AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 29–35

Partitioning sets of oriented triples into the smallest nontrivial oriented triple systems*

Hongtao Zhao

School of Mathematics and Physics
North China Electric Power University
Beijing 102206
China
ht_zhao@163.com

Yanfang Zhang

College of Mathematics and Statistics
Hebei University of Economics and Business
Shijiazhuang 050061
China
yanfang_zh@163.com

Abstract

We study partitions of the set of all cyclic (respectively, transitive) triples chosen from a v-set into pairwise disjoint MTS(4)s (respectively, DTS(4)s). We find necessary conditions for the partitions. Furthermore, we prove that the necessary conditions for the partitions are also sufficient.

1 Introduction

Let X be a finite set. In what follows, an ordered pair of X will always be an ordered pair (x, y), where $x \neq y \in X$. A *cyclic triple* on X is a set of three ordered pairs (x, y), (y, z) and (z, x) of X, which is denoted by $\langle x, y, z\rangle$ (or $\langle y, z, x\rangle$, or $\langle z, x, y\rangle$). Generally, a *cyclic k-cycle* on X is a set of k ordered pairs $(x_1, x_2), (x_2, x_3), \ldots,$ (x_{k-1}, x_k) and (x_k, x_1), which is denoted by $\langle x_1, x_2, \ldots, x_k\rangle$ (or $\langle x_2, x_3, \ldots, x_k, x_1\rangle, \ldots,$ or $\langle x_k, x_1, \ldots, x_{k-1}\rangle$). A *transitive triple* on X is a set of three ordered pairs (x, y), (y, z) and (x, z) of X, which is denoted by (x, y, z). It is easy to know that cyclic triple and transitive triple are the only two types of oriented triples. A *Mendelsohn* (respectively *directed*) *triple system* of order v and index λ, denoted by MTS(v, λ) (respectively DTS(v, λ)), is a pair $(X, \mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a collection of

* Research supported by NSFC Grant 10671055 and NSFHB Grant A2007000230.

cyclic (respectively transitive) triples on X, called *blocks*, such that each ordered pair of distinct elements of X is contained in exactly λ blocks of $\mathcal{B}$. Usually, MTS$(v,1)$ (respectively DTS$(v,1)$) is written as MTS(v) (respectively DTS(v)). A *Mendelsohn system* $M(v,k,\lambda)$ on X is a pair $(X,\mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a collection of cyclic k-cycles of X, called *blocks*, such that each ordered pair of distinct elements of X is contained in exactly λ blocks of $\mathcal{B}$. Obviously, the Mendelsohn system $M(v,k,\lambda)$ is a generalization of the Mendelsohn triple system MTS(v,λ). An $M(v,3,\lambda)$ is an MTS(v,λ).

The problem of partitioning larger combinatorial structures into copies of smaller ones has a long history. If $v \equiv 0,1 \pmod 3$ and $v \neq 6$, then a Mendelsohn triple system MTS(v) exists, and there is a partition of the simple MTS$(v,v-2)$ into $v-2$ MTS(v)s, that is, a *large set* of MTS(v)s (see [8, 7]). The blocks of a simple MTS$(v,v-2)$ are in fact all the cyclic triples from a set of size v. If $v \equiv 0,1,3,4,7,9$ (mod 12), then there is a partition of the simple MTS$(v+1,v-1)$ into $v+1$ MTS(v)s, that is, an *overlarge set* of MTS(v)s (see [9]). The set of all cyclic k-cycles chosen from a given v-set forms an $M(v,k,(v-2)\dots(v-k+1))$, which is simple. If $k=v$ and $v-1$ is a composite, then there is a partition of the simple $M(v,v,(v-2)!)$ into $(v-2)!$ $M(v,v,1)$s (see [5, 11]). The set of all transitive triples chosen from a given v-set forms a DTS$(v,3(v-2))$, which is simple. If $v \equiv 0,1 \pmod 3$, then a directed triple system DTS(v) exists, and there is a partition of the simple DTS$(v,3(v-2))$ into $3(v-2)$ DTS(v)s, that is, a large set of DTS(v)s (see [2, 6]). If $v \equiv 1,3$ (mod 6), $v \equiv 4,12$ (mod 24), $v \equiv 24$ (mod 120), then there is a partition of the simple DTS$(v+1,3(v-1))$ into $3(v+1)$ DTS(v)s, that is, an overlarge set of DTS(v)s (see [10]).

In this paper, we consider partitions of the set of all cyclic (respectively transitive) triples into the smallest nontrivial Mendelsohn (respectively directed) triple systems, i.e., MTS(4)s (respectively DTS(4)s). We find necessary conditions for the partitions. Furthermore, we prove that the necessary conditions for the partitions are also sufficient.

2 Necessary Conditions

Firstly, we consider the general case where one Mendelsohn system is partitioned into copies of another.

Theorem 2.1 *Let $D=(V,\mathcal{B})$ be a Mendelsohn system with parameters $M(v,k,\lambda)$, and $E=(W,\mathcal{C})$ be a Mendelsohn system with parameters $M(w,k,\mu)$. If D can be partitioned into pairwise disjoint copies of E, then the following divisibility conditions are necessary:*

$$\mu|\lambda, \tag{1}$$

$$(w-1)\mu|(v-1)\lambda, \tag{2}$$

$$\frac{w(w-1)\mu}{k}\Big|\frac{v(v-1)\lambda}{k}, \tag{3}$$

and D must be partitioned into n pairwise disjoint copies of E, where

$$n = \frac{v(v-1)\lambda}{w(w-1)\mu}. \tag{4}$$

Proof. Conditions (1) and (2) follow by counting, in each system, the occurrences of ordered pairs and the replications of elements respectively. Conditions (3) and (4) follow by counting numbers of blocks in each system. ■

Secondly, we consider the special case where $D = (V, \mathcal{B})$ is the simple $M(v, 3, v-2)$ consisting of all cyclic triples chosen from the v-set V. The next result is a direct consequence of Theorem 2.1.

Theorem 2.2 *Let E be the smallest nontrivial Mendelsohn system $MTS(4)$. If there is a partition of the simple $M(v, 3, v-2)$ into pairwise disjoint copies of $MTS(4)$s, then it is necessary that*

$$3|(v-1)(v-2) \text{ and } 12|v(v-1)(v-2)$$

or equivalently

$$v \equiv 1, 2, 4, 5, 8, 10 \ (mod\ 12) \tag{5}$$

Finally, we consider the partition of the set of all transitive triples chosen from a v-set into pairwise disjoint DTS(4)s. We know that the set of all transitive triples chosen from a given v-set forms a simple DTS$(v, 3(v-2))$. Similarly, we can obtain the necessary conditions of this kind of partition.

Theorem 2.3 *Let E be the smallest nontrivial directed system $DTS(4)$. If there is a partition of the simple $DTS(v, 3(v-2))$ into pairwise disjoint copies of $DTS(4)$s, then it is necessary that*

$$v \equiv 0, 1, 2 \ (mod\ 4) \tag{6}$$

3 Direct constructions for small cases

3.1 Partitioning into $MTS(4)$

We give direct constructions of partitions of the set of all the cyclic triples of a v-set for the cases $v = 4, 5, 13$.

$\underline{v=4}$: A large set of $MTS(4)$ will do.
$\underline{v=5}$: An overlarge set of $MTS(4)$ will do.
$\underline{v=13}$: Let $V = Z_{13}$ be the 13-set. Under the action of the group Z_{13}, all the cyclic triples on V are partitioned into 44 orbits $\mathcal{G}_i$, $1 \le i \le 44$, which are listed as follows with representatives

$\langle 0,2,1\rangle$, $\langle 0,11,1\rangle$, $\langle 0,1,3\rangle$, $\langle 0,1,2\rangle$, $\langle 0,3,1\rangle$, $\langle 0,10,1\rangle$, $\langle 0,1,4\rangle$, $\langle 0,1,11\rangle$,
$\langle 0,4,1\rangle$, $\langle 0,9,1\rangle$, $\langle 0,1,5\rangle$, $\langle 0,1,10\rangle$, $\langle 0,5,1\rangle$, $\langle 0,8,1\rangle$, $\langle 0,1,6\rangle$, $\langle 0,1,9\rangle$,

$$\begin{array}{llllllll}
\langle 0,6,1\rangle, & \langle 0,7,1\rangle, & \langle 0,1,7\rangle, & \langle 0,1,8\rangle, & \langle 0,4,2\rangle, & \langle 0,8,4\rangle, & \langle 0,2,8\rangle, & \langle 0,2,6\rangle,\\
\langle 0,9,2\rangle, & \langle 0,4,8\rangle, & \langle 0,2,4\rangle, & \langle 0,7,2\rangle, & \langle 0,5,2\rangle, & \langle 0,8,2\rangle, & \langle 0,2,7\rangle, & \langle 0,2,10\rangle,\\
\langle 0,10,2\rangle, & \langle 0,3,8\rangle, & \langle 0,2,5\rangle, & \langle 0,8,3\rangle, & \langle 0,6,2\rangle, & \langle 0,7,3\rangle, & \langle 0,2,9\rangle, & \langle 0,3,9\rangle,\\
\langle 0,9,3\rangle, & \langle 0,3,7\rangle, & \langle 0,3,6\rangle, & \langle 0,6,3\rangle. & & & &
\end{array}$$

Choosing one cyclic triple from each of the four orbits $\mathcal{G}_{4i-3},\ \mathcal{G}_{4i-2},\ \mathcal{G}_{4i-1}$, and $\mathcal{G}_{4i}$ to form a starter $MTS(4)$, denoted by π_i, where $1 \le i \le 11$, we obtain 11 orbits of $MTS(4)$s, each of which is of length 13, with the following starter MTS$(4)s$

$\pi_1 = \{\langle 0,2,1\rangle, \langle 2,0,3\rangle, \langle 1,3,0\rangle, \langle 3,1,2\rangle\}$; $\pi_2 = \{\langle 0,3,1\rangle, \langle 3,0,4\rangle, \langle 1,4,0\rangle, \langle 4,1,3\rangle\}$;
$\pi_3 = \{\langle 0,4,1\rangle, \langle 4,0,5\rangle, \langle 1,5,0\rangle, \langle 5,1,4\rangle\}$; $\pi_4 = \{\langle 0,5,1\rangle, \langle 5,0,6\rangle, \langle 1,6,0\rangle, \langle 6,1,5\rangle\}$;
$\pi_5 = \{\langle 0,6,1\rangle, \langle 6,0,7\rangle, \langle 1,7,0\rangle, \langle 7,1,6\rangle\}$; $\pi_6 = \{\langle 0,4,2\rangle, \langle 4,0,8\rangle, \langle 2,8,0\rangle, \langle 8,2,4\rangle\}$;
$\pi_7 = \{\langle 0,9,2\rangle, \langle 9,0,4\rangle, \langle 2,4,0\rangle, \langle 4,2,9\rangle\}$; $\pi_8 = \{\langle 0,5,2\rangle, \langle 5,0,7\rangle, \langle 2,7,0\rangle, \langle 7,2,5\rangle\}$;
$\pi_9 = \{\langle 0,10,2\rangle, \langle 10,0,5\rangle, \langle 2,5,0\rangle, \langle 5,2,10\rangle\}$; $\pi_{10} = \{\langle 0,6,2\rangle, \langle 6,0,9\rangle, \langle 2,9,0\rangle, \langle 9,2,6\rangle\}$;
$\pi_{11} = \{\langle 0,9,3\rangle, \langle 9,0,6\rangle, \langle 3,6,0\rangle, \langle 6,3,9\rangle\}$.

3.2 Partitioning into $DTS(4)$

We give direct constructions of partitions of the set of all the transitive triples of a v-set for the cases $v = 4, 5, 6, 9, 13$.

$\underline{v=4}$: A large set of DTS(4) will do.
$\underline{v=5}$: An overlarge set of DTS(4) will do.
$\underline{v=6}$: Let $V = Z_6$ be the 6-set. Under the action of the group Z_6, all the transitive triples on V are partitioned into 20 orbits $\mathcal{G}_i,\ 1 \le i \le 20$, which are listed as follows with representatives

$$\begin{array}{llllllll}
(0,2,4), & (0,4,5), & (0,3,2), & (0,3,1), & (0,4,2), & (0,2,1), & (0,3,4), & (0,3,5),\\
(0,1,2), & (0,5,2), & (0,1,4), & (0,5,4), & (0,1,3), & (0,5,1), & (0,5,3), & (0,1,5),\\
(0,2,3), & (0,4,3), & (0,2,5), & (0,4,1). & & & &
\end{array}$$

Choosing one transitive triple from each of the four orbits $\mathcal{G}_{4i-3},\ \mathcal{G}_{4i-2},\ \mathcal{G}_{4i-1}$, and $\mathcal{G}_{4i}$ to form a starter $DTS(4)$, denoted by π_i, where $1 \le i \le 4$, we obtain 4 orbits of $DTS(4)$s, each of which is of length 6, with the following starter $DTS(4)$s

$\pi_1 = \{(0,2,4),\ (2,0,1),\ (4,1,0),\ (1,4,2)\}$; $\pi_2 = \{(0,4,2),\ (4,0,5),\ (2,5,0),\ (5,2,4)\}$;
$\pi_3 = \{(0,1,2),\ (1,0,3),\ (2,3,0),\ (3,2,1)\}$; $\pi_4 = \{(0,1,3),\ (1,0,2),\ (3,2,0),\ (2,3,1)\}$.

Furthermore, let $\chi_i = \{(0,2,3),\ (3,5,0),\ (2,0,5),\ (5,3,2)\} + i$,
$\psi_i = \{(0,2,5),\ (3,5,2),\ (2,0,3),\ (5,3,0)\} + i,\ i = 0,1,2.$
The six $DTS(4)$s, χ_i and ψ_i $(i = 0,1,2)$, cover each of the transitive triples in orbits $\mathcal{G}_{17},\ \mathcal{G}_{18},\ \mathcal{G}_{19}$, and $\mathcal{G}_{20}$ exactly once. So all the 30 $DTS(4)$s give the required partition.
$\underline{v=9}$: Let $V = Z_9$ be the 9-set. Under the action of the group Z_9, all the transitive triples on V are partitioned into 56 orbits $\mathcal{G}_i,\ 1 \le i \le 56$, which are listed as follows with representatives

$$\begin{array}{llllllll}
(0,2,1), & (0,2,8), & (0,7,1), & (0,7,8), & (0,3,1), & (0,1,8), & (0,8,1), & (0,6,8),\\
(0,3,2), & (0,8,7), & (0,1,2), & (0,6,7), & (0,1,4), & (0,8,5), & (0,1,7), & (0,8,2),\\
(0,6,3), & (0,7,6), & (0,2,5), & (0,3,4), & (0,1,3), & (0,4,6), & (0,5,3), & (0,8,6),\\
(0,4,1), & (0,4,8), & (0,5,8), & (0,5,1), & (0,1,5), & (0,8,4), & (0,1,6), & (0,8,3),
\end{array}$$

$$\begin{array}{llllllll}(0,6,1), & (0,3,8), & (0,6,2), & (0,3,7), & (0,7,4), & (0,6,5), & (0,3,6), & (0,2,3), \\ (0,4,2), & (0,4,7), & (0,5,7), & (0,5,2), & (0,2,6), & (0,7,3), & (0,2,7), & (0,7,2), \\ (0,6,4), & (0,7,5), & (0,2,4), & (0,3,5), & (0,4,5), & (0,5,4), & (0,4,3), & (0,5,6).\end{array}$$

Choosing one transitive triple from each of the four orbits $\mathcal{G}_{4i-3}$, $\mathcal{G}_{4i-2}$, $\mathcal{G}_{4i-1}$, and $\mathcal{G}_{4i}$ to form a starter $DTS(4)$, denoted by π_i, where $1 \le i \le 14$, we obtain 14 orbits of DTS(4)s, each of which is of length 9, with the following starter $DTS(4)$s

$$\begin{array}{ll}
\pi_1 = \{(0,2,1),(1,3,0),(2,0,3),(3,1,2)\}; & \pi_2 = \{(0,3,1),(1,2,0),(2,1,3),(3,0,2)\}; \\
\pi_3 = \{(0,3,2),(2,1,0),(1,2,3),(3,0,1)\}; & \pi_4 = \{(0,1,4),(4,3,0),(3,4,1),(1,0,3)\}; \\
\pi_5 = \{(0,6,3),(3,1,0),(1,3,6),(6,0,1)\}; & \pi_6 = \{(0,1,3),(3,7,0),(7,3,1),(1,0,7)\}; \\
\pi_7 = \{(0,4,1),(1,5,0),(5,1,4),(4,0,5)\}; & \pi_8 = \{(0,1,5),(5,4,0),(4,5,1),(1,0,4)\}; \\
\pi_9 = \{(0,6,1),(1,4,0),(4,1,6),(6,0,4)\}; & \pi_{10} = \{(0,7,4),(4,1,0),(1,4,7),(7,0,1)\}; \\
\pi_{11} = \{(0,4,2),(2,6,0),(6,2,4),(4,0,6)\}; & \pi_{12} = \{(0,2,6),(6,4,0),(4,6,2),(2,0,4)\}; \\
\pi_{13} = \{(0,6,4),(4,2,0),(2,4,6),(6,0,2)\}; & \pi_{14} = \{(0,4,5),(5,1,0),(1,5,4),(4,0,1)\}.
\end{array}$$

$\underline{v = 13}$: From Section 3.1, we know that there is a partition of all the cyclic triples on Z_{13} into $13 \times 11 = 143$ pairwise disjoint MTS(4)s. For convenience, we call the 143 MTS(4)s $\mathcal{B}_i$, where $1 \le i \le 143$. Suppose each 4-set is $\{a,b,c,d\}$, then by the concrete construction of each $\mathcal{B}_i$, we know that each $\mathcal{B}_i$ $(1 \le i \le 143)$ consists of four cyclic triples $\langle a,b,c\rangle, \langle b,a,d\rangle, \langle c,d,a\rangle$ and $\langle d,c,b\rangle$, that is

$$\mathcal{B}_i = \{\langle a,b,c\rangle, \langle b,a,d\rangle, \langle c,d,a\rangle, \langle d,c,b\rangle\}.$$

For a cyclic triple $\langle a,b,c\rangle$, we call the transitive triples (a,b,c), (b,c,a), and (c,a,b) the three corresponding *shifts* of $\langle a,b,c\rangle$. Assigning shift to every block in $\mathcal{B}_i$, we get three collections $\mathcal{B}_i^j$ $(1 \le j \le 3)$ from $\mathcal{B}_i$, which are listed as follows:

$$\begin{array}{rcl}
\mathcal{B}_i^1 & = & \{(a,b,c),(b,a,d),(c,d,a),(d,c,b)\}; \\
\mathcal{B}_i^2 & = & \{(b,c,a),(a,d,b),(d,a,c),(c,b,d)\}; \\
\mathcal{B}_i^3 & = & \{(c,a,b),(d,b,a),(a,c,d),(b,d,c)\}.
\end{array}$$

It is not difficult to verify that each $\mathcal{B}_i^j$ $(1 \le i \le 143,\ 1 \le j \le 3)$ is a $DTS(4)$ since each $\mathcal{B}_i$ is an $MTS(4)$. So we get $143 \times 3 = 429$ $DTS(4)$s. Furthermore, because all the blocks in $\mathcal{B}_i$ form a partition of all the cyclic triples on Z_{13}, all the blocks in $\mathcal{B}_i^j$ form a partition of all the transitive triples on Z_{13}.

4 Recursive Constructions

A *t-wise balanced design* $S(t,K,v)$, is a pair $(X,\mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a collection of subsets of X, called *blocks*, such that the size of every block in the set $\mathcal{B}$ belongs to the set $K = \{k_1,\ldots,k_m\}$, and every t-subset of X is contained in exactly one block of $\mathcal{B}$. If $K = \{k\}$, then the design is called a *t-design*. Our recursive constructions depend on the existence of 3-*wise balanced design* $S(3,\{4,5\},v)$ for $v \equiv 1,2,4,5,8,10 \pmod{12}$ and $v \ne 13$ (see [3]), and the existence of $S(3,\{4,5,6\},v)$ for $v \equiv 0,1,2 \pmod 4$ and $v \ne 9,13$ (see [4]).

Theorem 4.1 *Let $(X, \mathcal{B})$ be an $S(3, K, v)$. If the set of all cyclic (respectively transitive) triples chosen from a k-set can be partitioned into pairwise disjoint copies of MTS(4)s (respectively DTS(4)s) for any $k \in K$, then there is a partition of the set of all cyclic (respectively transitive) triples chosen from a v-set into pairwise disjoint MTS(4)s (respectively DTS(4)s).*

Proof. For every block in $\mathcal{B}$, of size $k \in K$, there is a partition of all cyclic (respectively transitive) triples chosen from the k-set into pairwise disjoint copies of MTS(4)s (respectively DTS(4)s). Because each triple appears in a unique block of $\mathcal{B}$, the union of the partitions covers all cyclic (respectively transitive) triples chosen from the set X. ■

5 Main Results

Theorem 5.1 *There is a partition of the set of all cyclic triples chosen from a v-set into pairwise disjoint copies of MTS(4)s for all $v \equiv 1, 2, 4, 5, 8, 10 \pmod{12}$ and $v \geq 4$.*

Proof. For $v \equiv 1, 2, 4, 5, 8, 10 \pmod{12}$, $v \geq 4$ and $v \neq 13$, there exists an $S(3, \{4, 5\}, v)$. By Theorem 4.1 and the partitions for $v = 4, 5, 13$ constructed in Section 3.1, we can obtain the result. ■

Theorem 5.1 is equivalent with the result proved by Hartman and Phelps [1], and mentioned in reference [9], that a generalized idempotent 3-quasigroup whose conjugate invariant group contains the alternative group on 4 elements exists for exactly those same values of v. Our proof is different and shorter than the one given by Hartman and Phelps.

Theorem 5.2 *There is a partition of the set of all transitive triples chosen from a v-set into pairwise disjoint copies of DTS(4)s for all $v \equiv 0, 1, 2 \pmod{4}$ and $v \geq 4$.*

Proof. For $v \equiv 0, 1, 2 \pmod{4}$, $v \geq 4$ and $v \neq 9, 13$, there exists an $S(3, \{4, 5, 6\}, v)$. By Theorem 4.1 and the partitions for $v = 4, 5, 6, 9, 13$ constructed in Section 3.2, we can obtain the result. ■

Thus, we have proved that the necessary conditions for the partitions of the set of all cyclic (respectively transitive) triples chosen from a v-set into pairwise disjoint MTS(4)s (respectively DTS(4)s) are also sufficient.

Acknowledgements

The authors appreciate Professor Qingde Kang for his kind help and support. Also, the authors would like to thank the referees for their careful reading of the paper and helpful comments.

References

[1] A. Hartman and K.T. Phelps, Tetrahedral quadruple systems, *Utilitas Math.* 37 (1990), 181–189.

[2] S.H.Y. Hung and N.S. Mendelsohn, Directed triple systems, *J. Combin. Theory Ser. A* 14 (1973), 310–318.

[3] L. Ji, On the 3BD-closed set $B_3(\{4,5\})$, *Discrete Math.* 287 (2004), 55–67.

[4] L. Ji, On the 3BD-closed set $B_3(\{4,5,6\})$, *J. Combin. Des.* 12 (2004), 92–102.

[5] Q. Kang, A generalization of Mendelsohn triple systems, *Ars Combin.* 29C (1990), 207–215.

[6] Q. Kang and Y. Chang, A completion of the spectrum for large sets of transitive triple systems, *J. Combin. Theory Ser. A* 60 (1992), 287–294.

[7] Q. Kang, J. Lei and Y. Chang, The spectrum of large sets of disjoint Mendelsohn triple systems with any index, *J. Combin. Des.* 2 (1994), 351–358.

[8] N.S. Mendelsohn, A natural generalization of Steiner triple systems, in *Computers in Number Theory* (eds. A.O. Atkin and B.J. Birch), Academic Press, New York, 1971, pp. 323–338.

[9] L. Teirlinck, Generalized idempotent orthogonal arrays, in *Coding Theory and Design Theory, Part II, IMA Vol. Math. Appl.* 21, Springer-Verlag, 1990, pp. 368–378.

[10] Z. Tian, *On large sets and overlarge sets of combinatorial designs* (in Chinese), Hebei Normal University, Doctoral Thesis, 2003.

[11] H. Zhao and Q. Kang, Large sets of Hamilton cycle and path decompositions, *Discrete Math.* (to appear).

(Received 9 Aug 2006; revised 30 Dec 2007)