AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **41** (2008), Pages 283–290

Anti-magic labellings of a class of planar graphs

K. Ramanjaneyulu

Department of Mathematics
S C R Engineering College
Chilakaluripet, Guntur Dt.
India
kakkeraram@yahoo.co.in

V. Ch. Venkaiah Kishore Kothapalli*

Center for Security, Theory, and Algorithmic Research
International Institute of Information Technology
Gachibowli, Hyderabad – 500 032
India
{kkishore, venkaiah}@iiit.ac.in

Abstract

This paper focuses on anti-magic labellings of planar graphs of type (a, b, c) introduced by Lih (*Util. Math.* 24 (1983), 165–197). We show that a certain class of planar graphs defined using the complete graphs and a class of planar graphs defined using the complete bipartite graphs are (1,1,0) and (1,1,1)-anti-magic under certain mild conditions.

1 Introduction

We focus on connected undirected graphs $G = (V, E)$. In its simplest form a graph labeling is a bijection f from a subset of the elements of a graph to the set of positive integers. The elements of a graph are its vertices, edges, and faces if the graph is planar. For a vertex labeling, the domain of f is the set of vertices. For an edge labeling the domain of f is taken to be the set of edges. For simplicity, we let $n = |V|$ and $m = |E|$. For all other standard graph-theoretic notation we refer the reader to [17].

The origin of labellings can be attributed to Rosa [12] or Graham and Sloane [9]. Several properties of the bijection f are worth studying from a purely combinatorial perspective as well as from an application point of view. Graph labellings have found application in partitioning complete graphs into isomorphic subgraphs,

* Corresponding author.

in combinatorial and algebraic structures such as cyclic difference sets, characterizing neo-fields, generalized and near-complete mappings [5, 6]. Other applications of labellings include coding theory, network addressing, database management, and astronomy [4, 7, 5, 6, 16, 10].

A class of labellings called *magic labellings* motivated by magic squares was introduced by Sedlacek [13]. Lih [11] extended the notion of magic labellings of planar graphs to assign labels also to the faces of a planar graph. Lih defines a magic (anti-magic) labeling of type (1,1,1) of a planar graph $G = (V, E)$ as a bijection $f : V \cup E \cup F \to \{1, 2, \dots, |V| + |E| + |F|\}$ so that for each interior face the sum of the labels of the face, the edges of the face, and the vertices of the face, is fixed (*resp.* distinct). Similarly a magic (anti-magic) labeling of type (1,1,0) for a planar graph is defined as a bijection from $V \cup E$ to $\{1, 2, \dots, n+m\}$ so that for each interior face the sum of labels of the vertices and the edges of the face is fixed (*resp.* distinct). A subclass of these magic labellings involving faces is that of consecutive labellings where all the faces of a certain length get consecutive labels.

1.1 Our Results

In this paper, we show that a certain class of planar graphs obtained from the class of complete graphs are (1,1,0) and (1,1,1)–antimagic. We also define a class of planar graphs derived from the class of complete bipartite graphs and show that the class is (1,1,0) and (1,1,1)–antimagic.

Both the results are based on an embedding of the planar graphs in question. Hence the results require careful consideration to arrive at a proper embedding and then a labelling so as to show that the class is antimagic of a certain type. The labels we produce are also consecutive.

1.2 Organization of the Paper

The rest of the paper is organized as follows. Section 2 introduces the class of planar graphs Pl_n and shows that the class is (1,1,0) and (1,1,1)–antimagic. Section 3 defines the class of planar graphs $Pl_{m,n}$ obtained from complete bipartite graphs and shows that the class is (1,1,0) and (1,1,1)–antimagic. The paper ends with some concluding remarks.

2 The Class Pl_n of Planar Graphs

In [1], Babujee defines a class of planar graphs obtained by removing certain edges from the complete graphs. The class of planar graphs so obtained are denoted by Pl_n and contain the maximum number of edges possible in a planar graph on n vertices. We report the definition from [1].

Definition 2.1 *Let K_n be the complete graph on n vertices $V_n = \{1, 2, \dots, n\}$. The class of graphs Pl_n has the vertex set V_n and the edge set $E_n = E(K_n) \setminus \{(k, \ell) : 3 \leq k \leq n-2, k+2 \leq \ell \leq n\}$.*

The embedding we use for Pl_n is described as follows. Place the vertices $v_1, v_2, \ldots,$ v_{n-2} along a vertical line in that order with v_1 at the bottom and v_{n-2} at the top as shown in Figure 1. Now place the vertices v_{n-1} and v_n as the end points of a horizontal line segment (perpendicular to the line segment used for placing the other $n-2$ points) with v_{n-1} to the left of v_n so that the vertices v_n, v_{n-1}, and v_{n-2} form a triangular face. See Figure 1 for an illustration. The edges of the graph Pl_n can now be drawn without any crossings [17]. All the faces of this graph are of length 3. From now on, in this section, when we refer to the faces by the vertices of the face, we use the vertex numbers from the embedding described.

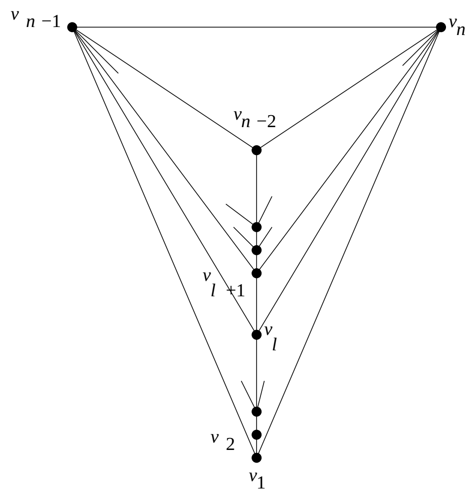

Figure 1: The class Pl_n.

Theorem 2.2 *The graph Pl_n is $(1,1,0)$–antimagic where $n \geq 5$.*

Proof. Consider a planar class $Pl_n(V, E)$ with n vertices $v_1, v_2, \ldots, v_n$ and $3n-6$ edges. Define a bijective function f from $V \cup E$ to $\{1, 2, \ldots, 4n-6\}$ as follows.

The labelling of the edges are:

$$\begin{array}{lcl} f(v_i, v_{i+1}) & = & i, \ \text{ for } 1 \leq i \leq n-1. \\ f(v_n, v_{n-2}) & = & n \\ f(v_{n-1}, v_i) & = & 2n-2-i, \ \text{ for } 1 \leq i \leq n-3. \\ f(v_n, v_i) & = & 3n-5-i, \ \text{ for } \ \leq i \leq n-3. \end{array}$$

The labelling of the vertices are:

$$f(v_i) = 3n - 6 + i \text{ for } 1 \leq i \leq n.$$

It may be seen that the number of interior faces in Pl_n is $2n-5$. The interior face consisting of vertices $v_n, v_{n-1}, v_{n-2}, v_n$ has sum equal to $15n-24$. Another

$n-3$ interior faces consisting of vertices $v_\ell, v_{\ell+1}, v_{n-1}, v_\ell$ have sum $14n+\ell-23$ for $1 \le \ell \le n-3$, and the rest of the $n-3$ interior faces consisting of vertices $v_\ell, v_{\ell+1}, v_n, v_\ell$ have sum $16n+\ell-28$, for $1 \le \ell \le n-3$. All these sums can be verified to be different.

Hence, the graph Pl_n, $n \ge 5$, is $(1,1,0)$–antimagic. □

Theorem 2.3 *The graph Pl_n is $(1,1,1)$–face antimagic where $n \ge 5$.*

Proof. Consider a planar class $Pl_n(V,E)$ with n vertices $v_1, v_2, \ldots, v_n$ and $3(n-2)$ edges and $|F| = 2n-5$ interior faces. Define a bijective function f from $V \cup E \cup F$ to $\{1, 2, \ldots, (6n-11)\}$ as follows. The labelling of the edges are:

$$\begin{aligned}
f(v_i, v_{i+1}) &= i, \quad \text{for } 1 \le i \le n-3.\\
f(v_{n-2}, v_{n-1}) &= n-1\\
f(v_{n-1}, v_n) &= n-2\\
f(v_n, v_{n-2}) &= n\\
f(v_{n-1}, v_{n-k}) &= n+2k-5, \quad \text{for } 3 \le k \le n-1.\\
f(v_n, v_{n-k}) &= n+2k-4, \quad \text{for } 3 \le k \le n-1
\end{aligned}$$

The labelling of the vertices are:

$$f(v_i) = 3n-6+i, \quad \text{for } 1 \le i \le n.$$

The labelling of the faces are as follows. For the face bounded by v_{n-1}, v_n, v_{n-2}, v_{n-1} the label is $4n-5$. For the face bounded by $v_{n-1}, v_{n-1-\ell}, v_{n-2-\ell}, v_{n-1}$ the label is $4n-6+2\ell$ for $1 \le \ell \le n-3$. For the face bounded by $v_n, v_{n-1-\ell}$, $v_{n-2-\ell}, v_n$ the label is $4n-5+2\ell$, $\ell = 1, 2, \ldots, (n-3)$.

It may be seen that the number of interior faces in Pl_n is $2n-5$. The interior face consisting of vertices $v_n, v_{n-1}, v_{n-2}, v_n$ has sum equal to $19n-29$. Another $n-3$ of these interior faces consisting of vertices $v_{n-1}, v_{n-1-\ell}, v_{n-2-\ell}, v_{n-1}$ have sum $19n+3\ell-34$ for $1 \le \ell \le n-3$. The rest of the $n-3$ interior faces consisting of vertices $v_n, v_{n-1-\ell}, v_{n-2-\ell}, v_n$ have sum $19n+3\ell-30$, for $1 \le \ell \le n-3$. All these sums can be verified to be different.

Hence, the graph Pl_n, $n \ge 5$ is $(1,1,1)$–face antimagic. □

3 The Class $Pl_{m,n}$ of Bipartite Planar Graphs

We define another class of planar graphs which is obtained from the complete bipartite graph $K_{m,n}$, $m, n \ge 3$ by removing some edges to make it planar graph, which is called a bipartite planar class and it is denoted by $Pl_{m,n}$. The graph $Pl_{m,n}$ has the maximum number of edges permissible in a planar bipartite graph.

Definition 3.1 *Let $K_{m,n}(V_m, U_n)$ be the complete bipartite graph on $V_m = \{v_1, v_2, \ldots, v_m\}$ and $U_n = \{u_1, u_2, \ldots, u_n\}$. The class of graphs $Pl_{m,n}(V,E)$ has the vertex set $V_m \cup U_n$ and the edge set $E = E(K_{m,n}(V_m, U_n)) \setminus \{(v_\ell, u_p) : 3 \le \ell \le m$ and $2 \le p \le n-1\}$.*

This graph is a bipartite planar graph with maximum number of edges $2m+2n-4$ and $m+n$ vertices[1]. We now describe the embedding we use for our proofs. Place the vertices $u_1, u_2, \ldots, u_n$ in that order along a horizontal line segment with u_1 as the left-endpoint and u_n as the right end-point as shown in Figure 2. Place the vertices $v_m, v_{m-1}, \ldots, v_3, v_1$ in that order along a vertical line segment with v_m as the top end-point and v_1 as the bottom end-point so that this entire line segment is above the horizontal line segment where the vertices u_1 through u_n are placed. Finally, place v_2 below the horizontal line segment so that the vertices v_1, u_k, v_2, u_{k+1} form a face of length 4 for $1 \leq k \leq n-1$. Notice that though we talk about placement along a line segment, no edges other than those mentioned in the definition are to be added. From now on, in this section, when we refer to the faces by listing the vertices forming that face, we use the vertex numbers given by the above embedding.

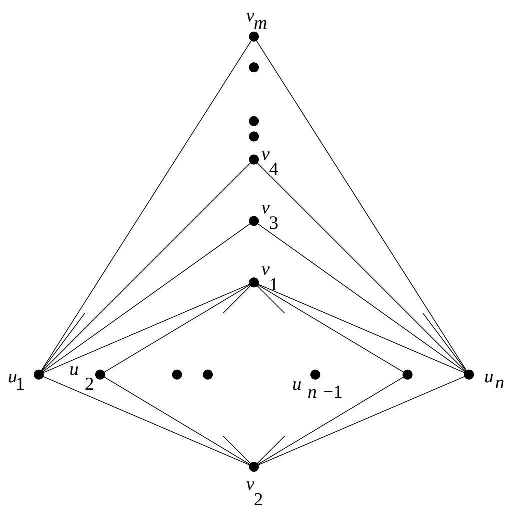

Figure 2: The class $Pl_{m,n}$.

Theorem 3.2 *For all $m, n \geq 3$ the graph $Pl_{m,n}$ is $(1,1,0)$–antimagic provided either m is odd or m is even and $4n+m \not\equiv 0 \pmod 6$.*

Proof. Consider the planar class $Pl_{m,n}(V, E)$ with $m+n$ vertices $v_1, v_2, \ldots, v_m$, $u_1, u_2, \ldots, u_n$ and $2m+2n-4$ edges. Define a bijective function f from $V \cup E$ to $\{1, 2, \ldots, (3m+3n-4)\}$ as follows.

[1]From Euler's formula concerning planar bipartite graphs $G=(V,E)$, $|E| \leq 2|V|-4$ as each face has length at least 4.

The labelling of the edges are:

$$\begin{aligned}
f(v_1,u_i) &= i, \quad \text{for } 1 \le i \le n.\\
f(v_2,u_i) &= n+i, \quad \text{for } 1 \le i \le n.\\
f(u_1,v_j) &= 2n+j-2, \quad \text{for } 3 \le j \le m.\\
f(u_n,v_j) &= 2n+m+j-4, \quad \text{for } 3 \le j \le m.
\end{aligned}$$

The labelling of the vertices are:

$$\begin{aligned}
f(u_i) &= 2n+2m-4+i, \quad \text{for } 1 \le i \le n.\\
f(v_i) &= 3n+2m-4+i, \quad \text{for } 1 \le i \le m.
\end{aligned}$$

The graph $Pl_{m,n}$ has $m+n-3$ interior faces. The interior face consisting of vertices u_1, v_1, u_n, v_3, u_1 has sum $16n + 9m - 10$. A further $n - 1$ faces consisting of vertices $v_1, u_k, v_2, u_{k+1}, v_1$ have sum $12n + 8m + 6k - 10$ for $1 \le k \le n - 1$. The remaining $m - 3$ faces consisting of vertices $u_1, v_{2+\ell}, u_n, v_{3+\ell}, u_1$ have sum $19n + 10m + 6\ell - 12$ for $1 \le \ell \le m - 3$. All these sums can be seen to be different if either m is odd or m is even and $4n + m \not\equiv 0 \pmod 6$.

So the graph $Pl_{m,n}$ is $(1, 1, 0)$–antimagic provided either m is odd or m is even and $4n + m \not\equiv 0 \pmod 6$. □

Theorem 3.3 *For all $m, n \ge 3$ the graph $Pl_{m,n}$ is $(1, 1, 1)$–face antimagic provided either $m > 2n - 7$ or $5n + m \not\equiv 0 \pmod 7$.*

Proof. Consider a planar class $Pl_{m,n}(V, E)$ with $m + n$ vertices and $2m + 2n - 4$ edges and $|F| = m+n-3$ interior faces. Define a bijective function f from $V \cup E \cup F$ to $\{1, 2, \ldots, 4n + 4m - 7\}$ as follows. The labels of the edges and the vertices are same as in the proof of Theorem 3.2 but are however repeated here for the sake of clarity.

The labelling of the edges are:

$$\begin{aligned}
f(v_1,u_i) &= i, \quad \text{for } 1 \le i \le n.\\
f(v_2,u_i) &= n+i, \quad \text{for } 1 \le i \le n.\\
f(u_1,v_j) &= 2n+j-2, \quad \text{for } 3 \le j \le m.\\
f(u_n,v_j) &= 2n+m+j-4, \quad \text{for } 3 \le j \le m.
\end{aligned}$$

The labelling of the vertices are:

$$\begin{aligned}
f(u_i) &= 2n+2m-4+i, \quad \text{for } 1 \le i \le n.\\
f(v_i) &= 3n+2m-4+i, \quad \text{for } 1 \le i \le m.
\end{aligned}$$

The labelling of the faces are as follows. For the face bounded by $v_1, u_k, v_2, u_{k+1}, v_1$ the label is $3n + 3m - 4 + k$ for $1 \le k \le n - 1$. For the face bounded by u_1, v_1, u_n, v_3, u_1 the label is $4n+3m-4$. For the face bounded by $u_1, v_{2+\ell}, u_n, v_{3+\ell}, u_1$ the label is $4n + 3m - 4 + \ell$ for $1 \le \ell \le m - 3$.

The graph $Pl_{m,n}$ has $m+n-3$ interior faces. The interior face consisting of vertices u_1, v_1, u_n, v_3, u_1 has sum $20n + 12m - 14$. A further $n - 1$ faces consisting of vertices

$v_1, u_k, v_2, u_{k+1}, v_1$ have sum $15n + 11m + 7k - 14$ for $1 \leq k \leq n-1$. The remaining $m-3$ faces consisting of vertices $u_1, v_{2+l}, u_n, v_{3+\ell}, h_1$ have sum $23n + 13m + 7\ell - 16$ for $1 \leq \ell \leq m-3$. All these sums can be seen to be different if either $m > 2n-7$ or $5n + m \not\equiv 0 \pmod 7$.

Hence, the graph $Pl_{m,n}$ is $(1,1,1)$–face antimagic provided either $m > 2n-7$ or $5n + m \not\equiv 0 \pmod 7$. □

4 Conclusions

We have shown that the class of planar graphs introduced in Section 2 is (1,1,0) and (1,1,1)–antimagic. Similar results are shown for the class of bipartite graphs introduced in Section 3.

Lih's definition of magic and anti-magic labellings of type (a, b, c) for planar graphs, was extended by Bača and Miller [2, 3] as follows. Define the weight of a face under a $(1,1,1)$ labeling as the sum of the labels of the face, the edges, and the vertices of the face. A labeling of plane graph G is called d–antimagic if for every ℓ the weights of faces of length ℓ form an arithmetic progression. It would be interesting to see if any of these classes are d–antimagic [3] for any value of d.

References

[1] J. Baskar Babujee, Planar graphs with maximum edges—antimagic property, *The Mathematics Education* 37(4) (2003), 194–198.

[2] M. Bača, E. Baskoro and M. Miller, Antimagic valuations for the special class of plane graphs, In *Combin. Geom. Graph Theory: Indonesia-Japan joint conf. (IJCCGGT)* vol. 3330, Springer (2003), pp. 58–64.

[3] M. Bača and M. Miller, On d-antimagic labelings of type $(1,1,1)$ for prisms, *J. Combin. Math. Combin. Comput.* 44 (2003), 199–207.

[4] G. S. Bloom and S. W. Golomb, Applications of numbered undirected graphs, *Proc. IEEE* 65 (1977), 562–570.

[5] G. S. Bloom and D. F. Hsu, On graceful digraphs and a problem in network addressing, *Congr. Numer.* 35 (1982), 91–103.

[6] G. S. Bloom and D. F. Hsu, *On graceful directed graphs that are computational models of some algebraic systems*, Wiley, New York, 1985.

[7] S. El-Zanati and C. Vanden Eynden, On α-valuations of disconnected graphs, *Ars Combin.* 61 (2001), 129–136.

[8] J. A. Gallian, A dynamic survey of graph labeling, *Electr. J. Combin.* 14, 2007.

[9] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, *SIAM J. Algebraic and Discrete Methods* 1 (1980), 382–404.

[10] Jeannette M. Janssen, *Channel assignment and graph labeling*, in *Handbook of Wireless Networks and Mobile Computing*, pp. 95–117. John Wiley & Sons, Inc., New York, NY, USA, 2002.

[11] K.-W. Lih, On magic and consecutive labelings of plane graphs, *Util. Math.* 24 (1983), 165–197.

[12] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs (Internat. Sympos., Rome, 1966)* pp. 349–355, (1967), Gordon and Breach, New York; Dunod, Paris.

[13] J. Sedlacek, Problem 27, In *Proc. Symposium on Theory of Graphs and its Applic.* (1963), 163–167.

[14] B. M. Stewart, Magic graphs, *Canad. J. Math.* 18 (1966), 1031–1059.

[15] B. M. Stewart, Supermagic complete graphs, *Canad. J. Math.* 19 (1967), 427–438.

[16] M. Sutton, *Summable graphs labellings and their applications*, PhD Thesis, Department of Computer Science, University of Newcastle, NSW, 2001.

[17] D. West, *Introduction to Graph Theory*, Prentice-Hall, 2001.

(Received 4 July 2007; revised 30 Jan 2008)