-1,0)$. Because of the periodicity, in this case all the nine A-sets reduce to the leftmost pattern in Figure 2, and all the nine $A^{\perp}$-sets reduce to the second pattern in Figure 2 — and obviously such patterns of non-codewords do not occur in our code. When $r\to\infty$, the density of this code tends to $1/4$. As we shall see, the density of every 2-edge-robust r-identifying code in K is at least $33/128$, so it is not sufficient to operate only with the sets $\mathcal{A}(i,j)$ and $\mathcal{A}^{\perp}(i,j)$.

Lemma 1 *i) If one of the points (i,j) and $(i+2r-1,j)$ is in C and the other one is not, then at least one of the sets $A_5(i,j)$ and $A_7(i,j)$ contains at least two codewords of C.*

ii) If one of the points (i,j) and $(i,j+2r-1)$ is in C and the other is not, then at least one of the sets $A_5^{\perp}(i,j)$ and $A_7^{\perp}(i,j)$ contains at least two codewords of C.

Proof. i) Assume first that $(i,j)\in C$ and $(i+2r-1,j)\notin C$. If $(i+2r,j)\in C$, then $A_5(i,j)$ has at least two codewords of C; so assume that $(i+2r,j)\notin C$. Because $A_8(i,j)$ must contain at least one codeword of C, we see that $(i,j+2r+1)\in C$ or $(i+2r,j+2r+1)\in C$, and this codeword together with (i,j) shows that there are at least two codewords of C in $A_5(i,j)$.

Assume second that $(i,j)\notin C$ and $(i+2r-1,j)\in C$. If $(i+2r,j)\in C$, then $A_7(i,j)$ contains at least two codewords of C; so assume that $(i+2r,j)\notin C$. Because $A_4(i,j)$ must contain at least one codeword of C, we see that $(i,j+2r+1)\in C$ or $(i+1,j+2r+1)\in C$, and this codeword together with $(i+2r-1,j)$ shows that there are at least two codewords of C in $A_7(i,j)$.

ii) This is proved in the same way. □

We denote

$$\mathcal{A}(i,j)=\{A_5(i,j),A_7(i,j),A_5^{\perp}(i,j),A_7^{\perp}(i,j)\}.$$

Lemma 2 *Assume that we have a code in the king grid. If the non-codewords in the pattern*

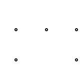

Figure 2: Four patterns of non-codewords. An open circle always denotes a non-codeword and a small dot a point which can be a codeword or not.

do not form any of the subpatterns in Figure 2 then the pattern must contain at least two codewords; and if there are only two, they must be the middle point of one of the two horizontal sides together with the middle point of one of the two vertical sides. □

Proof. If two diagonally opposite corners are codewords, there has to be at least one more codeword: otherwise the third or the fourth pattern in Figure 2 would occur. If two corners on one side are codewords, there has to be at least one more codeword, because otherwise the first or the second forbidden pattern in Figure 2 would occur. If exactly one corner is a codeword, then the first or second forbidden pattern occurs, unless we have two more codewords. Finally, if none of the corners is is codeword, then at least one of the middle points in the horizontal lines must be a codeword: otherwise, the first forbidden pattern appears. In the same way at least one of the middle points of the two vertical lines must be a codeword. □

Denote

$$S(i,j) = \{(i,j),(i,j+1),(i,j+2),(i+1,j+2),$$
$$(i+2,j+2),(i+2,j+1),(i+2,j),(i+1,j)\},$$

i.e., $S(i,j)$ is the pattern of Lemma 2 whose bottom-left corner is at (i,j), and

$$H(i,j) = S(i,j) \cup S(i,j+2r+1) \cup S(i+2r+1,j+2r+1) \cup S(i+2r+1,j). \quad (1)$$

We say that a pair $\{p, p+(0,2r-1)\}$ (where $p \in \mathbb{Z}^2$) forms an **upward mismatch** (resp. **downward mismatch**) if p is a not a codeword but $p+(0,2r-1)$ is (resp. if p is a codeword but $p+(0,2r-1)$ is not). Analogously, a pair $\{p, p+(2r-1,0)\}$ is a **leftward mismatch** (resp. **rightward mismatch**) if p is a codeword, but $p+(2r-1,0)$ is not (resp. if p is not a codeword, but $p+(2r-1,0)$ is). In all cases, the direction of the mismatch points to the direction of the codeword in the pair.

Consider the set $S(i,j)$. By Lemma 2, if the non-codewords in it do not exhibit any of the forbidden patterns of Figure 2, then $S(i,j)$ contains at least two codewords. Therefore, if $S(i,j)$ does not contain at least two codewords, at least one of the four forbidden patterns in Figure 2 occurs: assume that, for example, the first pattern

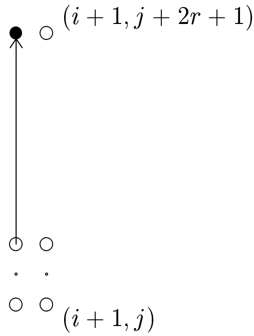

Figure 3: An example of finding an upward mismatch.

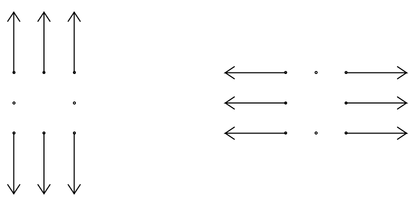

Figure 4: Possible locations for a mismatch for the first and second forbidden patterns.

occurs in the first two columns of $S(i, j)$. Now we can find an upward mismatch by considering the four points (i, j), $(i+1, j)$, $(i, j+2r+1)$ and $(i+1, j+2r+1)$ (cf. Figure 3). These four points form the set $A_1(i, j)$, which contains at least one codeword of C. Because (i, j) and $(i+1, j)$ are non-codewords, at least one of the points $(i, j+2r+1)$ and $(i+1, j+2r+1)$ must be in C, say (as in Figure 3) that the point $(i, j+2r+1)$ is in C. Then $\{(i, j+2), (i, j+2r+1)\}$ forms an upward mismatch. In the same way we see that $\{(i, j), (i, j-2r+1)\}$ or $\{(i+1, j), (i+1, j-2r+1)\}$ is a downward mismatch (or both are).

In the same way, if the second forbidden pattern in Figure 2 occurs, we find a leftward mismatch and a rightward mismatch (or more than one of each).

All in all, if the first forbidden pattern occurs, then at least one of the three pairs corresponding to the three upward arrays in Figure 4 gives an upward mismatch (each arrow has length $2r-1$), and at least one of the pairs corresponding to the three downward arrows gives a downward mismatch. In the same way, if the second

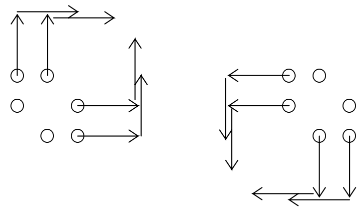

Figure 5: Possible locations for a mismatch for the third forbidden pattern.

forbidden pattern occurs, at least one of the rightward arrows and at least one of the leftward arrows in Figure 4 gives a mismatch.

Consider the third forbidden pattern in Figure 2, say, $\{(i, j+1), (i, j+2), (i+1, j), (i+1, j+2), (i+2, j), (i+2, j+1)\}$, and assume that none of these points is in C. We can find at least one upward or rightward mismatch, as illustrated in Figure 5, where each arrow has again length $2r-1$. Indeed, because at least one of the points in the set

$$\{(i, j+1), (i, j+2), (i+1, j), (i+2, j), (i+2r-1, j+2r+1),$$
$$(i+2r, j+2r+1), (i+2r+1, j+2r), (i+2r+1, j+2r-1)\}$$

is in C (as this is the symmetric difference between $B((i+r, j+r+1))$ and $B((i+r+1, j+r))$ in the graph where the edge from $(i+r, j+r+1)$ to $(i+r+2, j+r-1)$ and the edge from $(i+r+1, j+r)$ to $(i+r-1, j+r+2)$ have been added). The first four points are not in C, so one of the points $(i+2r-1, j+2r+1)$, $(i+2r, j+2r+1)$, $(i+2r+1, j+2r)$, $(i+2r+1, j+2r-1)$ is in C. Consider each in turn. If $(i+2r-1, j+2r+1)$ is in the code, then depending on whether the "intermediate" point $(i, j+2r+1)$ is in C or not, we find either an upward mismatch $\{(i, j+2), (i, j+2r+1)\}$ or a rightward mismatch $\{(i, j+2r+1), (i+2r-1, j+2r+1)\}$ (both cannot be mismatches): these correspond to two arrows in Figure 5. If $(i+2r, j+2r+1)$ is in C, then either $\{(i+1, j+2), (i+1, j+2r+1)\}$ is an upward mismatch or $\{(i+1, j+2r+1), (i+2r, j+2r+1)\}$ is a rightward mismatch. If $(i+2r+1, j+2r)$ is in C, then either $\{(i+2, j+1), (i+2r+1, j+1)\}$ is a rightward mismatch or $\{(i+2r+1, j+1), (i+2r+1, j+2r)\}$ is an upward mismatch. Finally, if $(i+2r+1, j+2r-1)$ is in C, then either $\{(i+2, j), (i+2r+1, j)\}$ is a rightward mismatch or $\{(i+2r+1, j), (i+2r+1, j+2r-1)\}$ is an upward mismatch. All in all, at least one of the eight pairs corresponding to the eight arrows in the left-hand side figure of Figure 5 is a mismatch.

If we decide to extend the figure down and left (instead of up and right as above), then we similarly find that at least one of the arrows in the right-hand side figure in Figure 5 gives us a downward mismatch or a leftward mismatch.

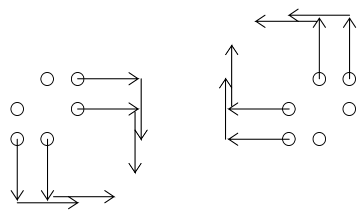

Figure 6: Possible locations for a mismatch for the fourth forbidden pattern.

In the same way, with the fourth pattern we look at the eight arrows in the left-hand side figure in Figure 6, and see that at least one of them gives a mismatch, and we look at the eight arrows in the right-hand side figure in Figure 6, and see that at least one of them gives a mismatch.

Given (i,j), we look at all the possible locations in Figures 4–6, and any mismatch found in those locations is called a **mismatch associated with the set** $S(i,j)$. Altogether, there are seven possible locations for an upward mismatch that we check, seven for a possible downward mismatch, seven for a possible rightward mismatch, and seven for a possible leftward mismatch. The number of mismatches among these 28 possible locations (each with a specific direction) is denoted by $f(i,j)$. In connection with $S(i,j)$ we are not interested in any other locations for possible mismatches: in particular, if a particular location marked with a right arrow, say, contains a left mismatch, then it is not counted.

All in all, if we go through all the sets $S(i,j)$, $(i,j)\in\mathbb{Z}^2$, each mismatch will be encountered exactly seven times.

The set $H(i,j)$ is a disjoint union of eight A-sets, and hence always contains at least eight codewords of C. We denote by $s(i,j)$ the number of codewords of C in $S(i,j)$, and

$$\begin{aligned} e(i,j) &= s(i,j)+s(i,j+2r+1)+s(i+2r+1,j+2r+1) \\ &\quad +s(i+2r+1,j)-8 \\ &= |H(i,j)\cap C|-8, \end{aligned}$$

and

$$h(i,j)=f(i,j)+f(i,j+2r+1)+f(i+2r+1,j+2r+1)+f(i+2r+1,j).$$

Lemma 3 *Let i and j be arbitrary.*

i) If $s(i,j)=0$, then $f(i,j)\geq 4$ and $f(i-1,j)\geq 2$ and $f(i+1,j)\geq 2$.

ii) Assume that $s(i,j)=1$. Then $f(i,j)\geq 2$. Moreover, $f(i,j)+f(i-1,j)\geq 4$ except possibly when the unique codeword in $S(i,j)$ is the middle point on either of *the two horizontal sides and* $s(i-1,j) \geq 3$. *Similarly,* $f(i,j)+f(i+1,j) \geq 4$ *except possibly when the unique codeword in* $S(i,j)$ *is the middle point on either of the two horizontal sides and* $s(i+1,j) \geq 3$.

iii) If $s(i,j) = s(i+1,j) = 1$, *then* $f(i,j) + f(i+1,j) \geq 6$.

iv) If $s(i,j) + s(i+1,j) = 4$, *then* $f(i,j) + f(i+1,j) \geq 2$.

v) If $s(i,j) + s(i+1,j) = 3$, *then* $f(i,j) + f(i+1,j) \geq 4$.

vi) If $s(i,j) + s(i+1,j) = 2$, *then* $f(i,j) + f(i+1,j) \geq 6$.

vii) If $s(i,j) + s(i+1,j) \leq 1$, *then* $f(i,j) + f(i+1,j) \geq 8$.

Proof. i) and ii) are easy.

iii) Assume that the unique codeword in $S(i,j)$ is the middle point on the left vertical side. By ii), $f(i,j) \geq 2$ and $f(i+1,j) \geq 2$. If the only codeword in $S(i+1,j)$ is the upper or lower right corner, the claim is clear. Assume that the only codeword in $S(i+1,j)$ is one of the middle points of the two vertical sides. We show that the upward mismatches associated with $S(i,j)$ and $S(i+1,j)$ contribute at least three to the sum $f(i,j) + f(i+1,j)$. The same is true for the downward mismatches and the claim then follows. If $\{(i+1,j+2),(i+1,j+2r+1)\}$ is an upward mismatch, it contributes to both $f(i,j)$ and $f(i+1,j)$ and at least one of $\{(i+2,j+2),(i+2,j+2r+1)\}$ and $\{(i+3,j+2),(i+3,j+2r+1)\}$ is an upward mismatch, and we are done. If $\{(i+1,j+2),(i+1,j+2r+1)\}$ is not an upward mismatch, then both $\{(i,j+2),(i,j+2r+1)\}$ and $\{(i+2,j+2),(i+2,j+2r+1)\}$ are, and as the latter contributes to both $f(i,j)$ and $f(i+1,j)$, we are again done. The case when the unique codeword is the middle point on the right vertical side goes through in the same way, and all the other cases are easy.

iv) If $s(i,j) = s(i,j+1) = 2$, this is clear by Lemma 2, because then $S(i,j)$ or $S(i+1,j)$ exhibits a forbidden pattern, and $f(i,j) \geq 2$ or $f(i+1,j) \geq 2$. If $s(i,j) \leq 1$ or $s(i+1,j) \leq 1$, then the claim follows from i) and ii).

v) Now $s(i,j)$ or $s(i+1,j)$ is 1 and the other is 2, and the claim follows from ii), or either of them is 0 and the other is 3, and the claim follows from i).

vi) Now $s(i,j)$ or $s(i+1,j)$ is 0 and the other is 2, and the claim follows from i), or both are 1's, and the claim follows from iii).

vii) If $s(i,j) = s(i+1,j) = 0$, the claim immediately follows from i). If one of $s(i,j)$ and $s(i+1,j)$ is 1 and the other is 0, the claim can easily be proved in the same way as iii). □

Lemma 4 *For all* $(i,j) \in \mathbb{Z}^2$,

$$e(i,j) + e(i+1,j) + \frac{1}{2}h(i,j) + \frac{1}{2}h(i+1,j) \geq 4.$$

Proof. We know that e and h are non-negative functions, so it suffices to consider the cases when $e(i,j) + e(i+1,j)$ is 3, 2, 1 and 0.

Consider the four sums

$$s(i,j) + s(i+1,j), s(i,j+2r+1) + s(i+1,j+2r+1),$$

$$s(i+2r+1,j+2r+1)+s(i+2r+2,j+2r+1), s(i+2r+1,j)+s(i+2r+2,j). \quad (2)$$

Assume that $e(i,j)+e(i+1,j)=3$. Because the sum of the four sums (2) is 19, at least one of them is at most 4, and the claim $h(i,j)+h(i+1,j)\geq 2$ follows from iv)–vii) of Lemma 3.

Assume that $e(i,j)+e(i+1,j)=2$. Now the sum of the sums (2) is 18. If one of the sums (2) is 3 or smaller, the claim follows from v)–vii) in Lemma 3; so assume that all the sums are 4 or bigger. Then there are at least two 4's, and the claim follows from iv) in Lemma 3.

Assume that $e(i,j)+e(i+1,j)=1$. We claim that $h(i,j)+h(i+1,j)\geq 6$. If one of the sums (2) is 2 or smaller, then the claim follows from vi) and vii) in Lemma 3; so assume that all the sums are 3 or bigger. If there is at least one 3, then (as the sum total is 17) there is another sum that is at most 4, and the claim follows from iv) and v) in Lemma 3; so assume that all the sums (2) are 4 or bigger. Then there are three 4's, and the claim follows from iv) in Lemma 3.

Assume finally that $e(i,j)+e(i+1,j)=0$. Then the sum of the sums (2) is 16. If one of the sums (2) is 0 or 1, the claim $h(i,j)+h(i+1,j)\geq 8$ follows from vii) in Lemma 3; so assume that all the sums are 2 or bigger. If one of the sums equals 2, then the fact that their sum is 16 implies that there is another sum which is at most 4, and the claim follows from iv)–vii) in Lemma 3; so assume that all the sums are 3 or bigger. If there are at least two 3's, the claim follows from v) in Lemma 3. If there is exactly one 3, then two of the remaining sums (2) are 4's, and the claim follows from iv)–v) in Lemma 3. If all the sums are 4 or bigger, then they are all 4's, and the claim follows from iv) in Lemma 3. □

Theorem 2 *The density of a 2-edge-robust r-identifying code in the king grid is at least* $\frac{33}{128}=0.2578125$.

Proof. Consider a fixed positive integer r. When we go through all the sets $H(i,j)$ for $(i,j)\in Q_{n-4r-2}$, each element of Q_n is counted at most 32 ($=|H(i,j)|$) times. We therefore get

$$\begin{aligned} 32|C\cap Q_n| &\geq \sum_{(i,j)\in Q_{n-4r-2}} |H(i,j)\cap C| \\ &\geq 8|Q_{n-4r-2}| + \sum_{(i,j)\in Q_{n-4r-2}} e(i,j). \end{aligned} \quad (3)$$

Similarly, if we separately count the number of codewords in the four sets in the families $\mathcal{A}(i,j)$ for all $(i,j)\in Q_{n-2r-1}$, each codeword in Q_n is counted at most 16 times. In each such collection of four sets each set is known to contain at least one codeword. There are at least $\frac{1}{7}\sum_{(i,j)\in Q_{n-4r-2}} f(i,j)$ mismatches in Q_{n-2r-1} (as each mismatch occurs at most 7 times) and therefore in the process of going through all the $4|Q_{n-2r-1}|$ sets in the collections $\mathcal{A}(i,j)$, we find that there are at least $\frac{1}{7}\sum_{(i,j)\in Q_{n-4r-2}} f(i,j)$ sets with more than one element by Lemma 1. We therefore get

$$\begin{aligned} 16|C \cap Q_n| &\geq \sum_{(i,j)\in Q_{n-2r-1}} \sum_{A \in \mathcal{A}(i,j)} |A \cap C| \\ &\geq 4|Q_{n-2r-1}| + \frac{1}{7} \sum_{(i,j)\in Q_{n-4r-2}} f(i,j). \end{aligned} \tag{4}$$

Summing the inequality of Lemma 4 over all $(i,j) \in Q_{n-6r-4}$ we get

$$2 \sum_{(i,j)\in Q_{n-4r-2}} e(i,j) + 4 \sum_{(i,j)\in Q_{n-4r-2}} f(i,j) \geq 4|Q_{n-6r-4}| \tag{5}$$

(because each $S(i,j)$ with $(i,j) \in Q_{n-4r-2}$ is counted at most eight times and we have the coefficient $1/2$, and each $H(i,j)$ with $(i,j) \in Q_{n-4r-2}$ is counted at most twice). Multiplying (3) by 2 and (4) by 28, adding them together and using (5) we get

$$\frac{|C \cap Q_n|}{|Q_n|} \geq \frac{|Q_{n-4r-2}|}{32|Q_n|} + \frac{7|Q_{n-2r-1}|}{32|Q_n|} + \frac{|Q_{n-6r-4}|}{128|Q_n|}.$$

Letting n tend to infinity, we obtain $D \geq 1/32 + 7/32 + 1/128$, as claimed. □

3 A construction of 2-edge-robust identifying codes in the king grid

Theorem 3 *Let $r \geq 3$. There is a 2-edge-robust r-identifying code in the king grid with density D_r such that $D_r \to \frac{3}{8}$ when $r \to \infty$.*

Proof. Let $r \geq 3$. To construct such a code we use a $(2r-1) \times (2r-1)$ tile. We construct the tile as follows. We start from the bottom-left and write in the same four by four constellation as in Figure 7, and keep repeating the same pattern (with periods $(0,4)$ and $(4,0)$) as long as possible. Finally, we take all the points in the one or three remaining rows and columns to the code. Figure 7 illustrates the case $r = 6$. This gives us a tile which is symmetric with respect to the diagonal with slope 1.

We place the bottom left-hand corner of the tile to $(0,0)$ and extend it to a doubly periodic tiling of $\mathbb{Z}^2$ with periods $(0, 2r-1)$ and $(2r-1, 0)$.

The resulting code C is symmetric with respect to the line $y = x$.

The density of the code clearly tends to $\frac{3}{8}$ when $r \to \infty$; so it suffices to prove that C is a 2-edge-robust r-identifying code.

Let first $v \in K \setminus C$ be arbitrary. Clearly, there are at least three codewords of C to which v is connected in the king grid with edge-disjoint paths. Hence in all relevant graphs, $I_r(v) \neq \emptyset$.

Assume therefore to the contrary that there is a graph K' that has been obtained from K by using at most two edge changes (each being an addition or deletion), and that there are two points $p = (x_p, y_p)$ and $q = (x_q, y_q)$ such that $I_r(K', p) = I_r(K', q)$.

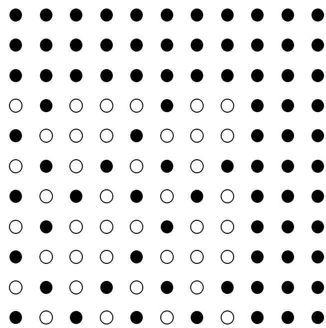

Figure 7: The tile for $r = 6$.

By symmetry, we can assume that $y_p > y_q$ (as we can always reflect in the line $x = y$).

We say that an edge e of K', which is not in the king grid, **helps** q with c if $c \in C$, $d_{K'}(c, p) \geq r - 1$, $d_K(c, q) > r$, and e is the last edge not in K on a shortest path in K' from q to c. In the same way we define what it means that an edge e' **helps** p with c' by reversing the roles of p and q.

We say that a horizontal line is **good**, if at least one of every two adjacent points is a codeword.

Our first claim is that there is a good line $y = s$ such that

$$y_p + r - 2 \leq s \leq y_p + r \text{ and } y_q < s - r, \tag{6}$$

or a good line $y = s$ such that

$$y_q - r \leq s \leq y_q - r + 2 \text{ and } y_p > s + r. \tag{7}$$

If $y_p = y_q + 1$, then $y_p + r \equiv (y_q - r) + 2 \pmod{2r - 1}$, and by the construction, $s = y_p + r$ or $s = y_q - r$ will do. If $y_p = y_q + 2$, then $s = y_p + r$ or $s = y_q - r + 1$ will do. Finally, if $y_p > y_q + 2$, then $s = y_p + r$, $s = y_p + r - 1$ or $s = y_p + r - 2$ will do.

Assume that $y = s$ is a good line such that (6) holds. The other case goes through in the same way.

We prove that at least one of the following is true:

i) There are at least two edges of the king grid in the half-plane $y \geq y_p$ that are missing from K'.

ii) There is at least one such missing edge, and at least one edge that helps q.

iii) There are at least two different edges that help q.

Because $y = s$ is a good line, there is a codeword, say c_1, in the set $\{(x_p - r, s), (x_p - r + 1, s)\}$. Then also $c_2 = c_1 + (2r - 1, 0)$ is in C.

If $d_{K'}(c_1, p) > r$, then at least one edge of K with an endpoint in the area where $x < x_p$ and $y > y_p$ is missing. Assume that this is not the case. If $d_{K'}(c_1, p) \leq r - 2$, then we look at the set $c_1 + N_1$, where

$$N_1 = \{(-2, 2), (-2, 1), (-1, 2), (-1, 1), (0, 2), (0, 1)\}.$$

By the construction, there is always a codeword in this set, and we replace c_1 with that codeword. We continue the same process, until the new codeword c_1 satisfies the condition $r \geq d_{K'}(c_1, p) \geq r - 1$. The process always terminates: otherwise more than two edges would have been added to K to obtain K'. The resulting codeword c_1 then belongs to $I_r(K', p) = I_r(K', q)$, and (using (6)) there is an edge that helps q with c_1.

We do the same for c_2, except that we now use the set

$$N_2 = \{(2, 2), (2, 1), (1, 2), (1, 1), (0, 2), (0, 1)\}$$

instead of N_1. Again, we see that 1) at least one edge of K with an endpoint in the area where $x > x_p$ and $y > y_p$ is missing, or 2) this is not the case, but there is an edge that helps q with c_2.

Clearly, at least one of i)-iii) holds unless there is a unique edge e that helps q with c_1, and a unique edge e' that helps q with c_2, and $e = e'$. We next show that this is impossible. Assume that P (resp. P') is a shortest path from q to c_1 (resp. to c_2) and that in both of them $e = uv$ is the last edge that does not belong to K. Clearly, P and P' cannot traverse e in different directions (say P from u to v, and P' from v to u): otherwise, if we denote the length of the subpath $q \ldots u$ of P by a and the length of the subpath $q \ldots v$ of P' by b, then $d_{K'}(q, u) = a = b + 1$ and $d_{K'}(q, v) = b = a + 1$, which is a contradiction. But P and P' cannot traverse $e = uv$ in the same direction either (say, from u to v), because the subpaths $v \ldots c_1$ and $v \ldots c_2$ of P and P' both have length at most $r - 1$ and all their edges belong to K, but by the definitions of c_1 and c_2 we know that $d_K(c_1, c_2) \geq 2r - 1$.

Because at most two edge changes have been made to obtain K', we know that no edge of K in the half-plane $y \leq y_q$ is missing from K'.

Take c_3 to be a codeword whose x-coordinate is at least $x_q - r$ and at most $x_q + r$ and whose y-coordinate equals $y_q - r$. Using the same replacing process as above (but now going downwards), we find a codeword c_3 which is at distance $r - 1$ or r from q in K', but not within distance r from p in K. Then there is an edge that helps p with c_3.

To obtain a contradiction, it now suffices to prove that an edge helping p with the codeword $c = c_3$ and an edge helping q with the codeword $c' = c_1$ or c_2 can never coincide. Assume that uv would be such an edge.

Let $P : p \ldots uv \ldots c$ be a shortest path in K' from p to c such that uv is the last edge not in K, and $P' : q \ldots uv \ldots c'$ or $P' : q \ldots vu \ldots c'$ be a shortest path in K' from q to c' such that $\{u, v\}$ is the last edge not in K.

We see that it is not possible that P' is of the form $q \dots uv \dots c'$. Assume to the contrary. The subpath $v \dots c$ of P has length at most $r-1$ and the subpath $v \dots c'$ of P' has length at most $r-1$, and all the edges on them belong to K. Hence $d_K(c,c') \le 2r-2$, which is a contradiction, because c' is on or above the horizontal line $y = y_p + r - 2$, whereas c is on or below the horizontal line $y = y_q - r$ (and $y_p > y_q$).

It therefore suffices to assume that P' is of the form $q \dots vu \dots c'$.

Because P is a shortest path from p to c in K' we see that

$$l(p \dots u) + 1 + l(v \dots c) \le r \tag{8}$$

(where $l(x \dots y)$ always denotes the length of the given path $x \dots y$, i.e., the number of edges on it); and likewise

$$l(q \dots v) + 1 + l(u \dots c') \le r. \tag{9}$$

Because uv helps p with c, we know that $d_{K'}(q,c) \ge r-1$, and hence

$$r - 1 \le l(q \dots v) + l(v \dots c); \tag{10}$$

and likewise

$$r - 1 \le l(p \dots u) + l(u \dots c'). \tag{11}$$

Adding (8)–(11) together we see that equality must hold in all of them. Now, if $q \dots v$ in P' only contains edges from K, then $d_K(c,q) \le r-1$ (because equality holds in the third inequality), but this contradicts the way $c = c_3$ was constructed. Hence the subpath $q \dots v$ must contain an edge, say st, which does not belong to K, and we can assume that the subpath is $q \dots st \dots v$. Now we know that uv and st have been added to K in K'; and hence no other changes have taken place.

By iii) (neither i) nor ii) can now hold) both uv and st help q, and therefore there is a shortest path $P'' : q \dots st \dots c''$ or $P'' : q \dots ts \dots c''$ where c'' is c_1 or c_2 and st is the last edge on P'' that does not belong to K. The latter is not possible: otherwise there would be shortest paths (in K') of the form $q \dots st$ (the subpath of P') and $q \dots ts$ (the subpath of P''), but the length $l(q \dots s)$ of the subpath of $q \dots st$ and the length $l(q \dots t)$ of the subpath of $q \dots ts$ satisfy $l(q \dots s) = l(q \dots t) + 1$ and $l(q \dots t) = l(q \dots s) + 1$ and they both cannot hold.

We can take $q \dots s$ in P'' to be the same subpath as in P' (since both are known to be shortest paths in K'). Now the subpaths $c'' \dots t$ of P'', $t \dots v$ of P' and $v \dots c$ of P together show that $d_K(c,c'') \le (r-1) + (r-2) = 2r-3$ (because equality holds in (10)). By the definition of c_3, and of c_1 and c_2 we know that $d_K(c,c'') \ge r + 1 + (r-2) = 2r-1$. This contradiction proves the claim. □

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