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# Perfect one-factorizations in line-graphs and planar graphs

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**Abstract**

A one-factorization of a regular graph $G$ is perfect if the union of any two one-factors is a Hamiltonian cycle. A graph $G$ is said to be P1F if it possess a perfect one-factorization. We prove that $G$ is a P1F cubic graph if and only if $L(G)$ is a P1F quartic graph. Moreover, we give some necessary conditions for the existence of a P1F planar graph.

## 1 Introduction

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. By $V(G)$ and $E(G)$ we denote the vertex-set and the edge-set, respectively, of a graph $G$. A one-factorization of $G$ is a partition of $E(G)$ into one-regular spanning subgraphs. A perfect one-factorization of $G$ is a one-factorization in which the union of any pair of one-factors is a Hamiltonian cycle of $G$. A graph is said to be perfectly one-factorable (P1F for short) if it admits a perfect one-factorization. In [5] P1F graphs are called Hamilton graphs, while in [6] they are called strongly Hamiltonian graphs and the perfect one-factorization is said to be a Hamilton decomposition, [5, 6]. The large number of papers dedicated to perfect one-factorizations of the complete graph $K_{2n}$ (see [10] for a survey) induces one to prefer the terminology of P1F graphs.

In [7] Kotzig and Labelle present a number of open problems for P1F regular graphs. The main result of this note is a partial solution to problem number 4 of their list. In our terminology their problem claims:
"Let $G$ be a graph with $|E(G)|$ even. Prove that $G$ is a P1F graph if and only if $L(G)$, the line-graph of $G$, is P1F".

As far as we know a proof of this result has not been published yet. One partial progress in this sense is the following result of Pike:

**Proposition 1** *[9] If $G$ is a $2k$-regular graph that has a perfect 1-factorization, then $L(G)$ is Hamilton decomposable.*

We propose a constructive proof of the problem posed by Kotzig and Labelle in the particular case in which $G$ is cubic and then we deduce two corollaries by the technique used in the proof. When $G$ is not cubic the question posed by Kotzig and Labelle remains an open problem.

Finally, in section 3, we consider the classes of planar P1F graphs of degree 3 and 4 and we produce some necessary conditions of Grinberg type for the existence of such graphs.

## 2 The main result

We denote by $L(G)$ the line-graph of $G$, namely the graph having the edges of $G$ as vertices and with two vertices adjacent whenever the corresponding edges are incident in $G$.

Moreover, we will make use of the standard operation on cubic graphs known as $Y$-reduction and of its inverse ($Y$-extension), defined as in Figure 1. It is straightforward

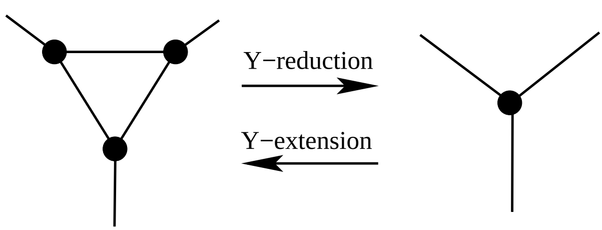


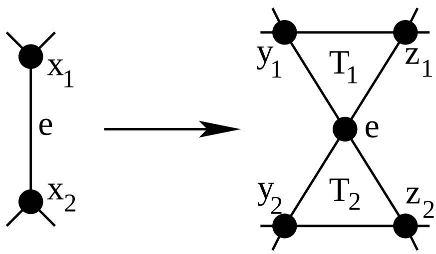

Figure 1: Y-operations

to see that $Y$-reduction and $Y$-extension are inner operations in the class of P1F cubic graphs (see for instance [5]).

**Proposition 2** *Let $G$ be a cubic graph with $|E(G)|$ even. Then the graph $G$ is P1F if and only if the graph $L(G)$ is P1F.*

*Proof.* Set $|E(G)| = 6n$. The graph $L(G)$ is 4-regular of order $6n$ and it admits a *one*-factor, say $f$, by the main result in [8]. By the very definition of $L(G)$, it can be partitioned into $4n$ triangles $T_i$, each triangle corresponding to a vertex of $G$, in such a way that no vertex of $L(G)$ lies in more than two of the triangles. Denote by $T$ the set of these triangles. Obviously each triangle in $T$ contains at most one edge of $f$. More precisely we can partition $T$ as the union of $S$ and $R$, where $S$ contains the $3n$ triangles with an edge of $f$ and $R$ contains the $n$ triangles without edges of $f$. The graph obtained by removing the edges of $f$ from $L(G)$ is cubic and contains copies of the $n$ triangles in $R$. We can reduce each of these $n$ triangles with a $Y$-reduction to obtain a smaller graph, say $H$. In what follows we prove that the graph $H$ is isomorphic to $G$.

Let $\phi$ be the natural bijection between the vertices $x_i$ of $G$ and the triangles $T_i$ in $T$. Define $\varphi_f$ as follows: if $T_i$ belongs to $S$ then $\varphi_f(T_i)$ is the unique vertex of $T_i$ not belonging to the edge of $f$ in $T_i$; if $T_i$ belongs to $R$ then $\varphi_f(T_i)$ is the vertex of $H$ in which the $Y$-operation reduced $T_i$. It is easy to see that $\varphi_f$ is a bijection between $T$ and $V(H)$ and then the composition $\varphi_f \circ \phi$ is a bijection between the vertices of $G$ and $H$. To prove that $\varphi_f \circ \phi$ is an isomorphism between $G$ and $H$ we have to prove that $[x_1, x_2]$ is an edge of $G$ if and only if $[\varphi_f \circ \phi(x_1), \varphi_f \circ \phi(x_2)]$ is an edge in $H$.




The vertices of the edge $[x_1, x_2]$ in $G$ correspond to triangles $T_1$ and $T_2$ in $L(G)$. Since the vertex $e$ in $L(G)$ belongs to an edge of $f$, exactly one of the edges $[e, y_1]$, $[e, z_1]$, $[e, y_2]$ and $[e, z_2]$ belongs to $f$. Without loss of generality we can suppose $[e, y_1] \in f$. This implies that $\varphi_f \circ \phi(x_1) = z_1$, by definitions of $\phi$ and $\varphi_f$. There are two possibilities: either $[y_2, z_2]$ belongs to $f$ or it does not. If $[y_2, z_2]$ belongs to $f$ then $\varphi_f \circ \phi(x_2) = e$; otherwise $[y_2, z_2]$ does not belong to $f$ and $\varphi_f \circ \phi(x_2)$ is the vertex of $H$ obtained by $T_2$ with a $Y$-reduction. In both cases the image of $[x_1, x_2]$ is an edge of $H$. The inverse argument can be used to prove the necessary condition. Since we have proved that $H$ is isomorphic to $G$, it admits a perfect one-factorization. As proved in [5] the use of a $Y$-extension (and its inverse) on a vertex of a cubic graph preserved the P1F property then the graph $L(G) \setminus E(f)$ inherits a perfect one-factorization $\mathcal{F} = \{f_1, f_2, f_3\}$ by $H$. Now we prove that $\overline{\mathcal{F}} = \{f_1, f_2, f_3\} \cup \{f\}$ is a perfect one-factorization of $L(G)$. It is straightforward to verify that $\overline{\mathcal{F}}$ is a one-factorization of $L(G)$, moreover the unions $f_i \cup f_j$, for $i \neq j$ and $i, j = 1, 2, 3$ are all Hamiltonian cycle by the fact that $\mathcal{F}$ is perfect for $L(G) \setminus E(f)$. It remains to prove that $f \cup f_i$ is a Hamiltonian cycle for $i = 1, 2, 3$. Note that if we walk on $f_1 \cup f_2$ from a prescribed vertex we meet the triangles $T_i$ in the same order as on $f_3 \cup f$. Since $f_1 \cup f_2$ is a Hamiltonian cycle, then we meet each $T_i$ and each vertex of $L(G)$ and then we return to the starting vertex, hence the same holds walking on $f_3 \cup f$ and this proves that $f_3 \cup f$ is a Hamiltonian cycle. Repeating the same argument for $f_1 \cup f$ and $f_2 \cup f$ we obtain that $\overline{\mathcal{F}}$ is perfect. The inverse argument proves that if $L(G)$ is P1F then we can produce a perfect one-factorization of $G$. $\square$

Note that in the proof of Proposition 2 we have used a completely arbitrary choice of the one-factor $f$ in $L(G)$, this naturally leads to the following results:

**Corollary 1** *Let $G$ be a P1F cubic graph with $|E(G)|$ even. Each one-factor of $L(G)$ belongs to a perfect one-factorization of $L(G)$.*

Recall that a graph $G$ is minimally one-factorable if every one-factor belongs to precisely one one-factorization (see [3]) and $G$ is uniquely edge colorable if it admits a unique proper coloring of the edges.

**Corollary 2** *Let $G$ be a uniquely edge colorable cubic graph with $|E(G)|$ even, then $L(G)$ is minimally one-factorable.*

Proposition 2 can be also used to construct examples of P1F 4-regular graphs starting from known P1F cubic graphs.

**Example 1** *The generalized Petersen graph $GP(10,2)$ is P1F (see [2]) then its line-graph is P1F*

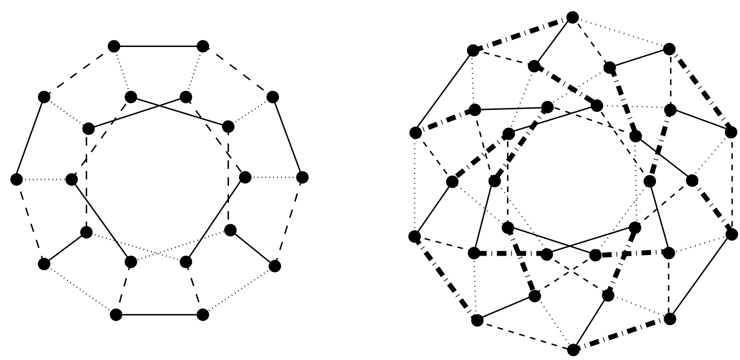


Figure 2: $GP(10,2)$ and its line-graph are P1F

## 3 P1F planar graphs

In this section, $G$ will always denote a planar regular graph.

In [4] Grindberg obtained a necessary condition for a planar graph $G$ to be Hamiltonian. Let $X$ be a set of faces of $G$ and let $d(F)$ be the degree of a face $F$ of $G$ (that is, the number of edges in its boundary), we define the function $g$ by $g(X) = \sum_{F \in X}(d(F) - 2)$.

**Theorem 1 (Grinberg)** *Let $G$ be a planar graph which contains a Hamiltonian cycle $C$. Denote by $X_1$ the set of faces of $G$ interior to $C$ and by $X_2$ the set of faces exterior to $C$. Then $g(X_1) = g(X_2)$.*

Bondy and Haggkvist (see [1]) establish, using the result of Grinberg, a similar necessary condition for a 4-regular planar graph to admit a Hamiltonian decomposition.

**Theorem 2 (Bondy-Haggkvist)** *Let $G$ be a 4-regular planar graph which is decomposable into two edge-disjoint Hamiltonian cycles $C$ and $D$. Denote by $X_{11}$, $X_{12}$, $X_{21}$, and $X_{22}$ the sets of faces of $G$ interior to both $C$ and $D$, interior to $C$ but exterior to $D$, interior to $D$ but exterior to $C$, and exterior to both $C$ and $D$, respectively. Then $g(X_{11}) = g(X_{22})$ and $g(X_{12}) = g(X_{21})$.*

Each 4-regular P1F planar graph is decomposable into edge-disjoint Hamiltonian cycles. In fact the union of two disjoint pairs of the four one-factors of a perfect one-factorization gives rise to a Hamilton decomposition. The converse is false. There exist examples of Hamilton decomposable graphs that are not P1F. For instance the graph in Figure 3 is Hamilton decomposable and is not P1F, since it is the line-graph of the cube that is not P1F (see [2] and Proposition 2).

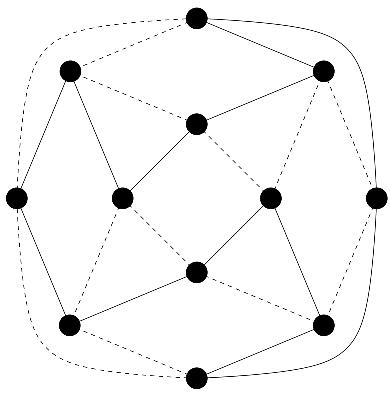


Figure 3: A 4-regular planar graph that is Hamilton decomposable but not P1F

In the following propositions we obtain two results similar to the statement of Bondy and Haggkvist.

Let $G$ be a cubic P1F planar graph. Denote by $\mathcal{F} = \{f_1, f_2, f_3\}$ a perfect one-factorization of $G$. Denote by $C_1, C_2, C_3$ the three Hamiltonian cycles obtained by pairwise union of one-factors of $\mathcal{F}$. Finally denote by $X_{i_1 i_2 i_3}$, where $i_1, i_2, i_3 \in \mathbb{Z}_2$, the sets of faces of $G$ such that a face is interior to $C_j$ if and only if $i_j = 1, (j = 1, 2, 3)$. For instance $X_{010}$ contains the faces of $G$ which are interior to $C_2$ and exterior to $C_1$ and $C_3$. In what follows we can suppose without loss of generality that the external face of $G$ is in $X_{000}$.

**Proposition 3** *Let $G$ be a cubic P1F planar graph of order $2n$.*
*Then $g(X_{i_1 i_2 i_3}) = n - 1$ when $i_1 + i_2 + i_3$ is even.*

*Proof.* A-priori there are eight possible sets of faces: nevertheless since we have supposed that the external face of $G$ is in $X_{000}$, then one can verify that $X_{100}, X_{010}, X_{001}$, $X_{111}$ are empty sets: let $F$ be a face and define a vector $(j_1, j_2, j_3)$ such that $F$ is external to cycle $C_i$ if and only if $j_i = 1$. Let $e$ be an edge of $F$ and let $F'$ be the face on the other side of $e$. Then for each of the 2 cycles that contain edge $e$, the vector for $F'$ will have its corresponding two entries be opposite of the respective values for the vector for $F$. The entry for the cycle that does not contain $e$ is the same for both $F$ and $F'$. It follows that the parity for every face's vector will be the same, and since the parity for the external face of the graph is an even number of 1's, then every face must have an even number of 1's in its vector. Thus each face is internal to an even number of cycles, and therefore $X_{100}, X_{010}, X_{001}, X_{111}$ are empty.

By application of Grinberg's Theorem on each of the Hamiltonian cycles $C_1$, $C_2$, $C_3$, we obtain the relations:

$$g(X_{000}) + g(X_{011}) = g(X_{101}) + g(X_{110})$$

$$g(X_{000}) + g(X_{101}) = g(X_{011}) + g(X_{110})$$

$$g(X_{000}) + g(X_{110}) = g(X_{011}) + g(X_{101})$$

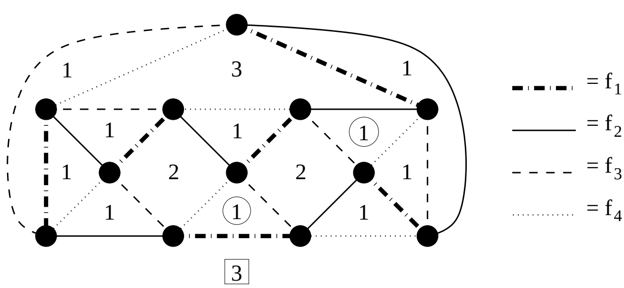
Combining them pairwise gives rise to $g(X_{000}) = g(X_{110})$, $g(X_{000}) = g(X_{101})$ and $g(X_{000}) = g(X_{011})$. Then $g(X_{i_1 i_2 i_3})$ is constant when $i_1+i_2+i_3$ is even. The assertion follows from the fact that for each planar regular graph $g(X) = 4(n-1)$ holds (where $X$ is the set of all faces of $G$). □

The same idea can be applied for 4-regular P1F planar graphs. We denote by $C_1 = f_1 \cup f_2$, $C_2 = f_1 \cup f_3$, $C_3 = f_1 \cup f_4$, $C_4 = f_2 \cup f_3$, $C_5 = f_2 \cup f_4$ and $C_6 = f_3 \cup f_4$ the six Hamiltonian cycles obtained by a perfect one-factorization of a graph $G$. We suppose also in this case that the external face of $G$ is in $X_{000000}$. The following proposition holds:

**Proposition 4** *Let $G$ be a 4-regular P1F planar graph of order $2n$. Then*
$g(X_{000000}) = g(X_{011110}) = g(X_{101101}) = g(X_{110011})$ *and*
$g(X_{111000}) = g(X_{100110}) = g(X_{010101}) = g(X_{001011})$.

*Proof.* This is by application of Grinberg's Theorem on each of the Hamiltonian cycles $C_j$. □

NOTE: The case $g(X_{000000}) \neq g(X_{111000})$ really occurs; see for instance the graph in the figure below where the faces in $X_{111000}$ are denoted by a circle and the unique face in $X_{000000}$ by a square; then $g(X_{000000}) = 3$ and $g(X_{111000}) = 2$.



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