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Large sets of disjoint directed and Mendelsohn packing triple systems on $6k+5$ points

Aihong Wu

Jingfan Middle School
Suzhou 215006
China
wu.aihong@163.com

Abstract

A Mendelsohn (or directed) packing triple system of order v, briefly MPT(v) (or DPT(v)), is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of cyclic (or transitive) triples on X such that every ordered pair of X belongs to at most one triple of $\mathcal{B}$. An LMPT(v) (or LDPT(v)) is a large set consisting of $v-3$ (or $3(v-3)$) disjoint compatible MPT(v)s (or DPT(v)s) with a 2-cycle as the common leave. In this paper, we show that an LMPT$(6k+5)$ and an LDPT$(6k+5)$ exist for any nonnegative integer k. Some small orders are based on the existent results of LR(9) and LR(15).

1 Introduction

Let X be a finite set. In what follows, an *ordered pair* of X will always be an ordered pair (x, y) where $x \neq y \in X$. A *cyclic triple* on X is a set of three ordered pairs (x, y), (y, z), and (z, x) of X, which is denoted by $\langle x, y, z\rangle$ (or $\langle y, z, x\rangle$, or $\langle z, x, y\rangle$). A *transitive triple* on X is a set of three ordered pairs (x, y), (y, z), and (x, z) of X, which is denoted by (x, y, z).

A Mendelsohn (respectively, directed) packing triple system of order v, written briefly MPT(v) (respectively, DPT(v)), is a pair $(X, \mathcal{B})$ where X is a v-set and $\mathcal{B}$ is a collection of cyclic (respectively, transitive) triples on X such that every ordered pair of X belongs to at most one triple of $\mathcal{B}$. An MPT(v) (respectively, DPT(v)) $(X, \mathcal{B})$ is *maximum* if there does not exist any MPT(v) (respectively, DPT(v)) $(X, \mathcal{A})$ with $|\mathcal{A}| > |\mathcal{B}|$. The *leave* of a Mendelsohn (respectively, directed) packing triple system $(X, \mathcal{B})$ is the graph (X, E), where E consists of all the ordered pairs which do not appear in any block of $\mathcal{B}$. In particular, when $v \equiv 0, 1 \pmod 3$, the leave of a maximum MPT (respectively, DPT) is an empty set and the MPT (respectively, DPT) is a Mendelsohn (respectively, directed) packing triple system, and denoted by MTS (respectively, DTS). It is well known that an MTS(v) exists if and only if $v \equiv 0, 1 \pmod 3$, $v \geq 3$, $v \neq 6$ [10], and a DTS(v) exists if and only if $v \equiv 0, 1 \pmod 3$ [4]. When $v \equiv 2 \pmod 3$, the leave of a maximum MPT (respectively, DPT) is a 2-cycle $\langle \infty_1 \infty_2 \rangle$. The existence of maximal MPT(v) is solved in [2] and the existence of maximal DPT(v) is solved in [11]. In this paper, an MPT(v) (respectively, DPT(v)) is always assumed to be maximal.

Two packings $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are called *disjoint* if $\mathcal{A} \cap \mathcal{B} = \emptyset$. If two packings have the same leave, then they are called *compatible*. A set of more than two packings is called disjoint (respectively, compatible) if each pair of them is disjoint (respectively, compatible).

Denote by $M_m(v)$ (respectively, $M_d(v)$) the maximum number of disjoint compatible Mendelsohn (respectively, directed) packing triple systems. A *large set* of Mendelsohn (respectively, directed) packing triple systems of order v, denoted by LMPT(v) (respectively, LDPT(v)), consists of $M_m(v)$ MPT(v)s (respectively, $M_d(v)$ DPT(v)s). It is known that for $v \equiv 0, 1 \pmod 3$, $M_m(v) = v - 2$ (respectively, $M_d(v) = 3(v-2)$) and the LMPT(v) (respectively, LDPT(v)) is an LMTS(v) (respectively, LDTS(v)) in fact. It is well known that an LMTS(v) exists if and only if $v \equiv 0, 1 \pmod 3), v \neq 6$ [8] and an LDTS(v) exists if and only if $v \equiv 0, 1 \pmod 3$ [7]. For $v \equiv 2 \pmod 3$, since the leaves of both an MPT(v) and a DPT(v) are 2-cycle $\langle \infty_1 \infty_2 \rangle$, by simple computation we have that $M_m(v) = v - 3$ and $M_d(v) = 3(v - 3)$.

In this paper, we shall focus on the existence of LMPT($6k + 5$) and LDPT($6k + 5$) which have the 2-cycle $\langle \infty_1 \infty_2 \rangle$ as their common leave and the following result will be proved.

Theorem 1.1 *There exist an LMPT(*$6k + 5$*) and an LDPT*($6k + 5$) *for any non-negative integer k.*

2 Recursive Constructions for LMPT and LDPT

In this section, we shall describe a construction to obtain an LMPT from a partitionable Mendelsohn candelabra system (PMCS) and a partitionable directed candelabra system (PDCS).

Let v be a non-negative integer. A *group divisible design* (or GDD) of order v and block size k denoted by GDD($2, k, v$) is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

1. X is a set of v elements (called *points*);

2. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a collection of non-empty subsets (called *groups*) of X which partition X;

3. $\mathcal{B}$ is a family of k-subsets of X (called *blocks*) such that each block intersects any given group in at most one point;

4. each pairs of points from two distinct groups is contained in exactly one block.

The *type* of the GDD is defined to be the list $(|G||G \in \mathcal{G})$ of group sizes.

A Mendelsohn (respectively, directed) GDD$(2,3,v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

1. X is a set of v elements (called *points*);
2. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a collection of non-empty subsets (called *groups*) of X which partition X;
3. $\mathcal{B}$ is a family of cyclic (respectively, transitive) triples of X (called *blocks*) such that each block intersects any given group in at most one point;
4. each ordered pair from two distinct groups is contained in exactly one block.

The *type* of the Mendelsohn (respectively, directed) GDD is defined to be the list $(|G||G \in \mathcal{G})$ of group sizes.

Let v be a non-negative integer. A *partitionable Mendelsohn* (respectively, *directed*) *candelabra system* (or PMCS (respectively, PDCS)) of order v and *type* $(g^n : s)$ is a quadruple $(X, S, \Gamma, \mathcal{A})$ that satisfies the following properties:

1. X is a set of $gn + s$ elements (called *points*);
2. S is a subset (called the *stem* of the candelabra) of X of size s;
3. $\Gamma = \{G_1, G_2, \ldots, G_n\}$ is a set of g-subsets (called *groups* or *branches*) of $X \backslash S$, which partition $X \backslash S$;
4. $\mathcal{A}$ is the set of all cyclic (respectively, transitive) triples (called *blocks*) of X except those cyclic (respectively, transitive) triples of $S \cup G_i$ for all i. It can be partitioned into $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_{gn}$, $\mathcal{A}_{gn+1}$, ..., $\mathcal{A}_{gn+s-2}$ with the following two properties: (i) for each group G, there are exactly g $\mathcal{A}_i$s $(1 \leq i \leq gn)$ such that $\mathcal{A}_i$ is the block set of a Mendelsohn (respectively, directed) GDD$(2,3,gn+s)$ of type $1^{g(n-1)}(g+s)^1$ with $G \cup S$ as the long group; (ii) for $gn+1 \leq i \leq gn+s-2$, $(X \setminus S, \mathcal{G}, \mathcal{A}_i)$ is a Mendelsohn (respectively, directed) GDD$(2,3,gn)$ of type g^n. Note that the condition (ii) is relevant only if $s \geq 3$.

In order to obtain an LMPT (respectively, LDPT) from a PMCS (respectively, PDCS), we need a holey large set. Let X be a set of v points and let $\mathcal{H}$ be an h-subset of X with $h \geq 2$. Let $\mathcal{A}_i$ $(1 \leq i \leq v-3)$ (respectively, $(1 \leq i \leq 3(v-3))$ be sets of cyclic (respectively, transitive) triples of X. $(X, \{\mathcal{A}_i) : 1 \leq i \leq v-3\}$ (respectively, $(X, \{\mathcal{A}_i) : 1 \leq i \leq 3(v-3)\})$ is called a *holey large set* of disjoint MPT(v) (respectively, DPT(v)) on X with a hole $\mathcal{H}$ (denoted by HLMPT(v,h) (respectively, HLDPT(v,h))) if $\mathcal{A}_i$ satisfy the following properties: (i) for $1 \leq i \leq v-h$ (respectively, $1 \leq i \leq 3(v-h)$), each $(X, \mathcal{A}_i)$ is an MPT(v) (respectively, DPT(v)) with the common leave of a 2-cycle $\langle\infty_1\infty_2\rangle$ and $\{\infty_1,\infty_2\}\subset\mathcal{H}$; (ii) for $v-h\le i\le v-3$ (respectively, $3(v-h)\le i\le 3(v-3)$), each $(X,\mathcal{A}_i)$ is a Mendelsohn (respectively, directed) GDD$(2,3,v)$ of type $1^{v-h}h^1$ with the long group $\mathcal{H}$; (iii) $(\cup_{i=1}^{v-3}\mathcal{A}_i)\cap\mathcal{H}^{(3)}=\emptyset$ (respectively, $(\cup_{i=1}^{3(v-3)}\mathcal{A}_i)\cap\mathcal{H}^{(3)}=\emptyset$), where $\mathcal{H}^{(3)}$ denotes the set of all cyclic (respectively, transitive) triples of $\mathcal{H}$.

Now, we are in a position to describe how to get an LMPT(v) (or an LDPT(v)] from a PMCS (or a PDCS).

Lemma 2.1 *Suppose there exists a PMCS$(g^n:5)$. If there exist an HLMPT$(g+5,5)$ and an LMPT$(g+5)$, then there is an LMPT$(gn+5)$.*

Proof: Suppose the given PMCS$(g^n:5)$ $(X,S,\Gamma,\mathcal{A})$ consists of gn Mendelsohn GDD$(2,3,gn+5)$s of type $1^{g(n-1)}(g+5)^1$ with the long group $G\cup S$ and block set $\mathcal{A}_y, y\in G$ and $G\in\Gamma$, and 3 Mendelsohn GDD$(2,3,gn)$s of type g^n with group set Γ and block set $\mathcal{A}_i, i=1,2,3$. Let $S=\{\infty_1,\infty_2,\dots,\infty_5\}$.

Take a group $G'\in\Gamma$. For each group $G\in\Gamma$, $G\ne G'$, suppose the given HLMPT$(g+5,5)$ on $G\cup S$ consists of g MPT$(g+5)$s with block sets $\mathcal{B}_y(y\in G)$ and $\langle\infty_1\infty_2\rangle$ as the common leave and 2 Mendelsohn GDD$(2,3,g+5)$ of type 1^g5^1 with the long group S and block sets $\mathcal{B}_i^G, i=1,2$.

For any $y\in G$, $G\in\Gamma$, $G\ne G'$, let $\mathcal{C}_y=\mathcal{A}_y\cup\mathcal{B}_y$. For $1\le i\le 2$, let $\mathcal{C}_i=\mathcal{A}_i\cup(\cup_{G\in\Gamma,G\ne G'}\mathcal{B}_i^G)$.

Then, each $(X,\mathcal{C}_y)$ is an MPT$(gn+5)$ with $\langle\infty_1\infty_2\rangle$ as the leave and each $\mathcal{C}_i$, $\mathcal{A}_y, y\in G'$ is the block set of a Mendelsohn GDD$(2,3,gn+5)$ of type $1^{g(n-1)}(g+5)^1$ with the long group $G'\cup S$. It is easy to see that these block sets are pairwise disjoint. So, they form an HLMPT$(gn+5,g+5)$.

Further, suppose the given LMPT$(g+5)$ on $G'\cup S$ consists of $g+2$ disjoint MPT$(g+5)$s with the common leave of a 2-cycle $\langle\infty_1\infty_2\rangle$. Denote these block sets by $\mathcal{B}_y(y\in G')$ and $\mathcal{B}_i, i=1,2$. Then, $\mathcal{A}_y\cup\mathcal{B}_y$ and $\mathcal{C}_i\cup\mathcal{B}_i$ are all MPT$(gn+5)$, and all $gn+2$ MPTs form an LMPT$(gn+5)$. □

Lemma 2.2 *Suppose there exists a PDCS$(g^n:5)$. If there exist a HLDPT$(g+5,5)$ and an LDPT$(g+5)$, then there is an LDPT$(gn+5)$.*

Proof: The proof is similar to the proof of Lemma 2.1. □

To obtain the required PMCS and PDCS, we describe constructions for them from a PCS, where PCS is introduced in [3] and plays an important in the construction of a large set of packing on $6k+5$ points.

Let v be a non-negative integer. A *partitionable candelabra system* (or PCS) of order v and *type* $(g^n:s)$ is a quadruple $(X,S,\Gamma,\mathcal{A})$ that satisfies the following properties:

1. X is a set of $gn+s$ elements (called *points*);

2. S is a subset (called the *stem* of the candelabra) of X of size s;

3. $\Gamma = \{G_1, G_2, \ldots, G_n\}$ is a set of g-subsets (called *groups* or *branches*) of $X \backslash S$, which partition $X \backslash S$;

4. $\mathcal{A}$ is the set of triples (called *blocks*) of X except all triples of $S \cup G_i$ for all i. It can be partitioned into $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_{gn}$, $\mathcal{A}_{gn+1}$, ..., $\mathcal{A}_{gn+s-2}$ with the following two properties: (i) for each group G, there are exactly g $\mathcal{A}_i$s $(1 \leq i \leq gn)$ such that $\mathcal{A}_i$ is the block set of a GDD$(2, 3, gn + s)$ of type $1^{g(n-1)}(g + s)^1$ with $G \cup S$ as the long group; (ii) for $gn + 1 \leq i \leq gn + s - 2$, $(X \setminus S, \mathcal{G}, \mathcal{A}_i)$ is a GDD$(2, 3, gn)$ of type g^n. Note that the condition (ii) is relevant only if $s \geq 3$.

Let $(X, S, \Gamma, \mathcal{A})$ be a PCS$(g^n : s)$. Define

$$\mathcal{B} = \{\langle x, y, z\rangle;\ \{x, y, z\} \in \mathcal{A}\} \cup \{\langle z, y, x\rangle; \{x, y, z\} \in \mathcal{A}\}.$$

Then $(X, S, \Gamma, \mathcal{B})$ is a PMCS$(g^n : s)$. And, define transitive triples

$$(x, y, z), (z, y, x) \in \mathcal{B}^1, \quad (y, z, x), (x, z, y) \in \mathcal{B}^2, \quad (z, x, y), (y, x, z) \in \mathcal{B}^3,$$

for each $\{x, y, z\} \in \mathcal{A}$. Then it is easy to see that $(X, S, \Gamma, \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3)$ is a PDCS$(g^n : s)$.

From [5], we know that there exists a PCS$(6^k : 5)$ for any integer $k \geq 3$. Then we can get the following results.

Lemma 2.3 *There exists a PDCS*$(6^k : 5)$ *for any integer* $k \geq 3$.

Lemma 2.4 *There exists a PMCS*$(6^k : 5)$ *for any integer* $k \geq 3$.

From the above lemmas, we easily know that the existence of HLMPT$(11, 5)$ and LMPT(5) implies the existence of LMPT$(6k+5)$ and the existence of HLDPT$(11, 5)$ and LDPT(5) implies the existence of LDPT$(6k + 5)$. These small orders will be discussed in Section 3.

To construct required holey large sets we need an LR design. An LR design is introduced in [9], and plays a very important role in the construction of LKTS. Here we shall use LR designs to construct LMPT and LDPT of some small orders. A GDD$(2, 3, v)$ of type 1^v $(X, \mathcal{G}, \mathcal{B})$ is often called a *Steiner triple system* and denoted by STS(v) $(X, \mathcal{B})$. A STS $(X, \mathcal{B})$ is *resolvable* if its block set $\mathcal{B}$ admits a partition into *parallel classes*, each parallel class being a partition of the point set X. A resolvable STS(v) is called a *Kirkman triple system* and is denoted by KTS(v). Let X be a v-set; an *LR design* of order v (briefly an LR(v)) is a collection $\{(X, \mathcal{A}_k^j); 1 \leq k \leq (v-1)/2, j = 0, 1\}$ of KTS(v)s with the following properties:

1. Let the resolution of $\mathcal{A}_k^j$ be $\Gamma_k^j = \{A_k^j(h); 1 \leq h \leq (v-1)/2\}$. There is an element in each Γ_k^j , which without loss of generality, we can suppose is $A_k^j(1)$, such that
$$\bigcup_{k=1}^{(v-1)/2} A_k^0(1) = \bigcup_{k=1}^{(v-1)/2} A_k^1(1) = \mathcal{A}$$
and $(X, \mathcal{A})$ is a KTS(v).

2. For any triple $T = \{x, y, z\} \subset X, x \neq y \neq z$, there exist k, j such that $T \in \mathcal{A}_k^j$.

Lemma 2.5 *If there exists an LR*(v)*, then there exist an HLMPT*$(v+2, 5)$ *and an HLDPT*$(v+2, 5)$*.*

Proof: Let $\{(X, \mathcal{A}_k^j); 1 \leq k \leq (v-1)/2, j = 0, 1\}$ be the given LR(v). Let $\{\infty_1, \infty_2\} \cap X = \emptyset$. We shall construct the required design on $X \cup \{\infty_1, \infty_2\}$ with $\langle \infty_1, \infty_2 \rangle$ as the common leave.

For any $1 \leq k \leq (v-1)/2$ and each block $A = \{x, y, z\} \in A_k^0(1)$, we construct an LMPT(5) on $\{x, y, z, \infty_1, \infty_2\}$ which consists of two MPT(5)s with the block sets B_A^j, $j = 0, 1$. For each block $A' = \{x', y', z'\} \in \mathcal{A}_k^j$ with $A' \notin \mathcal{A}$, we construct an LMTS(3) on $\{x', y', z'\}$ with the block set $B_{A'}$. Then for $1 \leq k \leq (v-1)/2$ and $j = 0, 1$, each $(\cup_{A' \in A_k^j(h), 2 \leq h \leq (v-1)/2} B_{A'}) \cup (\cup_{A \in A_k^j(1)} B_A^j)$ is an MPT$(v+2)$. These obtained $v-1$ MPT$(v+2)$s form an LMPT$(v+2)$. It is easy to see that if we delete an LMPT(5) on $\{x, y, z, \infty_1, \infty_2\}$ ($\{x, y, z\} \in A_1^0(1)$) we can get $v-3$ MPT$(v+2)$s and two Mendelsohn GDD$(2, 3, v+2)$s of type $1^{v-3}5^1$ which form an HLMPT$(v+2, 5)$. In the same way, an HLDPT$(v+2, 5)$ can be obtained. ▯

3 Some small orders

In this section, we shall construct HLMPT$(11, 5)$, LMPT(5) and HLDPT$(11, 5)$, LDPT(5). The 2-cycle $\langle \infty_1 \infty_2 \rangle$ is always assumed to be the common leave.

Lemma 3.1 *There exists an LMPT*(5)*.*

Proof: Take the point set $X = \{0, 1, 2, \infty_1, \infty_2\}$. Let

$$\begin{array}{llll} \mathcal{B}_1 : & <\infty_1, 0, 1> & <1, \infty_1, 2> & <0, 2, \infty_1> \\ & <\infty_2, 2, 0> & <2, \infty_2, 1> & <1, 0, \infty_2>. \end{array}$$

$$\begin{array}{llll} \mathcal{B}_2 : & <1, 0, \infty_1> & <2, \infty_1, 1> & <\infty_1, 2, 0> \\ & <0, 2, \infty_2> & <1, \infty_2, 2> & <\infty_2, 0, 1>. \end{array}$$

Then it is easy to check that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are the required two MPTs which form an LMPT(5). ▯

Lemma 3.2 *There exists an LDPT(5).*

Proof: Take the point set $X = \{0, 1, 2, \infty_1, \infty_2\}$. Let

$$\mathcal{B}_1: \quad (\infty_1, 0, 1) \quad (1, \infty_1, 2) \quad (0, 2, \infty_1)$$
$$(\infty_2, 2, 0) \quad (2, \infty_2, 1) \quad (1, 0, \infty_2).$$

$$\mathcal{B}_2: \quad (1, 0, \infty_1) \quad (2, \infty_1, 1) \quad (\infty_1, 2, 0)$$
$$(0, 2, \infty_2) \quad (1, \infty_2, 2) \quad (\infty_2, 0, 1).$$

Then it is easy to check that $\mathcal{B}_1$ and $\mathcal{B}_2$ together generate the required six DPTs mod 3. ∎

Lemma 3.3 *There exist an HLMPT$(v, 5)$ and an HLDPT$(v, 5)$ for $v = 11$ and $v = 17$.*

Proof: From [6, 9], there exist an LR(9) and an LR(15). By Lemma 2.5, we get the required results. ∎

4 Proof of Theorem 1.1

In this section we prove the main result stated in Theorem 1.1.

Proof of Theorem 1.1: For $k = 0, 1, 2$, there exists an LMPT$(6k + 5)$ and an LDPT$(6k + 5)$ by Lemmas 3.1, 3.2 and 3.3. For $k \geq 3$, the result follows from Lemmas 2.1–2.4 and Lemmas 3.2 and 3.3.

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