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C_5-decompositions of the tensor product of complete graphs

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Abstract

In this paper, it has been proved that the necessary conditions for the existence of C_5-decomposition of $K_m \times K_n$ are sufficient, where $\times$ denotes the tensor product of graphs. Using these necessary and sufficient conditions, it can be shown that every even regular complete multipartite graph G can be decomposed into 5-cycles if the number of edges of G is divisible by 5.

1 Introduction

All graphs considered here are simple and finite. Let C_n denote a cycle of length n. If the edge set of G can be partitioned into cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_r}$, then we say that $C_{n_1}, C_{n_2}, \ldots, C_{n_r}$ *decompose* G. If $n_1 = n_2 = \ldots = n_r = k$, then we say that G has a C_k*-decomposition* and in this case we write $C_k \mid G$. If G has a 2-factorization and each 2-factor of it has only cycles of length k, then we say that G has a C_k*-factorization*, in notation $C_k \,\|\, G$. We write $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, if $H_1, H_2, \ldots, H_k$ are edge-disjoint subgraphs of G and $E(G) = E(H_1) \cup E(H_2) \cup \ldots \cup E(H_k)$. The complete graph on m vertices is denoted by K_m and its complement is denoted by $\overline{K}_m$. For a positive integer k and the graph H, the graph kH denotes k vertex disjoint copies of H. For a graph G, $G(\lambda)$ denotes the graph obtained from G by replacing each of its edges by λ edges. A cycle of length k is called a k*-cycle* and it is denoted by C_k. A path on k vertices is denoted by P_k.

For two graphs G and H, their *wreath product* $G * H$ has vertex set $V(G) \times V(H)$ in which (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Similarly, $G \times H$, the *tensor product* of the graphs G and H has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1g_2 \in E(G)$ *and* $h_1h_2 \in E(H)$. Clearly, the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \ldots \oplus H_k$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \ldots \oplus (H_k \times H)$. For $h \in V(H)$, $V(G) \times h = \{(v, h) \,|\, v \in V(G)\}$ is called the *column* of vertices of $G \times H$ corresponding to h. Further, for $x \in V(G)$, $x \times V(H) = \{(x, v) \,|\, v \in V(H)\}$ is called the *layer* of vertices of $G \times H$ corresponding to x. Similarly, we can define column and layer for the wreath product of graphs also. It is easy to observe that $K_m * \overline{K}_n$ is isomorphic to the complete m-partite graph in which each partite set has exactly n vertices. It is easy to see that $K_m \times K_n$ is a subgraph of $K_m * \overline{K}_n$; in fact, $K_m \times K_n = (K_m * \overline{K}_n) - nK_m$.

A *latin square* of order n is an $n \times n$ array, each cell of which contains exactly one of the symbols in $\{1, 2, \ldots, n\}$, such that each row and each column of the array contains each of the symbols in $\{1, 2, \ldots, n\}$ exactly once. A latin square is said to be *idempotent* if the cell (i, i) contains the symbol i, $1 \le i \le n$.

Let G be a bipartite graph with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. If G contains the set of edges $F_i(X, Y) = \{x_j y_{i+j} \mid 1 \le j \le n\}$, $0 \le i \le n-1$, where addition in the subscript is taken modulo n with residues $1, 2, \ldots, n$, then we say that G has the *1-factor of distance i from X to Y*. Note that $F_i(Y, X) = F_{n-i}(X, Y)$, $0 \le i \le n-1$. Clearly, if $G = K_{n,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$. In a bipartite graph with bipartition (X, Y) with $|X| = |Y|$, if $x_i y_j$ is an edge, then $x_i y_j$ is called an edge of *distance* $j - i$ if $i \le j$, or $n - (i - j)$, if $i > j$, *from X to Y*. (The same edge is said to be of *distance* $i - j$ if $i \ge j$ or $n - (j - i)$, if $i < j$, *from Y to X*.)

Recently, it has been proved that if n is odd and $m \mid \binom{n}{2}$ or n is even and $m \mid (\binom{n}{2} - \frac{n}{2})$, then $C_m \mid K_n$ or $C_m \mid K_n - I$, where I is a 1-factor of K_n [1, 16]. A similar problem is also considered for regular complete multipartite graphs; Cavenagh and Billington [8] and Mahmoodian and Mirzakhani [12] have considered C_5-decompositions of complete tripartite graphs, see also [6]. Moreover, Billington [3] has studied the decompositions of complete tripartite graphs into cycles of length 3 and 4. Further, Cavenagh and Billington [7] have studied the 4-cycle, 6-cycle and 8-cycle decompositions of complete multipartite graphs. Billington et al. [5] have solved the problem of decomposing $(K_m * \overline{K}_n)(\lambda)$ into 5-cycles. The present authors also have proved [13, 14] that the necessary conditions for the existence of C_p-decompositions of $K_m * \overline{K}_n$ and $K_m \times K_n$ are also sufficient, where $p \ge 7$ is a prime. A detailed account of cycle decompositions of complete graphs and complete multipartite graphs can be seen in [9] and [4], respectively. Also a complete solution to C_k-factorization of even regular complete multipartite graphs can be seen in [11].

In this paper, we prove that the obvious necessary conditions for $K_m \times K_n$, $m, n \ge 3$, to have a C_5-decomposition are also sufficient. Among other results, here we prove the following main result.

Theorem 1.1 *For $m, n \ge 3$, $C_5 \mid K_m \times K_n$ if and only if* (1) $5 \mid nm(m-1)(n-1)$ *and* (2) *either m or n is odd.*

Using Theorem 1.1, we can prove the following theorem, which is also proved in [5].

Theorem 1.2 *[5] For* $m \geq 3$, $C_5 \mid K_m * \overline{K}_n$ *if and only if* (1) $n(m-1)$ *is even and* (2) $5 \mid m(m-1)n^2$. ■

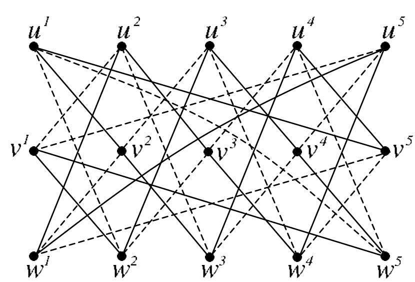
2 C_5-Decomposition of $C_3 \times K_m$

Lemma 2.1 $C_5 \,|\, C_3 \times K_{10}$.

Proof. Let the partite sets (layers) of the tripartite graph $C_3 \times K_{10}$ be $U = \{u_1, u_2, \ldots, u_{10}\}$, $V = \{v_1, v_2, \ldots, v_{10}\}$ and $W = \{w_1, w_2, \ldots, w_{10}\}$; we assume that the vertices of U, V and W having same subscripts are the 'corresponding vertices' of the partite sets. By the definition of the tensor product, $\{u_i, v_i, w_i\}$, $1 \leq i \leq 10$, are independent sets and the subgraph induced by each of the subsets of vertices $U \cup V$, $V \cup W$ and $W \cup U$ is isomorphic to $K_{10,10} - F_0$, where F_0 is the 1-factor of distance zero in $K_{10,10}$.

We obtain a new graph from $C_3 \times K_{10}$ as follows: for each i, $1 \leq i \leq 5$, identify the subsets of vertices $\{u_{2i-1}, u_{2i}\}$, $\{v_{2i-1}, v_{2i}\}$ and $\{w_{2i-1}, w_{2i}\}$ into new vertices u^i, v^i and w^i, respectively, and two of these vertices are adjacent if and only if the corresponding subsets of vertices in $C_3 \times K_{10}$ induce a $K_{2,2}$. The resulting graph is isomorphic to $C_3 \times K_5$ with partite sets $U' = \{u^1, u^2, \ldots, u^5\}$, $V' = \{v^1, v^2, \ldots, v^5\}$ and $W' = \{w^1, w^2, \ldots, w^5\}$; note that $\{u^i, v^i, w^i\}$, $1 \leq i \leq 5$, are independent sets of $C_3 \times K_5$.

Now $C_3 \times K_5 = C_3 \times (C_5 \oplus C_5) = (C_3 \times C_5) \oplus (C_3 \times C_5)$. The graph $C_3 \times C_5$ can be decomposed into 5-cycles, see Figure 1.



$C_3 \times C_5$

A 5-cycle decomposition of $C_3 \times C_5$

Figure 1

By "lifting back" these 5-cycles of $C_3 \times K_5$ to $C_3 \times K_{10}$, we get edge-disjoint subgraphs isomorphic to $C_5 * \overline{K}_2$. But $C_5 * \overline{K}_2$ can be decomposed into cycles of length 5, see [15]. Thus the subgraphs of $C_3 \times K_{10}$ obtained by "lifting back" the 5-cycles of $C_3 \times K_5$ can be decomposed into cycles of length 5. The edges of $C_3 \times K_{10}$ which are not covered by these 5-cycles are shown in Figure 2.

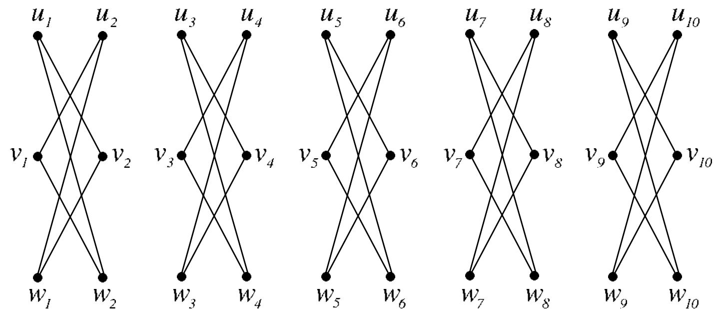
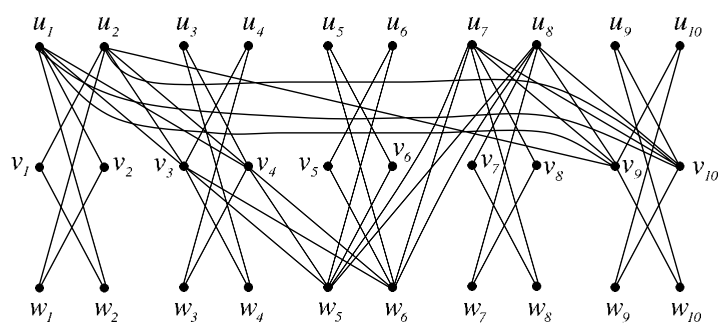


Figure 2

To complete the proof, we fuse some of the 5-cycles obtained above with the graph of Figure 2 and decomposing the resulting graph into cycles of length 5. Let H' be the graph obtained by the union of the graph of Figure 2 and the subgraph of $C_3 \times K_{10}$ which is obtained by "lifting back" a 5-cycle of $C_3 \times K_5$, namely, $(u^1, v^2, w^3, u^4, v^5)$ shown in Figure 1. The subgraph H' of $C_3 \times K_{10}$ is shown in Figure 3. A 5-cycle decomposition of H' is given below.

$(u_1, v_2, w_1, u_2, v_3)$, $(u_1, w_2, v_1, u_2, v_4)$, $(v_3, w_4, u_3, v_4, w_5)$, $(v_3, u_4, w_3, v_4, w_6)$, $(w_5, v_6, u_5, w_6, u_7)$, $(w_5, u_6, v_5, w_6, u_8)$, $(u_7, w_8, v_7, u_8, v_9)$, $(u_7, v_8, w_7, u_8, v_{10})$, $(v_9, u_{10}, w_9, v_{10}, u_1)$, $(v_9, w_{10}, u_9, v_{10}, u_2)$. ■



H'

Figure 3

Lemma 2.2 $C_5 \,|\, C_3 \times K_6$.

Proof. Let the partite sets (layers) of the tripartite graph $C_3 \times K_6$ be $\{u_1, u_2, \ldots, u_6\}$, $\{v_1, v_2, \ldots, v_6\}$ and $\{w_1, w_2, \ldots, w_6\}$. We assume that the vertices having the same subscript are the corresponding vertices of the partite sets. A 5-cycle decomposition of $C_3 \times K_6$ is given below:

$(u_1, w_4, u_6, v_3, w_2)$, $(u_1, v_6, u_2, w_5, v_3)$, $(u_6, v_2, w_4, v_6, w_5)$, $(u_3, v_1, w_5, u_4, w_2)$, $(u_4, v_6, w_2, v_1, w_6)$, $(u_4, v_5, w_6, u_5, v_1)$, $(u_3, v_6, w_3, u_5, w_4)$, $(u_3, v_5, w_4, v_3, w_1)$, $(u_4, v_3, u_5, v_6, w_1)$, $(u_1, v_5, w_3, v_2, w_6)$, $(u_1, w_5, v_2, u_4, w_3)$, $(u_1, v_4, w_5, u_3, v_2)$, $(u_2, w_4, v_1, u_6, w_1)$, $(u_2, v_1, w_3, v_4, w_6)$, $(u_2, v_4, u_3, w_6, v_3)$, $(u_2, v_5, w_2, u_6, w_3)$, $(u_5, v_2, w_1, v_4, w_2)$ and $(u_5, v_4, u_6, v_5, w_1)$. ■

Theorem 2.3 [2]. *Let t be an odd integer and p be a prime so that $3 \le t \le p$. Then $C_t * \overline{K}_p$ has a 2-factorization so that each 2-factor is composed of t cycles of length p.* ■

Theorem 2.4 [1]. *If $n \equiv 1$ or $t \pmod{2t}$, where $t \geq 3$ is an odd integer, then $C_t \,|\, K_n$.* ■

Remark 2.5 *Let the partite sets (layers) of the complete tripartite graph $C_3 * \overline{K}_m$, $m \geq 1$, be $\{u_1, u_2, \ldots, u_m\}$, $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_m\}$. Consider a latin square $\mathcal{L}$ of order m. We associate a triangle (3-cycle) of $C_3 * \overline{K}_m$ with each entry of $\mathcal{L}$ as follows: if k is the (i, j)th entry of $\mathcal{L}$, then the triangle of $C_3 * \overline{K}_m$ corresponding to k is (u_i, v_j, w_k). Clearly, the triangles corresponding to the entries of $\mathcal{L}$ decompose $C_3 * \overline{K}_m$; see e.g.* [3]. ■

Theorem 2.6 *$C_5 \,|\, C_3 \times K_m$ if and only if $m \equiv 0$ or $1 \pmod 5$.*

Proof. The necessity is obvious. We prove the sufficiency in two cases.

Case 1. $m \equiv 1 \pmod 5$.
Let $m = 5k + 1$.

Subcase 1.1. $k \neq 2$.
Let the partite sets (layers) of the tripartite graph $C_3 \times K_m$ be

$$U = \{u_0\} \cup (\bigcup_{i=1}^{k} \{u_1^i, u_2^i, \ldots, u_5^i\}), \quad V = \{v_0\} \cup (\bigcup_{i=1}^{k} \{v_1^i, v_2^i, \ldots, v_5^i\})$$

$$\text{and} \qquad W = \{w_0\} \cup (\bigcup_{i=1}^{k} \{w_1^i, w_2^i, \ldots, w_5^i\});$$

we assume that the vertices of U, V and W having the same subscript and superscript are the corresponding vertices of the partite sets. By the definition of the tensor product, $\{u_0, v_0, w_0\}$ and $\{u_j^i, v_j^i, w_j^i\}$, $1 \le j \le 5$, $1 \le i \le k$, are independent sets and the subgraph induced by each of the sets $U \cup V$, $V \cup W$ and $W \cup U$ is isomorphic to $K_{m,m} - F_0$,where F_0 is the 1-factor of distance zero in $K_{m,m}$.

We obtain a new graph from $H = (C_3 \times K_m) - \{u_0, v_0, w_0\} \cong C_3 \times K_{5k}$ as follows: for each i, $1 \le i \le k$, identify the sets of vertices $\{u_1^i, u_2^i, \ldots, u_5^i\}$, $\{v_1^i, v_2^i, \ldots, v_5^i\}$ and $\{w_1^i, w_2^i, \ldots, w_5^i\}$ with new vertices u^i, v^i and w^i, respectively; two new vertices are adjacent if and only if the corresponding sets of vertices in H induce a complete bipartite subgraph $K_{5,5}$ or $K_{5,5} - F$, where F is a 1-factor of $K_{5,5}$. This defines the graph isomorphic to $C_3 * \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$ and $\{w^1, w^2, \ldots, w^k\}$. Consider an idempotent latin square $\mathcal{L}$ of order k, $k \neq 2$ (which exists; see [10]). To complete the proof of this subcase, we associate with entries of $\mathcal{L}$ edge-disjoint subgraphs of $C_3 * \overline{K}_m$ which are decomposable by C_5. The ith diagonal entry of $\mathcal{L}$ corresponds to the triangle (u^i, v^i, w^i), $1 \leq i \leq k$, of $C_3 * \overline{K}_k$; see Remark 2.5. The subgraph of H corresponding to the triangle of $C_3 * \overline{K}_k$ is isomorphic to $C_3 \times K_5$. For each triangle (u^i, v^i, w^i), $1 \leq i \leq k$, of $C_3 * \overline{K}_k$ corresponding to the ith diagonal entry of $\mathcal{L}$, associate the subgraph of $C_3 \times K_m$ induced by vertices $\{u_0, u_1^i, u_2^i, \ldots, u_5^i\} \cup \{v_0, v_1^i, v_2^i, \ldots, v_5^i\} \cup \{w_0, w_1^i, w_2^i, \ldots, w_5^i\}$; since this subgraph is isomorphic to $C_3 \times K_6$, it can be decomposed into cycles of length 5, by Lemma 2.2. Again, if we consider the subgraph of H corresponding to the triangle of $C_3 * \overline{K}_k$ which corresponds to a non-diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 * \overline{K}_5$. By Theorem 2.3, $C_3 * \overline{K}_5$ can be decomposed into cycles of length 5. Thus we have decomposed $C_3 \times K_m$ into cycles of length 5 when $k \neq 2$.

Subcase 1.2. $k = 2$.
By Theorem 2.4, $C_5 \mid K_{11}$ and hence we write $C_3 \times K_{11} = (C_3 \times C_5) \oplus (C_3 \times C_5) \oplus \ldots \oplus (C_3 \times C_5)$. Now $C_3 \times C_5$ can be decomposed into cycles of length 5, see Figure 1. This proves that $C_5 \mid C_3 \times K_{11}$.

Case 2. $m \equiv 0 \pmod 5$.
Let $m = 5k$. If $k = 2$, then the result follows from Lemma 2.1. Hence we may assume that $k \neq 2$. Let the partite sets of the tripartite graph $C_3 \times K_m$ be $U = \bigcup_{i=1}^{k} \{u_1^i, u_2^i, \ldots, u_5^i\}$, $V = \bigcup_{i=1}^{k} \{v_1^i, v_2^i, \ldots, v_5^i\}$ and $W = \bigcup_{i=1}^{k} \{w_1^i, w_2^i, \ldots, w_5^i\}$. We assume that the vertices of U, V and W having the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, from $C_3 \times K_m = C_3 \times K_{5k}$ we obtain the graph $C_3 * \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$ and $\{w^1, w^2, \ldots, w^k\}$.

Consider an idempotent latin square $\mathcal{L}$ of order k, $k \neq 2$. The diagonal entries of $\mathcal{L}$ correspond to the triangles (u^i, v^i, w^i), $1 \leq i \leq k$, of $C_3 * \overline{K}_k$. If we consider the subgraph of $C_3 \times K_m$ corresponding to a triangle of $C_3 * \overline{K}_k$, which corresponds to a diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 \times K_5$. Clearly, $C_3 \times K_5 = (C_3 \times C_5) \oplus (C_3 \times C_5)$. Now $C_5 \mid C_3 \times C_5$, see Figure 1. Again, as in the previous case, the triangle of $C_3 * \overline{K}_k$ corresponding to a non-diagonal entry of $\mathcal{L}$, corresponds to a subgraph of $C_3 \times K_m$ isomorphic to $C_3 * \overline{K}_5$; by Theorem 2.3, $C_5 \mid C_3 * \overline{K}_5$. ■

Lemma 2.7 *If* $C_{2k-1} \mid C_{2k-1} \times K_m$, $k \geq 2$, *then* $C_{2k+1} \mid C_{2k+1} \times K_m$.

Proof. Let the partite sets of the $(2k-1)$-partite graph $C_{2k-1} \times K_m$ be $\{u_1^i, u_2^i, \ldots, u_m^i\}$, $1 \leq i \leq 2k-1$. We assume that the vertices having the same subscript are the corresponding vertices of the partite sets. Let the partite sets of the $(2k+1)$-partite graph $C_{2k+1} \times K_m$ be $\{v_1^i, v_2^i, \ldots, v_m^i\}$, $1 \leq i \leq 2k+1$. Let $\mathcal{C}$ be a C_{2k-1}-decomposition of $C_{2k-1} \times K_m$. Now we obtain a C_{2k+1}-decomposition $\mathcal{C}'$ of $C_{2k+1} \times K_m$ as follows:

$$\mathcal{C}' = \left\{ (v_{j_1}^1, v_{j_2}^2, \ldots, v_{j_{2k-2}}^{2k-2}, v_{j_{2k-1}}^{2k-1}, v_{j_{2k-2}}^{2k}, v_{j_{2k-1}}^{2k+1}) \mid (u_{j_1}^1, u_{j_2}^2, \ldots, u_{j_{2k-2}}^{2k-2}, u_{j_{2k-1}}^{2k-1}) \in \mathcal{C} \right\}.$$

Clearly, $\mathcal{C}'$ is a C_{2k+1}-decomposition of $C_{2k+1} \times K_m$. This completes the proof. ■

Lemma 2.8 *For* $m \geq 3,\ C_3 \,|\, C_3 \times K_m$.

Proof. The triangles corresponding to the non-diagonal entries of an idempotent latin square $\mathcal{L}$ decompose $C_3 \times K_m$, see Remark 2.5. An idempotent latin squares of order $m,\ m \neq 2$, exists [10] and hence $C_3 \,|\, C_3 \times K_m,\ m \geq 3$. ■

Lemma 2.9 *For* $k \geq 1$ *and* $m \geq 3,\ C_{2k+1} \,|\, C_{2k+1} \times K_m$.

Proof. Proof follows from Lemma 2.8 and successive application of Lemma 2.7. ■

3 C_5-Decomposition of $K_m \times K_n$

The following theorem can be found in [10].

Theorem 3.1 *Let* m *be an odd integer and* $m \geq 3$.
(1) *If* $m \equiv 1$ *or* 3 $(mod\ 6)$, *then* $C_3 \,|\, K_m$.
(2) *If* $m \equiv 5$ $(mod\ 6)$, *then* K_m *can be decomposed into* $(m(m-1)-20)/6$ *3-cycles and two 5-cycles.* ■

Proof of Theorem 1.1. The proof of the necessity is obvious and we prove the sufficiency in two cases. Since the tensor product is commutative, we may assume that m is odd and so $m \equiv 1, 3$ or 5 (mod 6).

Case 1. $n \equiv 0$ or 1 (mod 5).

Subcase 1.1. $m \equiv 1$ or 3 (mod 6).
By Theorem 3.1, $C_3 \,|\, K_m$ and hence $K_m \times K_n = (C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \ldots \oplus (C_3 \times K_n)$. By Theorem 2.6, $C_5 \,|\, C_3 \times K_n$ and hence $C_5 \,|\, K_m \times K_n$.

Subcase 1.2. $m \equiv 5$ (mod 6).
By Theorem 3.1, $K_m = \underbrace{C_3 \oplus C_3 \oplus \ldots \oplus C_3}_{(m(m-1)-20)/6-\ times} \oplus (C_5 \oplus C_5)$.

Now $K_m \times K_n = \Big((C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \ldots \oplus (C_3 \times K_n) \Big) \oplus \Big((C_5 \times K_n) \oplus (C_5 \times K_n) \Big)$.
Since $C_5 \,|\, C_3 \times K_n$, by Theorem 2.6, and $C_5 \,|\, C_5 \times K_n$, by Lemma 2.9, $C_5 \,|\, K_m \times K_n$.

Case 2. $n \not\equiv 0$ (mod 5) and $n \not\equiv 1$ (mod 5).
Since $n(n-1) \not\equiv 0$ (mod 5), condition (1) implies that $m \equiv 0$ or 1 (mod 5). As m is odd we have $m \equiv 1$ or 5 (mod 10). Because $C_5 \,|\, K_m$, by Theorem 2.4, $K_m \times K_n = (C_5 \times K_n) \oplus (C_5 \times K_n) \oplus \ldots \oplus (C_5 \times K_n)$. Now $C_5 \,|\, C_5 \times K_n$, by Lemma 2.9, and so $C_5 \,|\, K_m \times K_n$. This completes the proof. ■

We do not supply the proof of Theorem 1.2 since it is similar to the proof of the following Theorem 3.2 given in [14]. Further, Billington et al. [5] have obtained a stronger result than this, namely, the necessary conditions for the existence of a C_5-decomposition of$(K_m * \overline{K}_n)(\lambda)$ are also sufficient.

Theorem 3.2 [14]. *For a prime* $p \geq 11$ *and* $m \geq 3$, $C_p \,|\, K_m * \overline{K}_n$ *if and only if* (1) $n(m-1)$ *is even and* (2) $p \,|\, m(m-1)n^2$. ■

Although the proof of Theorem 1.1 has independent interest, the proof of Theorem 1.2 (similar to the proof of Theorem 3.2 of [14]) explains the effectiveness of Theorem 1.1 in completing the proof of Theorem 1.2. From this paper and [13] and [14], we conclude that the C_k-cycle decomposition problem of $K_m \times K_n$ may shed some light on C_k-cycle decomposition problem of $K_m * \overline{K}_n$.

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