

The unimodality of independence polynomials of some graphs

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Abstract

Levit and Mandrescu showed that independence polynomials of centipedes are unimodal and further conjectured that polynomials have only real zeros. In the present paper we verify this conjecture. And we also show that the independence polynomials of caterpillars are unimodal.

1 Introduction

Let $G = (V, E)$ be a simple graph with the vertex set V and the edge set E . An *independent set* in G is a set of pairwise non-adjacent vertices. The *independence number* of G , denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G . Let $i_k(G)$ denote the number of independent sets of cardinality k in G . The polynomial

$$\sum_{k=0}^{\alpha(G)} i_k(G)x^k, \quad i_0 = 1$$

is called the *independence polynomial* of G , denoted by $I(G, x)$ or $G(x)$.

Let a_0, a_1, \dots, a_n be a sequence of nonnegative numbers. It is *unimodal* if there is some m , called the *mode* of the sequence, such that

$$a_0 \leq a_1 \leq \dots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \dots \geq a_n.$$

It is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n-1$. It is *symmetric* if $a_i = a_{n-i}$ for $0 \leq i \leq n$. Clearly, a log-concave sequence of positive terms is unimodal and a symmetric unimodal sequence has its maximum at the middle terms.

We say that a polynomial $\sum_{k=0}^n a_k x^k$ is *unimodal* if its coefficients a_0, a_1, \dots, a_n form a unimodal sequence. The mode of the sequence a_0, a_1, \dots, a_n is also called the *mode* of the polynomial $\sum_{k=0}^n a_k x^k$. The following result is well-known (see [7, p.104] for instance).

Newton Inequality. Let a_0, a_1, \dots, a_n be a sequence of nonnegative numbers. Suppose that its generating function $\sum_{k=0}^n a_k x^k$ has only real zeros. Then

$$a_k^2 \geq a_{k-1} a_{k+1} \frac{k+1}{k} \frac{n-k+1}{n-k}, \quad k = 1, 2, \dots, n-1.$$

(The sequence is therefore log-concave and unimodal with at most two modes.)

The positive matching polynomial of a graph G can be, in a certain sense, regarded as the independence polynomial of the line graph of G [5, Proposition 1]. It is well-known that the matching polynomial of any graph has only real zeros [8, 11]. Wilf asked whether independence polynomials also enjoy the same property. Hamidoune [6] showed that the independence polynomial of claw-free graph is unimodal (a graph is claw-free if it has no subgraph isomorphic to $K_{1,3}$). Chudnovsky and Seymour [4] showed further that the independence polynomials of claw-free graphs have only real zeros. However, Alavi et al. [1] provided examples to demonstrate that independence polynomials are not even unimodal in general. They also proposed the following conjecture.

Conjecture 1. [1] *The independence polynomial of any tree or forest is unimodal.*

So far, the conjecture has been proved for some special classes of trees. For example, Levit and Mandrescu [9] settled the conjecture for centipedes. A *centipede* W_n is a tree with the vertex set $V = A \cup B$, where $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$, $A \cap B = \emptyset$, and the edge set $E = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n-1\}$ (see Fig. 1). They further conjectured that the independence polynomials of centipedes have only real zeros. The first object of the present paper is to verify this conjecture.

Theorem 1. *The independence polynomials of centipedes have only real zeros.*

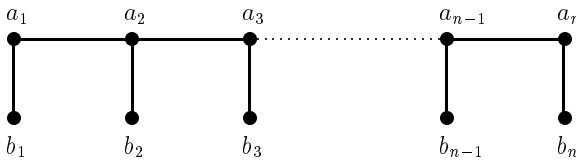


Fig. 1. The centipede W_n

The second object of this paper is to settle Conjecture 1 for caterpillars. A *caterpillar* H_n is a tree with the vertex set $V = A \cup (\cup_{i=1}^n B_i)$ where $A = \{a_1, \dots, a_n\}$, $B_i = \{b_1^{(i)}, b_2^{(i)}\}$, and the edges set $E = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_i b_j^{(i)} : 1 \leq i \leq n, 1 \leq j \leq 2\}$ (see Fig. 2). Our result is the following.

Theorem 2. *The independence polynomials of caterpillars are symmetric and unimodal.*

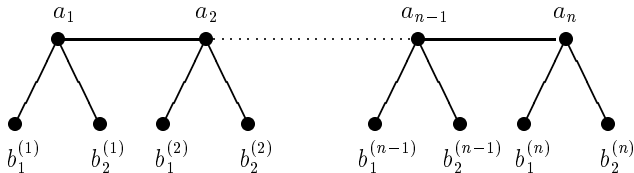


Fig. 2. The caterpillar H_n

2 Proof of Theorem 1

To prove Theorem 1, we need the following results.

Lemma 1. (Levit and Mandrescu [9]) *Let $W_n(x)$ be the independence polynomial of the centipede W_n . Then $W_n(x)$ satisfies the recurrence relation*

$$W_n(x) = (1 + x)[W_{n-1}(x) + xW_{n-2}(x)],$$

with $W_0(x) = 1$ and $W_1(x) = 1 + 2x$.

Let $f(x)$ and $g(x)$ be two real polynomials with $\deg f = n - 1$ and $\deg g = n$. Suppose that both $f(x)$ and $g(x)$ have only real zeros and that the zeros r_1, \dots, r_{n-1} of $f(x)$ and the zeros s_1, \dots, s_{n-1}, s_n of $g(x)$ satisfy

$$s_n \leq r_{n-1} \leq s_{n-1} \leq \dots \leq s_2 \leq r_1 \leq s_1.$$

Then we say that $f(x)$ interlaces $g(x)$.

Lemma 2. (Heilmann and Lieb [8]) *Let $f(x), g(x)$ be two real polynomials with positive leading coefficients and $\deg f = \deg g - 1$. Suppose that both $f(x)$ and $g(x)$ have only real zeros and that $f(x)$ interlaces $g(x)$. Then the polynomial $f(x) + g(x)$ has only real zeros. Moreover, the zeros s_1, \dots, s_n of $g(x)$ and the zeros t_1, \dots, t_n of $f(x) + g(x)$ satisfy*

$$t_n \leq s_n \leq t_{n-1} \leq s_{n-1} \leq \dots \leq t_2 \leq s_2 \leq t_1 \leq s_1.$$

Proof of Theorem 1. Let $J_n(x) = x^n W_n(1/x)$ be the reciprocal polynomial of $W_n(x)$. It follows from Lemma 1 that the sequence $\{J_n(x)\}$ satisfies the recurrence relation

$$J_n(x) = (x + 1)[J_{n-1}(x) + J_{n-2}(x)] \tag{1}$$

with $J_0(x) = 1$ and $J_1(x) = x + 2$. Obviously, $J_n(x)$ is a monic polynomial of degree n . It is also clear that $W_n(x)$ has only real zeros if and only if $J_n(x)$ does. So, to show the statement, it suffices to show that $J_n(x)$ has only real zeros. In what follows, we show that for $n \geq 1$, $J_n(x)$ has only real zeros and J_{n-1} interlaces $J_n(x)$. We proceed by induction on n .

It is obvious that $J_1(x)$ interlaces $J_2(x)$ since $J_2(x) = (x+1)(x+3)$ by (1). Now assume that both $J_{n-2}(x)$ and $J_{n-1}(x)$ have only real zeros and that $J_{n-2}(x)$ interlaces $J_{n-1}(x)$. Then $J_{n-1}(x) + J_{n-2}(x)$ has only real zeros by Lemma 2, and so does $J_n(x)$ by (1). By Lemma 2 again, the zeros s_1, \dots, s_{n-1} of $J_{n-1}(x)$ and the zeros t_1, \dots, t_{n-1} of $J_{n-1}(x) + J_{n-2}(x)$ satisfy

$$t_{n-1} \leq s_{n-1} \leq t_{n-2} \leq s_{n-2} \leq \dots \leq s_2 \leq t_1 \leq s_1. \quad (2)$$

However, $J_n(x) > 0$ for $x > -1$ from the recurrence relation (1). Hence -1 is the largest zero of $J_n(x)$ for $n \geq 2$. Thus $J_{n-1}(x)$ interlaces $J_n(x)$ by (2), as desired. \square

Remark 1. Let G be a forest all of whose components are centipedes. Then the independence polynomial of G is the product of the independence polynomials of such centipedes. It follows from Theorem 1 that $G(x)$ has only real zeros. Thus $G(x)$ is unimodal. It is worth noticing that this result cannot be followed from Levit-Mandrescu's result about the unimodality of independence polynomials of centipedes.

3 Proof of Theorem 2

Let $G = (V, E)$. For $X \subset V$, we denote by $G[X]$ the subgraph of G spanned by X , and by $G - X$ the subgraph $G[V - X]$. For $v \in V$, let $N[v] = \{w : w \in V \text{ and } vw \in E\} \cup \{v\}$.

The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \sqcup G_2$ with a vertex set the disjoint union of $V(G_1), V(G_2)$ and an edge set the disjoint union of $E(G_1), E(G_2)$. It is well-known that

$$I(G_1 \sqcup G_2, x) = I(G_1, x) \cdot I(G_2, x) \quad (3)$$

and

$$I(G, x) = I(G - \{v\}, x) + xI(G - N[v], x), \quad (4)$$

See [3, 5] for more properties of independence polynomials of graphs.

Lemma 3. *The independence polynomial $H_n(x)$ of the caterpillar H_n satisfies the recurrence relation:*

$$H_n(x) = (1+x)^2[H_{n-1}(x) + xH_{n-2}(x)], \quad (5)$$

with $H_0(x) = 1$ and $H_1(x) = x + (1+x)^2$.

Proof. By definition we have $H_0(x) = 1$ and $H_1(x) = 1 + 3x + x^2$. Now let $n \geq 2$. Then it is clear that $H_n - \{a_n\} = H_{n-1} \sqcup \overline{K_2}$ and $H_n - N[a_n] = H_{n-2} \sqcup \overline{K_2}$ (see Fig. 2), where $\overline{K_2}$ is the empty graph of order 2. Note that $I(\overline{K_2}, x) = (1+x)^2$. Hence we have by (3) and (4)

$$\begin{aligned} I(H_n, x) &= I(H_n - \{a_n\}, x) + xI(H_n - N[a_n], x) \\ &= I(H_{n-1} \sqcup \overline{K_2}, x) + xI(H_{n-2} \sqcup \overline{K_2}, x) \\ &= (1+x)^2 I(H_{n-1}, x) + x(1+x)^2 I(H_{n-2}, x) \\ &= (1+x)^2 [I(H_{n-1}, x) + xI(H_{n-2}, x)]. \end{aligned}$$

Thus the proof is complete. \square

Remark 2. From (5) and by induction, we have $\deg H_n(x) = 2n$, which can also be obtain by definition.

Lemma 4. (Andrews [2]) *If $p(x)$ and $q(x)$ are two polynomials with nonnegative, symmetric and unimodal coefficients, then the same is true of their product.*

Proof of Theorem 2. We use induction on n . The result is obviously true for $n = 0, 1$ since $H_0(x) = 1$ and $H_1(x) = 1 + 3x + x^2$. Now suppose it to be true for $n \leq k - 1$ and consider the case $n = k$.

Let $H_{k-1}(x) = \sum_{i=0}^{2k-2} a_i x^i$ and $H_{k-2}(x) = \sum_{i=0}^{2k-4} b_i x^i$. Then by the induction hypothesis, $H_{k-1}(x)$ and $H_{k-2}(x)$ are symmetric and unimodal:

$$a_0 = a_{2k-2} \leq a_1 = a_{2k-3} \leq \dots \leq a_{k-2} = a_k \leq a_{k-1}$$

and

$$b_0 = b_{2k-4} \leq b_1 = b_{2k-5} \leq \dots \leq b_{k-3} = b_{k-1} \leq b_{k-2}.$$

Thus the polynomial

$$H_{k-1}(x) + xH_{k-2}(x) = a_0 + \sum_{i=1}^{2k-3} (a_i + b_{i-1})x^i + a_{2k-2}x^{2k-2}$$

is symmetric and unimodal. It is well-known that the polynomial $(1+x)^2$ is symmetric and unimodal. Hence $H_k(x) = (1+x)^2[H_{k-1}(x) + xH_{k-2}(x)]$ is also symmetric and unimodal by Lemma 4. This completes the proof of the theorem. \square

4 Remarks

A *vertebrate* $V_n^{(m)}$ is a tree with the vertex set $V = A \cup (\bigcup_{i=1}^n B_i)$, where $A = \{a_1, \dots, a_n\}$, $B_i = \{b_1^{(i)}, \dots, b_m^{(i)}\}$, and the edges set $E = \{a_i a_{i+1} : 1 \leq i \leq n-1\} \cup \{a_i b_j^{(i)} : 1 \leq i \leq n, 1 \leq j \leq m\}$.

For $m = 0, 1, 2$, $V_n^{(m)}$ is the path P_n with n vertices, the centipede W_n and the caterpillar H_n respectively. It is known that their independence polynomials are unimodal (see [6, Proposition 2.5], [9, Theorem 2.5] and Theorem 2).

By the same method used in the proof of Lemma 3, we obtain that the independence polynomials of vertebrates satisfy the recurrence relation

$$V_n^{(m)}(x) = (1+x)^m [V_{n-1}^{(m)}(x) + xV_{n-2}^{(m)}(x)] \tag{6}$$

with $V_0^{(m)}(x) = 1$ and $V_1^{(m)}(x) = x + (1+x)^m$.

We can also give an explicit expression of the number $i_k(V_n^{(m)})$ of independent sets of cardinality k in $V_n^{(m)}$ by the following method. For each k , select k independent elements from the vertex set of size $(n+1)m$ of $V_n^{(m)}$ in a two-stage process. First, let us

choose j independent elements from the n -set $\{a_1, \dots, a_n\}$. Then select the remaining $(k - j)$ independent elements from the nm -set $\{b_1^{(1)}, \dots, b_m^{(1)}, \dots, b_1^{(n)}, \dots, b_m^{(n)}\}$. The number of ways in which we make two choices are $\binom{n-j+1}{j}$ and $\binom{m(n-j)}{k-j}$ respectively. So, the total number of ways of ending up with k -independent sets from the $(n + 1)m$ -set by this process is

$$i_k(V_n^{(m)}) = \sum_{j=0}^k \binom{n-j+1}{j} \binom{m(n-j)}{k-j}. \quad (7)$$

In the case $m = 2$, we have deduced that $V_n^{(m)}(x)$ is unimodal from (6) (Theorem 2). This result can also be followed directly from (7). However, for $m \neq 2$ it is not easy to show that $V_n^{(m)}(x)$ is unimodal from (6) and (7), even for $m = 1$ ([9]). In fact, the independence polynomials of the vertebrates with $m > 2$ have not only real zeros, so we must be satisfied with unimodality.

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