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Note: A new proof of an inequality for finite planar spaces

Vito Napolitano*

Dipartimento di Matematica
Università della Basilicata
Contrada Macchia Romana
85100 Potenza
Italy
vito.napolitano@unibas.it

Abstract

In this note we give a short and geometric proof of a famous result on finite planar spaces which states that the number of planes is greater than or equal to the number of points, and that equality holds if, and only if, the planar space is either a finite 3–dimensional projective space, or there are two disjoint lines containing all the points of the planar space, or all the points but one belong to a finite projective plane.

1 Introduction

Let $S = (\mathcal{P}, \mathcal{L})$ be a linear space. A subset X of points is a *subspace* if any line connecting two points of X is wholly contained in X. Then $\emptyset$ and S are subspaces. Also, the intersection of any set of subspaces is again a subspace, and so the notion of spanning subspace makes sense.

A *plane* is a subspace spanned by three non–collinear points, that is, the intersection of all subspaces of S containing the three given points.

A *planar space* is a linear space with a family of planes such that given any three non–collinear points there is exactly one plane containing them, there are at least two planes and each plane contains at least three non–collinear points.

Examples of planar spaces are the affine and projective spaces of dimension at least 3 with respect to their lines and their planes.

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Let $\mathcal{H}$ denote the family of planes of a planar space.

Assume that S is a finite planar space; that is, $|\mathcal{P}| < \infty$, and put $v = |\mathcal{P}|$, $b = |\mathcal{L}|$, $c = |\mathcal{H}|$. For any point p, let π_p denote the number of planes passing through p, and if π is a plane let v_π and b_π be the numbers of its points and lines.

Finding (a) the number of subspaces determined by a subset of v points, (b) a bound for this number, and (c) a characterization of the extremal cases are interesting problems in finite geometry. It has been also the starting point for some characterization results for finite geometries with a prescribed arithmetic conditions on their parameters. In the fifties, starting from a famous result of de Bruijn and Erdős [2] which states that in a finite linear space there are as many lines as points, and that equality holds if the linear space is a projective one, a number of papers devoted to the corresponding question for finite planar spaces appeared; see, for example, [1, 3, 4].

In [3] Hanani gave a new proof of the de Bruijn and Erdős theorem for finite linear spaces, and he also showed that, for a finite planar space on v points with c planes, $c \geq v$, and that the equality holds if the linear space is a finite 3-dimensional projective space, or the union of two disjoint and non-coplanar *skew* lines, or all points but one belong to a finite projective plane.

In this note we give a new and short proof of this result. Our proof uses the same technique used in a recent proof [5] of the de Bruijn–Erdos theorem on finite linear spaces.

The following property is easy to show.

If π is a plane and p is a point outside of π, then $b_\pi \leq \pi_p$.

Also, we recall that a finite projective space is a finite linear space whose planes are all projective.

As usual, we denote a planar space as a triple $(\mathcal{P}, \mathcal{L}, \mathcal{H})$.

Theorem 1.1 *Let $(\mathcal{P}, \mathcal{L}, \mathcal{H})$ be a finite planar space with v points and c planes. Then*

(i) $c \geq v$;

(ii) *equality holds if, and only if, $(\mathcal{P}, \mathcal{L}, \mathcal{H})$ is either a 3–dimensional finite projective space, or there are two skew lines ℓ and ℓ' such that $\mathcal{P} = \ell \cup \ell'$, or all the points of $(\mathcal{P}, \mathcal{L}, \mathcal{H})$ but one form a finite projective plane.*

Proof. Assume that $c \leq v$; then we prove that $c = v$ and the planar space is one of those described in the statement.

Double counting gives the following:

$$\sum_{p \in \mathcal{P}} \pi_p = \sum_{\pi \in \mathcal{H}} v_\pi. \tag{1.1}$$

Let

$$m = \min_{p \in \mathcal{P}} \pi_p, \quad k = \max_{\pi \in \mathcal{H}} v_\pi;$$

then (1.1) implies that

$$vm \leq ck.$$

From $c \leq v$ it follows that $k \geq m$.

If $k = m$, then $c = v$, the planes have the same size m, through any point there pass m planes, and any two planes have non–empty intersection.

Let π be a plane, and p a point outside π. Each line of π gives rise to a plane through p; so $m = \pi_p \geq b_\pi$ and, since $b_\pi \geq v_\pi = m$, it follows that $b_\pi = v_\pi = m$. Also, any two planes meet in a line, since the number of lines in a plane is equal to the number of planes through a point.

If every plane is a projective plane, the linear space is a projective space, and since any two planes meet in a line it is $PG(3, q)$.

If there is a plane π which is a near–pencil $L \cup \{p\}$, each plane through L is a near–pencil. Let q be a point not in π. Through q pass all near–pencils, one for each line of π; so the planar space is the union of L and the line pq.

Next, consider the case that $k > m$.

Let π be a plane of size k; then $b_\pi \geq k > m$. So each point with $\pi_p = m$ is in π. Let p be a point of degree m; then $p \in \pi$. Each plane α not containing p has size at most m, since $v_\alpha \leq b_\alpha \leq \pi_p = m$. Each point x outside π has $\pi_x \geq k$. Hence (1.1) and $c \leq v$ imply the following:

$$km + (v - k)k \leq \sum_{p \in \mathcal{P}} \pi_p = \sum_{\pi \in \mathcal{H}} v_\pi \leq mk + (c - m)m \leq mk + (v - m)m. \quad (1.2)$$

It follows that

$$v \leq k + m.$$

In π there are at least two lines on exactly two planes; otherwise

$$c \geq 2 + (k - 1)2 = 2k,$$

and so from $c \leq k + m$ it follows that $k \leq m$, a contradiction.

If on every line of π there are exactly two planes, then there is a single point x outside of π, so $v = k + 1$. Thus $c \leq k + 1$. But $c \geq k + 1$, since π has at least k lines, and there are at least two planes. Hence, $v = c = k + 1$.

Since $\pi_x = k$ it follows that $b_\pi = k$, and so π is either a near–pencil or a projective plane.

In the former case, the planar space is the union of two skew lines, one of length 2 and the other with at least three points. In the latter case, the planar space is the union of a projective plane and a point.

Finally, we may assume that there is at least a line of π on at least three planes. Thus $v \geq c \geq k+2$.

Let ℓ_1 and ℓ_2 be two lines of π on exactly two planes, and let π_i denote the plane containing ℓ_i, $i = 1, 2$, different from π.

If $\ell_1 \cap \ell_2 = \emptyset$, since $v \geq k+2$ the planes π_1 and π_2 meet in a line t disjoint from π. So $v = k + |t|$. Each line meeting both ℓ_1 and ℓ_2 is on $|t|+1$ planes; so $c \geq k+t = v$. This means that there is a single line of π meeting both ℓ_1 and ℓ_2, a contradiction.

Therefore $\ell_1 \cap \ell_2 \neq \emptyset$. In this case, π_1 and π_2 meet in a line t outside π and intersecting π in $\ell_1 \cap \ell_2$, and all the points of the planar space not in π are those of t. If $|t| = 2$ any line of π is on exactly two planes, a contradiction to our assumptions. Thus $|t| \geq 3$ and $v = k + |t| - 1$.

Any line connecting a point of ℓ_1 and ℓ_2 is on $|t|$ planes; so $c \geq k + |t| - 1$. Since $c \leq v$ it follows that $c = v = k + |t| - 1$. Hence there is a single line L in π on at least three planes, since each line give rise to at least one plane different from π, and so ℓ_1 and ℓ_2 have length 2, as well as each line of π through $\ell_1 \cap \ell_2$. So π is a near–pencil, and also each plane through t is a near–pencil. Hence S is the union of two skew lines, t and L.

Thus the theorem is completely proved. □

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