The Intersection Problem for Minimum Coverings of K_n by Triples

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1 Introduction.

A *Steiner triple system* (more simply, triple system) is a pair (S, T) where S is the vertex set of the complete undirected graph K_n on n vertices and T is a collection of edge-disjoint triangles (triples) which partition the edge set of K_n. The *number* n is called the *order* of the triple system (S, T) and it has been known forever (= since 1847 [2]) that the spectrum of triple systems (= set of all n such that a triple system of order n exists) is precisely the set of all $n \equiv 1$ or $3 \pmod 6$. In this case $|T| = n(n-1)/6$.

In [3] C. C. Lindner and A. Rosa gave a complete solution of the *intersection problem* for triple systems by determining all pairs (n, k) such that there exists a pair of triple systems (S, T_1) and (S, T_2) of order n such that $|T_1 \bigcap T_2| = k$. In particular, if we set $I[n] = \{k|$ there exist a pair of triple systems of order n intersecting in k triples $\}$, then $I[3] = \{1\}, I[7] = \{0, 1, 3, 7\}, I[9] = \{0, 1, 2, 3, 4, 6, 12\}$ and for $n \geq 13, I[n] = \{0, 1, 2, \ldots, n(n-1)/6 = t\} \backslash \{t-1, t-2, t-3, t-5\}$.

A *maximum packing* of K_n with triples (MPT) is a pair (S, P), where S is the vertex set of K_n and P is a collection of edge-disjoint triangles (or triples) of the edge set of K_n so that $|P|$ is a large as possible. As with triple systems, the number n is called the *order*. The collection of unused edges is called the *leave* of the MPT (S, P). So, a triple system is a MPT with leave $L = \emptyset$. Just as with triple systems, nonisomorphic MPTs are like grains of sand on the beach. However, MPTs of the same order all have one thing in common; the leave! In particular if (S, P) is a MPT of order n, then the leave is (i) a 1-factor if $n \equiv 0$ or

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2 (mod 6), (ii) a 4-cycle if $n \equiv 5 \pmod 6$, and (iii) a *tripole* = a spanning graph with each vertex having odd degree and containing $(n+2)/2$ edges if $n \equiv 4 \pmod 6$. The intersection problem for MPTs is the following: Determine all pairs (n, k) such that there exists a pair of MPTs (S, P_1) and (S, P_2) of order n *with the same leave* such that $|P_1 \bigcap P_2| = k$. So, the intersection problem for triple systems is the intersection problem for MPTs with leave the empty set. The intersection problem for MPTs has been solved completely in a series of two papers [1, 4]. In particular, if (i) $n \equiv 0$ or $2 \pmod 6$, $I[6] = \{0, 4\}, I[8] = \{0, 2, 8\}$, and $I[n] = \{0, 1, 2, \ldots, n(n-2)/6 = t\} \backslash \{t-1, t-2, t-3, t-5\}$ for $n \geq 12$; (ii) $n \equiv 4 \pmod 6$, $I[4] = \{1\}$ and $I[n] = \{0, 1, 2, \ldots, (n^2 - 2n - 2)/6 = t\} \backslash \{t-1, t-2, t-3, t-5\}$ for $n \geq 10$; and (iii) $n \equiv 5 \pmod 6$, $I[5] = \{2\}$ and $I[n] = \{0, 1, 2, \ldots, (n^2 - n - 8)/6 = t\} \backslash$
$\{t-1, t-2, t-3, t-5\}$ for $n \geq 11$.

The purpose of this paper is to give a complete solution of the intersection problem for *minimum coverings* of K_n with triples. Some preliminaries are in order. Quite a few actually.

2 Coverings.

A *covering* of K_n (with *padding* X) with triples is a pair (S, C), where S is the vertex set of K_n and C is a collection of edge disjoint triples which *partition* $E(K_n) + X$, where $X \subseteq (E(\lambda K_n))$. So that there is no confusion: an edge $\{a, b\}$ belongs to exactly $x + 1$ triples of C, where x is the number of times $\{a, b\}$ belongs to the padding X. If the padding X is as small as possible, then (S, C) is called a *minimum* covering of K_n with triples (MCT). So, a Steiner triple system is a MCT with padding $X = \emptyset$.

Example 2.1 (MCTs). (1) $n = 4$ with padding $X = \{\{1,2\}, \{2,3\}, \{2,4\}\}$ and $C = \{\{1,2,3\}, \{2,3,4\}, \{1,2,4\}\}$.

(2) $n = 5$ with padding $X = \{\{1,2\}, \{1,2\}\}$ and $C = \{\{1,2,3\}, \{1,2,4\}, \{1,2,5\}, \{3,4,5\}\}$.

(3) $n = 6$ with padding $X = \{\{1,2\}, \{3,4\}\{5,6\}\}$ and $C = \{\{1,2,3\}, \{3,4,5\}, \{1,5,6\}, \{1,2,4\}, \{3,4,5\}, \{2,5,6\}\}$.

(4) $n = 8$ with padding $X = \{\{1,2\}, \{1,3\}, \{1,4\}, \{5,6\}, \{7,8\}\}$ and $C = \{\{1,2,7\}, \{1,4,5\}, \{3,5,6\}, \{1,2,3\}, \{2,4,8\}, \{5,7,8\}, \{1,3,8\}, \{2,5,6\}, \{6,7,8\}, \{1,4,6\}, \{3,4,7\}\}$.

Just as the case for MPTs, the padding of a MCT is determined by its order. In particular, if (S, C) is a MCT of order n, then the padding is (i) a 1-factor if $n \equiv 0 \pmod 6$, (ii) a tripole if $n \equiv 2$ or $4 \pmod 6$, and (iii) a double edge $= \{\{a,b\},\{a,b\}\}$ if $n \equiv 5 \pmod 6$. The intersection problem for MCTs is the following: Determine all pairs (n, k) such that there exists a pair of MCTs (S, C_1) and (S, C_2) of order n with *the same padding* such that $|C_1 \cap C_2| = k$. We will give a *complete solution* of this problem. In particular, if (i) $n \equiv 0 \pmod 6$, $I[n] = \{0,1,2,\ldots,n^2/6 = t\} \backslash \{t-1,t-2,t-3,t-5\}$; (ii) $n \equiv 2$ or $4 \pmod 6$, $I[4] = \{3\}$ and $I[n] = \{0,1,2,\ldots,(n^2+2)/6 = t\} \backslash \{t-1,t-2,t-3,t-5\}$ for $n \geq 8$; and (iii) $n \equiv 5 \pmod 6$, $I[5] = \{4\}$ and $I[n] = \{0,1,2,\ldots,(n^2-n+4)/6 = t\} \backslash \{t-1,t-2,t-3,t-5\}$ for $n \geq 11$. The fact that $|C_1 \cap C_2| \in J[n] = \{0,1,2,\ldots,(n(n-2)+2|X|) \backslash 6 = t\}$ $\backslash \{t-1,t-2,t-3,t-5\}$ is trivial and is left to the reader. It's the non-trivial converse we will concern ourselves with here; i.e., $J[n] \subseteq I[n]$.

Finally, as might be expected, we will handle the cases according to the padding.

3 $n \equiv 0 \pmod 6$.

We begin with an example. In this section $J[n] = \{0,1,2,\ldots,n^2/6 = t\} \backslash \{t-1,t-2,t-3,t-5\}$.

Example 3.1 MCTs of order 6 with padding $X = \{\{1,2\},\{3,4\},\{5,6\}\}$:

$$C_1 = \left\{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 4 & 5 \\ 3 & 4 & 6 \\ 2 & 5 & 6 \\ 1 & 5 & 6 \end{array}\right. \qquad C_2 = \left\{\begin{array}{ccc} 1 & 2 & 5 \\ 1 & 2 & 6 \\ 1 & 3 & 4 \\ 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 5 & 6 \end{array}\right. \qquad C_3 = \left\{\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 4 & 6 \\ 2 & 5 & 6 \\ 3 & 5 & 6 \end{array}\right.$$

Then $|C_1 \cap C_2| = 0$, $|C_1 \cap C_3| = 2$, and (of course) $|C_1 \cap C_1| = 6$, so that $J[6] = \{0,2,6\} \subseteq I[6]$.

With the above example in hand we can now concern ourselves with the case where $n \equiv 0 \pmod 6 \geq 12$. The following lemma reduces our work considerably.

Lemma 3.2 $J[n] \backslash \{0,1,2,\ldots,n/2\} \subseteq I[n]$.

Proof: Write $n-1 = 6m+5$ and let $S = \{1,2,3,4,5\} \bigcup \{x_1, x_2, \ldots, x_{3m}\} \bigcup \{y_1, y_2, \ldots, y_{3m}\}$. Let (S, P_1) and (S, P_2) be a pair of MPTs of order $n-1 = 6m+5$ with leave the 4-cycle $L = (1,2,3,4)$. Now let T^* be the collection of $3m+4$ triples defined by $T^* = \{\{2,3,4\}, \{\infty,1,4\}, \{\infty,1,2\}, \{\infty,3,5\}, \{\infty,x_1,y_1\}, \{\infty,x_2,y_2\}, \ldots, \{\infty,x_{3m},y_{3m}\}\}$. Then $(S^*, P_1 \bigcup T^*)$ and $(S^*, P_2 \bigcup T^*)$ are MCTs of order n with padding
$X = \{\{\infty,1\},\{2,4\},\{3,5\},\{x_1,y_1\},\{x_2,y_2\},\ldots,\{x_{3m},y_{3m}\}\}$ where $S^* = \{\infty\} \bigcup S$. Then $|(P_1 \bigcup T^*) \bigcap (P_2 \bigcup T^*)| = k + 3m + 4 = k + (n/2+1)$, where $k \in I_p[n-1]$ = the intersection spectrum for MPTs for order $n-1$. An easy calculation shows that $I_p[n-1] + \{n/2+1\} = J[n] \backslash \{0,1,2,\ldots,n/2\} \subseteq I[n]$. □

It remains to show that $\{0,1,2,3,\ldots,n/2\} \subseteq I[n]$ for every $n \equiv 0 \pmod 6 \geq 12$. The following constructions will do the job. But first a few preliminaries.

The *partial* triple systems (S, T_1) and (S, T_2) are said to be *mutually balanced* provided T_1 and T_2 cover precisely the same edges and are *disjoint* provided $T_1 \bigcap T_2 = \emptyset$.

Lemma 3.3 *(Stern and Lenz [5]). Let $n \geq 4$ and let S be the vertex set of K_{2n}. Then there exist mutually balanced disjoint partial triple systems (S, T_1) and (S, T_2) of order $2n$ such that $|T_1| = |T_2| = 2n$ and such that there exists a 1-factorization $F = \{f_1, f_2, f_3, \ldots, f_{2n-7}\}$ of $E(K_{2n}) \backslash T_1 = E(K_{2n}) \backslash T_2$.* □

We will call such a 1-factorization a *Stern and Lenz 1-factorization.*

The $12n$ Construction. Let (S, C) be a MCT of order $6n$ with padding X and $F = \{f_1, f_2, \ldots, f_{6n-1}\}$ a 1-factorization of $E(K_{6n})$ based on W, where $S \bigcap W = \emptyset$. Let α be a mapping from S *onto* F. Then $x\alpha = y\alpha$ for *exactly one* 2-element subset $\{x, y\}$ of S. Without loss in generality we can assume $x\alpha = y\alpha = f_1$. Define a collection of triples C^* as follows:

(1) $C \subseteq C^*$, and

(2) $\{x, y, z\} \in C^*$ if and only if $x\alpha = f_i$ and $\{y, z\} \in f_i$.

Then $(S \bigcup W, C^*)$ is a MCT of order $12n$ with padding $X \bigcup f_1$. □

The $12n+6$ Construction. Let (S, C) be a MCT of order $6n$ with padding X and $F = \{f_1, f_2, \ldots, f_{6n-1}\} \bigcup T$ a Stern and Lenz 1-factorization of $E(K_{6n+6})$ based on W, where $S \bigcap W = \emptyset$. Define α as in the $12n$ Construction and define a collection of triples C^* by:

(1) $C \subseteq C^*$,

(2) $T \subseteq C^*$, and

(3) $\{x, y, z\} \in C^*$ if and only if $x\alpha = f_i$ and $\{y, z\} \in f_i$.

Then $(S \bigcup W, C^*)$ is a MCT of order $12n + 6$ with padding $X \bigcup f_1$. □

Lemma 3.4 $\{0, 1, 2, \ldots, n/2\} \subseteq I[n]$.

Proof: The cases $n = 12$ and 18 are taken care of in the appendix, so we can assume $n \geq 24$ and that $J[t] \subseteq I[t]$ for all $12 \leq t < n$. Write $n = 12m$ or $12m + 6$ and let (S, C_1) and (S, C_2) be a pair of MCTs of order $6m$ with padding X with intersection number $k \in \{0, 1, 2, \ldots, n/2\}$. (Since $6m \geq 12, (6m)^2/6 - 6 \geq 6m + 3 \geq n/2$. Hence $k \in I[6m]$.) There are two cases to consider.

$n = 12m$. Let F be a 1-factorization of $E(K_{6m})$ based on W, $W \bigcap S = \emptyset$, and let α and β be defined as above with the additional property that $x\alpha \neq x\beta$ for all $x \in S$. Let $(S \bigcup W, C_1^*)$ and $(S \bigcup W, C_2^*)$ be constructed from the $12n$ Construction using α for C_1^* and β for C_2^*. Then $(S \bigcup W, C_1^*)$ and $(S \bigcup W, C_2^*)$ are a pair of MCTs of order $12m$ with padding $X \bigcup f_1$ and intersection number k.

$n = 12m + 6$. Let $F_1 = \{f_1, f_2, \ldots, f_{6m-1}\} \bigcup T_1$ and $F_2 = \{f_1, f_2, \ldots, f_{6m-1}\} \bigcup T_2$ be Stern and Lenz 1-factorizations of $E(K_{6m+6})$ based on W, $W \bigcap S = \emptyset$, where T_1 and T_2 are mutually balanced and disjoint partial triple systems. Define α and β as above and let $(S \bigcup W, C_1^*)$ and $(S \bigcup W, C_2^*)$ be constructed from the $12n + 6$ Construction using α and T_1 for C_1^* and β and T_2 for C_2^*. Then $(S \bigcup W, C_1^*)$ and $(S \bigcup W, C_2^*)$ are a pair of MCTs of order $12m + 6$ with padding $X \bigcup f_1$ and intersection number k.

Combining the above two cases completes the proof. □

Combining Example 3.1 and Lemmas 3.2 and 3.4 gives the following theorem.

Theorem 3.5 *$I[n] = J[n]$ for all $n \equiv 0$ (mod 6).* □

4 $n = 2$ or 4 (mod 6).

It is trivial to see that $I[4] = \{3\}$. So, in what follows $n \equiv 2$ or 4 (mod 6) ≥ 8 and, of course, $J[n] = \{0, 1, 2, 3, \ldots, (n^2 + 2)/6 = t\} \backslash \{t-1, t-2, t-3, t-5\}$.

Lemma 4.1 $J[n] \backslash \{0, 1, 2, \ldots, n/2 - 1\} \subseteq I[n]$.

Proof: The proof is similar to the proof of Lemma 3.2. Since $n \equiv 2$ or $4 \pmod 6$, $n - 1 \equiv 1$ or $3 \pmod 6$ which is the order of a Steiner triple system. So, write $n - 1 = 3 + 2m$ and let $S = \{1, 2, 3\} \bigcup \{x_1, x_2, \ldots, x_m\} \bigcup \{y_1, y_2, \ldots, y_m\}$. Let (S, T_1) and (S, T_2) be a pair of Steiner triple systems of order $n - 1 = 3 + 2m$ and let T^* be the collection of $m + 2$ triples defined by $T^* = \{\{\infty, 1, 2\}, \{\infty, 2, 3\}\}, \{\infty, x_1, y_1\}, \{\infty, x_2, y_2\}, \ldots, \{\infty, x_m, y_m\}\}$. Then $(S^*, T_1 \bigcup T^*)$ and $(S^*, T_2 \bigcup T^*)$ are MCTs of order n with padding the tripole $X = \{\{2, \infty\}, \{2, 1\}, \{2, 3\}, \{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}\}$ where $S^* = \{\infty\} \bigcup S$. Then $|(T_1 \bigcup T^*) \bigcap (T_2 \bigcup T^*)| = k + m + 2 = k + n/2$, where $k \in I_p[n - 1] =$ the intersection spectrum for Steiner triple systems of order $n - 1$. It is straight forward to see that $I_p[n - 1] + \{n/2\} = J[n] \backslash \{0, 1, 2, \ldots, n/2 - 1\} \subseteq I[n]$ *except* for $n = 10$. In this case $I_p[9] + \{5\} = \{0, 1, 2, 3, 4, 6, 12\} + \{5\} = \{5, 6, 7, 8, 9, 11, 17\}$. However, the cases $k = 10$ and 13 are taken care of by example in the appendix. □

The $12n + (4$ or $8)$ Construction. Let (S, C) be a MCT of order $6n + 2$ or $6n + 4$ with padding the tripole X. Let F be a 1-factorization of $E(K_{6n+2})$ or $E(K_{6n+4})$ based on W, $W \bigcap S = \emptyset$. If we define $(S \bigcup W, C^*)$ as in the $12n$ Construction, then $(S \bigcup W, C^*)$ is a MCT of order $12n + (4$ or $8)$ with padding the tripole $X \bigcup f_1$. □

The $12n + (2$ or $10)$ Construction. Let (S, C) be a MCT of order $6n - 2$ or $6n + 2$ with padding X. Let $F \bigcup T$ be a Stern and Lenz 1-factorization of $E(K_{6n+4})$ or $E(K_{6n+8})$ based on W, $W \bigcap S = \emptyset$. If we define $(S \bigcup W, C^*)$ as in the $12n + 6$ Construction, then $(S \bigcup W, C^*)$ is a MCT of order $12n + (2$ or $10)$ with padding the tripole $X \bigcup f_1$. □

Lemma 4.2 $\{0, 1, 2, \ldots, n/2 - 1\} \subseteq I[n]$.

Proof: The cases $n = 8, 10, 14$, and 16 are handled in the appendix so that we can assume $n \geq 20$ and that $J[t] \subseteq I[t]$ for all $8 \leq t < n$. There are two cases to consider.

$n = 12m + (4$ or $8)$. Let (S, C_1) and (S, C_2) be a pair of MCTs of order $6m + (2$ or $4)$ with padding the tripole X with intersection number $k \in \{0, 1, 2, \ldots, n/2 - 1\}$. (Since $n \geq 8$, $((6m + 2)^2 + 2)/6 - 6 \geq 6m + 3 \geq n/2 - 1$. Hence $k \in I[6m + (2$ or $3)]$.) Let F be a 1-factorization of $E(K_{6m+(2 \text{ or } 4)})$ based on W, $W \bigcap S = \emptyset$, and let α and β be defined in the usual way. Let $(S \bigcup W, C_1^*)$ and $(S \bigcup W, C_2^*)$ be constructed from the $12n + (4$ or $8)$

Construction using α for C_1^* and β for C_2^*. Then $(S\bigcup W, C_1^*)$ and $(S\bigcup W, C_2^*)$ are a pair of MCTs of order $12m+(4 \text{ or } 8)$ with padding the tripole $X\bigcup f_1$ and intersection number k.

$n = 12m+(2 \text{ or } 10)$. Let (S, C_1) and (S, C_2) be a pair of MCTs of order $6m+(-2 \text{ or } 2)$ with padding the tripole X and intersection number $k \in \{0, 1, 2, \ldots, n/2-1\}$. (An argument similar to the argument in the $12m+(4 \text{ or } 8)$ case shows this is always possible.) Let $F\bigcup T_1$ and $F\bigcup T_2$ be Stern and Lenz 1-factorizations of $E(K_{6m+(4 \text{ or } 8)})$ based on W, $W\bigcap S = \emptyset$, where T_1 and T_2 are mutually balanced and disjoint partial triple systems. Define α and β as above and let $(S\bigcup W, C_1^*)$ and $(S\bigcup W, C_2^*)$ be constructed from the $12n+(2 \text{ or } 10)$ Construction using α and T_1 for C_1^* and β and T_2^* for C_2^*. Then $(S\bigcup W, C_1^*)$ and $(S\bigcup W, C_2^*)$ are a pair of MCTs of order $12m+(2 \text{ or } 10)$ with padding the tripole $X\bigcup f_1$ and intersection number k.

Combining the above two cases completes the proof. □

Now combining Lemmas 4.1 and 4.2 gives the following theorem.

Theorem 4.3 *$I[4] = \{3\}$ and $I[n] = J[n]$ for all $n \equiv 2$ or 4 (mod 6).* □

5 $n \equiv 5$ (mod 6).

There is no difficulty in showing that $I[5] = \{4\}$ and so we will assume $n \equiv 5 \pmod 6 \geq 11$. In this case $J[n] = \{0, 1, 2, \ldots, (n^2-n+4)/6 = t\}\backslash\{t-1, t-2, t-3, t-5\}$.

Lemma 5.1 $J[n]\backslash\{0,1\} \subseteq I[n]$.

Proof: Let (S, P_1) and (S, P_2) be a pair of MPTs of order $n \equiv 5 \pmod 6$ with leave the 4-cycle $L = (1,2,3,4)$. Let T^* be the collection of 2 triples defined by $T^* = \{\{1,2,4\},\{2,3,4\}\}$. Then $(S, P_1\bigcup T^*)$ and $(S, P_2\bigcup T^*)$ are MCTs of order n with padding the double edge $X = \{\{2,4\},\{2,4\}\}$. Then $|(P_1\bigcup T^*)\bigcap(P_2\bigcup T^*)| = k+2$, where $k \in I_p[n]$ = the intersection spectrum for MPTs of order n. It is less that trivial to see that $I_p[n]+\{2\} = J[n]\backslash\{0,1\} \subseteq I[n]$. □

The $12n+(5 \text{ or } 11)$ Construction. Let (S, C) be a MCT of order $6n+5$ or $6n-1$ with padding the double edge X. Let F $(F\bigcup T)$ be a 1-factorization (Stern and Lenz 1-factorization) of $E(K_{6n+6})$ based on W, $W\bigcap S = \emptyset$. Let α be any $1-1$ mapping of S onto F and define $(S\bigcup W, C^*)$ in the usual way. Then $(S\bigcup W, C^*)$ is a MCT of order $12n+(5 \text{ or } 11)$ with padding the double edge X. □

Lemma 5.2 $\{0,1\} \subseteq I[n]$.

Proof: The cases $n = 11$ and 17 are handled in the appendix so that we can assume $n \geq 23$ and that $J[t] \subseteq I[t]$ for all $11 \leq t < n$. The $12n + (5 \text{ or } 11)$ Construction incorporated into a by now familiar argument produces a pair of MCTs of order $12n+(5 \text{ or } 11)$ with padding a double edge and intersection number 0 or 1. □

Combining Lemmas 5.1 and 5.2 gives the following theorem.

Theorem 5.3 *$I[5] = \{4\}$ and $I[n] = J[n]$ for all $n \equiv 2$ or 4 (mod 6).* □

6 Conclusion.

We collect the results in Theorems 3.5, 4.3, and 5.3 in the accompanying easy to read table.

K_n	padding	intersection spectrum
$n \equiv 1$ or 3 (mod 6) Steiner triple system	$\emptyset$	$I[3] = \{1\}, I[7] = \{0,1,3,7\}$, $I[9] = \{0,1,2,3,4,6,12\}$ and $I[n] = \{0,1,2,\ldots,n(n-1)/6 = t\}$ $\backslash\{t-1,t-2,t-3,t-5\}$ for $n \geq 13$ [2].
$n \equiv 0$ (mod 6)	1-factor	$I[n] = \{0,1,2,\ldots,n^2/6 = t\}$ $\backslash\{t-1,t-2,t-3,t-5\}$ for all $n \equiv 0$ (mod 6).
$n \equiv 2$ or 4 (mod 6)	tripole	$I[4] = \{3\}$ and $I[n] = \{0,1,2,\ldots,(n^2+2)/6 = t\}$ $\backslash\{t-1,t-2,t-3,t-5\}$ for all $n \geq 8$.
$n \equiv 5$ (mod 6)	double edge	$I[5] = \{4\}$ and $I[n] = \{0,1,2,\ldots,(n^2-n+4)/6 = t\}$ $\backslash\{t-1,t-2,t-3,t-5\}$ for all $n \geq 11$

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APPENDIX

The following systems provide the remaining intersections of minimum coverings not produced by the constructions in the text.

SYSTEM 1. Order 8, intersection number 0.

1	2	7	1	2	3	1	3	8	1	4	6
1	4	5	2	4	8	2	5	6	3	4	7
3	5	6	5	7	8	6	7	8			
1	2	4	1	2	5	1	3	6	1	3	4
1	7	8	2	3	7	2	6	8	3	5	8
4	5	6	4	7	8	5	6	7			

SYSTEM 2. Order 8, intersection number 1.

1	2	4 *	1	2	5	1	3	4	1	3	6
1	7	8	2	3	7	2	6	8	3	5	8
4	5	6	4	7	8	5	6	7			
1	2	3	1	2	4 *	1	3	8	1	4	7
1	5	6	2	5	6	2	7	8	3	4	5
3	6	7	4	6	8	5	7	8			

SYSTEM 3. Order 8, intersection number 2.

1	2	4	1	2	5 *	1	3	4 *	1	3	8
1	6	7	2	3	6	2	7	8	3	5	7
4	5	6	4	7	8	5	6	8			
1	2	3	1	2	5 *	1	3	4 *	1	4	6
1	7	8	2	4	7	2	6	8	3	5	6
3	7	8	4	5	8	5	6	7			

SYSTEM 4. Order 8, intersection number 3.

1 2 3	1 2 4 *	1 3 7 *	1 4 8
1 5 6	2 5 6	2 7 8	3 4 5
3 6 8	4 6 7	5 7 8 *	

1 2 4 *	1 2 5	1 3 4	1 3 7 *
1 6 8	2 3 8	2 6 7	3 5 6
4 5 6	4 7 8	5 7 8 *	

SYSTEM 5. Order 11, intersection number 0.

1 2 5	1 2 3	1 2 10	1 4 9
1 6 8	1 7 11	2 4 8	2 6 11
2 7 9	3 4 10	3 5 7	3 6 9
3 8 11	4 5 11	4 6 7	5 6 10
5 8 9	7 8 10	9 10 11	

1 2 8	1 2 9	1 2 11	1 3 6
1 4 10	1 5 7	2 3 4	2 5 10
2 6 7	3 5 9	3 7 8	3 10 11
4 5 8	4 6 9	4 7 11	5 6 11
6 8 10	7 9 10	8 9 11	

SYSTEM 6. Order 11, intersection number 1.

1 2 9	1 2 4	1 2 11	1 3 5
1 6 7	1 8 10	2 3 6	2 5 8 *
2 7 10	3 4 10	3 7 8	3 9 11
4 5 6	4 7 11	4 8 9	5 7 9
5 10 11	6 8 11	6 9 10	

1 2 3	1 2 7	1 2 6	1 4 10
1 5 11	1 8 9	2 4 9	2 5 8 *
2 10 11	3 4 11	3 5 9	3 6 7
3 8 10	4 5 7	4 6 8	5 6 10
6 9 11	7 8 11	7 9 10	

To each of the following systems, add the four triples.

1 2 4	1 3 5	2 5 6	3 4 6

Since there are 4 disjoint triples that are mutually balanced with these four triples, each system produces 2 intersection numbers.

SYSTEM 7. Order 10, intersection numbers 0 and 4.

1 2 7	1 3 10	1 4 8	1 6 9
2 3 9	2 8 10	3 7 8	4 5 7
4 9 10	5 6 10	5 8 9	6 7 8
7 9 10			

1 2 8	1 3 9	1 4 10	1 6 7
2 3 7	2 9 10	3 8 10	4 5 9
4 7 8	5 6 8	5 7 10	6 9 10
7 8 9			

SYSTEM 8. Order 10, intersection number 1.

1 2 9	1 3 7	1 4 8	1 6 10
2 3 10	2 7 8	3 8 9	4 5 7
4 9 10	5 6 9	5 8 10 *	6 7 8
7 9 10			

1 2 7	1 3 9	1 4 10	1 6 8
2 3 8	2 9 10	3 7 10	4 5 9
4 7 8	5 6 7	5 8 10 *	6 9 10
7 8 9			

SYSTEM 9. Order 10, intersection number 2.

1 2 3	1 4 8	1 6 10	1 7 9
2 7 8	2 9 10	3 7 8 *	3 9 10
4 5 9	4 7 10	5 6 7 *	5 8 10
6 8 9			

1 2 7	1 3 9	1 4 10	1 6 8
2 3 10	2 8 9	3 7 8 *	4 5 8
4 7 9	5 6 7 *	5 9 10	6 9 10
7 8 10			

SYSTEM 10. Order 10, intersection number 3.

1 2 3	1 4 9 *	1 6 7	1 8 10
2 7 10	2 8 9	3 7 8 *	3 9 10
4 5 10	4 7 8	5 6 8	5 7 9
6 9 10 *			

1 2 7	1 3 10	1 4 9 *	1 6 8
2 3 9	2 8 10	3 7 8 *	4 5 8
4 7 10	5 6 7	5 9 10	6 9 10 *
7 8 9			

SYSTEM 11. Order 10, intersection number 10.

1 2 9 *	1 3 8 *	1 4 7 *	1 6 10 *
2 3 7	2 8 10	3 9 10	4 5 10
4 8 9	5 6 9 *	5 7 8	6 7 8 *
7 9 10			

1 2 9 *	1 3 8 *	1 4 7 *	1 6 10 *
2 3 10	2 7 8	3 7 9	4 5 8
4 9 10	5 6 9 *	5 7 10	6 7 8 *
8 9 10			

SYSTEM 12. Order 10, intersection numbers 13.

1 2 10	1 3 8	1 4 9 *	1 6 7 *
2 3 9 *	2 7 8	3 7 10	4 5 7 *
4 8 10 *	5 6 8 *	5 9 10 *	6 9 10 *
7 8 9 *			
1 2 8	1 3 10	1 4 9 *	1 6 7 *
2 3 9 *	2 7 10	3 7 8	4 5 7 *
4 8 10 *	5 6 8 *	5 9 10 *	6 9 10 *
7 8 9 *			

SYSTEM 13. Order 12, intersection numbers 0 and 4.

1 2 10	1 6 12	1 7 8	1 9 11
2 3 7	2 8 9	2 11 12	3 4 11
3 8 10	3 9 12	4 5 7	4 8 12
4 9 10	5 6 8	5 9 10	5 11 12
6 7 9	6 10 11	7 8 11	7 10 12

1 2 8	1 6 11	1 7 10	1 9 12
2 3 9	2 7 12	2 10 11	3 4 8
3 7 11	3 10 12	4 5 10	4 7 9
4 11 12	5 6 12	5 7 8	5 9 11
6 7 8	6 9 10	8 9 10	8 11 12

SYSTEM 14. Order 12, intersection numbers 1 and 5.

1 2 10	1 6 7	1 8 9	1 11 12
2 3 11	2 7 8	2 9 12	3 4 9
3 7 8	3 10 12	4 5 8	4 7 10
4 11 12 *	5 6 11	5 7 12	5 9 10
6 8 12	6 9 10	7 9 11	8 10 11

1 2 8	1 6 11	1 7 9	1 10 12
2 3 7	2 9 10	2 11 12	3 4 10
3 8 11	3 9 12	4 5 9	4 7 8
4 11 12 *	5 6 10	5 7 11	5 8 12
6 7 12	6 8 9	7 8 10	9 10 11

SYSTEM 15. Order 12, intersection numbers 2 and 6.

1 2 9 *	1 6 10	1 7 12	1 8 11
2 3 12	2 7 8	2 10 11	3 4 10
3 7 8	3 9 11	4 5 8	4 7 11
4 9 12	5 6 7	5 9 10 *	5 11 12
6 8 9	6 11 12	7 9 10	8 10 12

1 2 9 *	1 6 8	1 7 11	1 10 12
2 3 7	2 8 10	2 11 12	3 4 8
3 9 12	3 10 11	4 5 11	4 7 12
4 9 10	5 6 12	5 7 8	5 9 10 *
6 7 10	6 9 11	7 8 9	8 11 12

SYSTEM 16. Order 12, intersection number 3.

1 2 8	1 6 12	1 7 11	1 9 10 *
2 3 10	2 7 12	2 9 11	3 4 9
3 7 8	3 11 12	4 5 7	4 8 10 *
4 11 12	5 6 9	5 8 11	5 10 12
6 7 8	6 10 11	7 9 10	8 9 12 *

1 2 11	1 6 8	1 7 12	1 9 10 *
2 3 9	2 7 8	2 10 12	3 4 12
3 7 10	3 8 11	4 5 9	4 7 11
4 8 10 *	5 6 10	5 7 8	5 11 12
6 7 9	6 11 12	8 9 12 *	9 10 11

SYSTEM 17. Order 14, intersection numbers 0 and 4.

1 2 11	1 3 9	1 4 13	1 6 14
1 7 8	1 10 12	2 3 13	2 7 10
2 8 14	2 9 12	3 7 8	3 10 14
3 11 12	4 5 10	4 7 11	4 8 9
4 12 14	5 6 8	5 7 12	5 9 11
5 13 14	6 7 13	6 9 10	6 11 12
7 9 14	8 10 11	8 12 13	9 10 13
11 13 14			

1 2 13	1 3 8	1 4 11	1 6 9
1 7 12	1 10 14	2 3 14	2 7 9
2 8 10	2 11 12	3 7 11	3 9 13
3 10 12	4 5 12	4 7 8	4 9 10
4 13 14	5 6 13	5 7 8	5 9 10
5 11 14	6 7 14	6 8 12	6 10 11
7 10 13	8 9 14	8 11 13	9 11 12
12 13 14			

SYSTEM 18. Order 14, intersection numbers 1 and 5.

1 2 14	1 3 12	1 4 11	1 6 13
1 7 8	1 9 10	2 3 10	2 7 9
2 8 13	2 11 12	3 7 11	3 8 9
3 13 14	4 5 8	4 7 14	4 9 13
4 10 12	5 6 7	5 9 10	5 11 13
5 12 14	6 8 14	6 9 12 *	6 10 11
7 8 10	7 12 13	8 11 12	9 11 14
10 13 14			

1 2 11	1 3 9	1 4 8	1 6 7
1 10 12	1 13 14	2 3 8	2 7 10
2 9 13	2 12 14	3 7 13	3 10 14
3 11 12	4 5 12	4 7 11	4 9 14
4 10 13	5 6 10	5 7 9	5 8 11
5 13 14	6 8 13	6 9 12 *	6 11 14
7 8 12	7 8 14	8 9 10	9 10 11
11 12 13			

SYSTEM 19. Order 14, intersection numbers 2 and 6.

1 2 7	1 3 9	1 4 13	1 6 8
1 10 14	1 11 12	2 3 13	2 8 10
2 9 14	2 11 12 *	3 7 8	3 10 12
3 11 14	4 5 7	4 8 9	4 10 11
4 12 14	5 6 14	5 8 11	5 9 10
5 12 13	6 7 11	6 9 12	6 10 13
7 8 12	7 9 10 *	7 13 14	8 13 14
9 11 13			

1 2 13	1 3 12	1 4 11	1 6 14
1 7 8	1 9 10	2 3 9	2 7 8
2 10 14	2 11 12 *	3 7 11	3 8 10
3 13 14	4 5 10	4 7 14	4 8 12
4 9 13	5 6 11	5 7 12	5 8 13
5 9 14	6 7 13	6 8 9	6 10 12
7 9 10 *	8 11 14	9 11 12	10 11 13
12 13 14			

SYSTEM 20. Order 14, intersection number 3.

1 2 7	1 3 10 *	1 4 11	1 6 14 *
1 8 9	1 12 13	2 3 8	2 9 10
2 11 13	2 12 14	3 7 13	3 9 14
3 11 12 *	4 5 10	4 7 14	4 8 12
4 9 13	5 6 9	5 7 8	5 11 12
5 13 14	6 7 11	6 8 13	6 10 12
7 8 10	7 9 12	8 11 14	9 10 11
10 13 14			

1 2 8	1 3 10 *	1 4 9	1 6 14 *
1 7 13	1 11 12	2 3 9	2 7 12
2 10 11	2 13 14	3 7 8	3 11 12 *
3 13 14	4 5 12	4 7 10	4 8 14
4 11 13	5 6 7	5 8 13	5 9 10
5 11 14	6 8 12	6 9 11	6 10 13
7 8 11	7 9 14	8 9 10	9 12 13
10 12 14			

SYSTEM 21. Order 16, intersection numbers 0 and 4.

1 2 14	1 3 9	1 4 12	1 6 11
1 7 8	1 10 15	1 13 16	2 3 10
2 7 15	2 8 16	2 9 13	2 11 12
3 7 13	3 8 14	3 11 15	3 12 16
4 5 13	4 7 9	4 8 11	4 10 14
4 15 16	5 6 9	5 7 8	5 10 11
5 12 15	5 14 16	6 7 10	6 8 12
6 13 14	6 15 16	7 11 16	7 12 14
8 9 10	8 13 15	9 10 16	9 11 12
9 14 15	10 12 13	11 13 14	

1 2 10	1 3 8	1 4 16	1 6 9
1 7 15	1 11 13	1 12 14	2 3 7
2 8 13	2 9 14	2 11 16	2 12 15
3 9 11	3 10 15	3 12 13	3 14 16
4 5 15	4 7 8	4 9 10	4 11 12
4 13 14	5 6 14	5 7 12	5 8 11
5 9 13	5 10 16	6 7 16	6 8 15
6 10 13	6 11 12	7 8 9	7 10 11
7 13 14	8 10 14	8 12 16	9 10 12
9 15 16	11 14 15	13 15 16	

SYSTEM 22. Order 16, intersection numbers 1 and 5.

1	2	7	1	3	12	1	4	16	1	6	8
1	9	10	1	11	13	1	14	15	2	3	14
2	8	12	2	9	13	2	10	15	2	11	16
3	7	11	3	8	13	3	9	15	3	10	16
4	5	7	4	8	9	4	10	11	4	12	15
4	13	14 *	5	6	13	5	8	14	5	9	10
5	11	12	5	15	16	6	7	9	6	10	14
6	11	12	6	15	16	7	8	10	7	8	16
7	12	14	7	13	15	8	11	15	9	11	14
9	12	16	10	12	13	13	14	16			

1	2	8	1	3	7	1	4	9	1	6	11
1	10	14	1	12	13	1	15	16	2	3	9
2	7	10	2	11	14	2	12	16	2	13	15
3	8	14	3	10	15	3	11	12	3	13	16
4	5	8	4	7	12	4	10	16	4	11	15
4	13	14 *	5	6	16	5	7	15	5	9	13
5	10	11	5	12	14	6	7	8	6	9	10
6	12	15	6	13	14	7	8	9	7	11	13
7	14	16	8	10	13	8	11	12	8	15	16
9	10	12	9	11	16	9	14	15			

SYSTEM 23. Order 16, intersection numbers 2 and 6.

1	2	13	1	3	10	1	4	14	1	6	11
1	7	12	1	8	16	1	9	15	2	3	12
2	7	8	2	9	10	2	11	14	2	15	16
3	7	9	3	8	14	3	11	16	3	13	15
4	5	7	4	8	10	4	9	12	4	11	13
4	15	16 *	5	6	14	5	8	15	5	9	11
5	10	13	5	12	16	6	7	15	6	8	12
6	9	10	6	13	16	7	8	11	7	10	16 *
7	13	14	8	9	13	9	14	16	10	11	12
10	14	15	11	12	15	12	13	14			

1	2	15	1	3	14	1	4	10	1	6	13
1	7	8	1	9	11	1	12	16	2	3	16
2	7	12	2	8	9	2	10	11	2	13	14
3	7	11	3	8	13	3	9	10	3	12	15
4	5	12	4	7	14	4	8	11	4	9	13
4	15	16 *	5	6	15	5	7	13	5	8	14
5	9	10	5	11	16	6	7	8	6	9	16
6	10	14	6	11	12	7	9	15	7	10	16 *
8	10	12	8	15	16	9	12	14	10	13	15
11	12	13	11	14	15	13	14	16			

SYSTEM 24. Order 16, intersection numbers 3 and 7.

1 2 8	1 3 14	1 4 9 *	1 6 13
1 7 11	1 10 12	1 15 16	2 3 15 *
2 7 10	2 9 11	2 12 16	2 13 14
3 7 9	3 8 12	3 10 11	3 13 16
4 5 13	4 7 8	4 10 15	4 11 12
4 14 16	5 6 15	5 7 14	5 8 10
5 9 16	5 11 12	6 7 12	6 8 9
6 10 16	6 11 14	7 8 16	7 13 15 *
8 11 13	8 14 15	9 10 13	9 10 14
9 12 15	11 15 16	12 13 14	

1 2 11	1 3 16	1 4 9 *	1 6 8
1 7 10	1 12 15	1 13 14	2 3 15 *
2 7 8	2 9 14	2 10 12	2 13 16
3 7 8	3 9 10	3 11 12	3 13 14
4 5 15	4 7 14	4 8 10	4 11 16
4 12 13	5 6 13	5 7 16	5 8 12
5 9 11	5 10 14	6 7 11	6 9 10
6 12 16	6 14 15	7 9 12	7 13 15 *
8 9 13	8 11 15	8 14 16	9 15 16
10 11 13	10 15 16	11 12 14	

SYSTEM 25. Order 18, intersection numbers 2 and 6.

1 2 16	1 6 13	1 7 9	1 8 11
1 10 14	1 12 18	1 15 17	2 3 18
2 7 15	2 8 10	2 9 13	2 11 14
2 12 17	3 4 9	3 7 8	3 10 15
3 11 12	3 13 14 *	3 16 17	4 5 18
4 7 10	4 8 14	4 11 16	4 12 15
4 13 17	5 6 9	5 7 14	5 8 15
5 10 17	5 11 12	5 13 16 *	6 7 12
6 8 17	6 10 11	6 14 18	6 15 16
7 8 16	7 11 13	7 17 18	8 9 18
8 12 13	9 10 12	9 10 16	9 11 15
9 14 17	10 13 18	11 17 18	12 14 16
13 14 15	15 16 18		

1 2 17	1 6 14	1 7 11	1 8 18
1 9 12	1 10 16	1 13 15	2 3 12
2 7 9	2 8 16	2 10 11	2 13 14
2 15 18	3 4 10	3 7 16	3 8 17
3 9 18	3 11 15	3 13 14 *	4 5 14
4 7 17	4 8 15	4 9 16	4 11 12
4 13 18	5 6 11	5 7 8	5 9 10
5 12 18	5 13 16 *	5 15 17	6 7 18
6 8 12	6 9 15	6 10 13	6 16 17
7 8 14	7 10 15	7 12 13	8 9 10
8 11 13	9 11 14	9 13 17	10 12 14
10 17 18	11 12 17	11 16 18	12 15 16
14 15 16	14 17 18		

SYSTEM 26. Order 18, intersection numbers 3 and 7.

1 2 8	1 6 13	1 7 18	1 9 10
1 11 12	1 14 17	1 15 16	2 3 11
2 7 17	2 9 15	2 10 12	2 13 18
2 14 16	3 4 15	3 7 10	3 8 16
3 9 14	3 12 18 *	3 13 17	4 5 12
4 7 9	4 8 13	4 10 18	4 11 14
4 16 17	5 6 15	5 7 14	5 8 9
5 10 17	5 11 18	5 13 16	6 7 8
6 9 10	6 11 17	6 12 14	6 16 18
7 8 15 *	7 11 16 *	7 12 13	8 10 11
8 12 17	8 14 18	9 11 13	9 12 16
9 17 18	10 13 14	10 15 16	11 12 15
13 14 15	15 17 18		

1 2 17	1 6 16	1 7 13	1 8 12
1 9 15	1 10 11	1 14 18	2 3 7
2 8 16	2 9 14	2 10 13	2 11 18
2 12 15	3 4 8	3 9 11	3 10 14
3 12 18 *	3 13 15	3 16 17	4 5 9
4 7 10	4 11 12	4 13 14	4 15 16
4 17 18	5 6 11	5 7 12	5 8 13
5 10 16	5 14 15	5 17 18	6 7 14
6 8 17	6 9 12	6 10 15	6 13 18
7 8 18	7 8 15 *	7 9 17	7 11 16 *
8 9 10	8 11 14	9 10 18	9 13 16
10 12 17	11 12 13	11 15 17	12 14 16
13 14 17	15 16 18		

For the following two systems, in addition to the four triples listed at the beginning of the appendix, the following four triples should be added.

7 8 10	7 9 11	8 11 12	9 10 12

Again, there are 4 disjoint triples that are mutually balanced with these four triples, so each system now produces 3 intersection numbers.

SYSTEM 27. Order 18, intersection numbers 0,4 and 8.

1 2 14	1 6 8	1 7 13	1 9 15
1 10 18	1 11 17	1 12 16	2 3 18
2 7 8	2 9 17	2 10 11	2 12 15
2 13 16	3 4 17	3 7 14	3 8 9
3 10 15	3 11 16	3 12 13	4 5 7
4 8 15	4 9 18	4 10 16	4 11 13
4 12 14	5 6 14	5 8 16	5 9 10
5 11 15	5 12 17	5 13 18	6 7 17
6 9 16	6 10 13	6 11 12	6 15 18
7 12 18	7 15 16	8 13 14	8 17 18
9 13 14	10 14 17	11 14 18	13 15 17
14 15 16	16 17 18		

1	2	11	1	6	9	1	7	16	1	8	13
1	10	15	1	12	14	1	17	18	2	3	12
2	7	18	2	8	15	2	9	16	2	10	17
2	13	14	3	4	11	3	7	15	3	8	18
3	9	10	3	13	16	3	14	17	4	5	10
4	7	8	4	9	13	4	12	18	4	14	16
4	15	17	5	6	15	5	7	17	5	8	9
5	11	14	5	12	13	5	16	18	6	7	12
6	8	14	6	10	16	6	11	18	6	13	17
7	13	14	8	16	17	9	14	15	9	17	18
10	11	13	10	14	18	11	12	17	11	15	16
12	15	16	13	15	18						

SYSTEM 28. Order 18, intersection numbers 1,5 and 9.

1	2	17	1	6	18	1	7	8	1	9	14
1	10	16	1	11	15	1	12	13	2	3	12
2	7	15	2	8	13	2	9	18	2	10	14
2	11	16	3	4	8 *	3	7	17	3	9	15
3	10	13	3	11	18	3	14	16	4	5	16
4	7	18	4	9	10	4	11	12	4	13	14
4	15	17	5	6	15	5	7	12	5	8	9
5	10	17	5	11	14	5	13	18	6	7	16
6	8	17	6	9	13	6	10	11	6	12	14
7	13	14	8	14	15	8	16	18	9	16	17
10	15	18	11	13	17	12	15	16	12	17	18
13	15	16	14	17	18						

1	2	8	1	6	12	1	7	13	1	9	10
1	11	14	1	15	16	1	17	18	2	3	7
2	9	16	2	10	13	2	11	12	2	14	17
2	15	18	3	4	8 *	3	9	14	3	10	18
3	11	15	3	12	13	3	16	17	4	5	18
4	7	16	4	9	15	4	10	14	4	11	13
4	12	17	5	6	10	5	7	8	5	9	17
5	11	16	5	12	15	5	13	14	6	7	15
6	8	9	6	11	18	6	13	17	6	14	16
7	12	14	7	17	18	8	13	16	8	14	18
8	15	17	9	13	18	10	11	17	10	15	16
12	16	18	13	14	15						

SYSTEM 29. Order 17, intersection number 0.

1	2	8	1	2	12	1	6	7	1	9	14
1	10	11	1	13	15	1	16	17	2	3	11
2	7	17	2	9	13	2	10	15	2	14	16
3	7	12	3	8	9	3	10	16	3	13	17
3	14	15	4	5	9	4	7	16	4	8	14
4	10	17	4	11	15	4	12	13	5	7	15
5	8	17	5	10	13	5	11	16	5	12	14
6	8	15	6	9	16	6	10	14	6	11	13
6	12	17	7	13	14	8	13	16	9	15	17
11	14	17	12	15	16						

1	2	15	1	2	7		1	6	12	1	8	14
1	9	13	1	10	17		1	11	16	2	3	8
2	9	16	2	10	11		2	12	13	2	14	17
3	7	16	3	9	17		3	10	13	3	11	15
3	12	14	4	5	10		4	7	14	4	8	9
4	11	13	4	12	17		4	15	16	5	7	13
5	8	15	5	9	14		5	11	17	5	12	16
6	7	17	6	8	13		6	9	15	6	10	16
6	11	14	7	12	15		8	16	17	10	14	15
13	14	16	13	15	17							

SYSTEM 30. Order 17, intersection number 1.

1	2	11	1	2	17		1	6	7	1	8	16
1	9	15	1	10	13		1	12	14	2	3	12
2	7	14	2	8	15		2	9	13	2	10	16
3	7	13	3	8	9	*	3	10	17	3	11	15
3	14	16	4	5	9		4	7	16	4	8	13
4	10	15	4	11	14		4	12	17	5	7	12
5	8	17	5	10	14		5	11	13	5	15	16
6	8	14	6	9	16		6	10	11	6	12	15
6	13	17	7	15	17		9	14	17	11	16	17
12	13	16	13	14	15							

1	2	15	1	2	10		1	6	8	1	7	17
1	9	14	1	11	13		1	12	16	2	3	13
2	7	12	2	8	17		2	9	16	2	11	14
3	7	14	3	8	9	*	3	10	16	3	11	17
3	12	15	4	5	12		4	7	15	4	8	16
4	9	13	4	10	11		4	14	17	5	7	13
5	8	14	5	9	15		5	10	17	5	11	16
6	7	16	6	9	17		6	10	13	6	11	15
6	12	14	8	13	15		10	14	15	12	13	17
13	14	16	15	16	17							