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The neighbourhood polynomial of a graph

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Abstract

We examine *neighbourhood polynomials*, which are generating functions for the number of faces of each cardinality in the neighbourhood complex of a graph. We provide explicit polynomials for hypercubes, for graphs not containing a four-cycle and for the graphs resulting from joins and Cartesian products. We also show that the closure of the roots are dense in the complex plane except possibly in the disc $|z+1|<1$.

1 The Neighbourhood Polynomial of a Graph

There are a number of graph polynomials that have been widely studied. Chromatic polynomials count the number of proper colourings of a graph. Matching polynomials enumerate matchings. Independence polynomials are generating polynomials for the number of independent sets of each cardinality. (All Terminal) reliability polynomials provide the probability of communication between all pairs of vertices given that edges are independently operation with the same probability. For each polynomial there is an underlying complex, which puts all of these polynomials in a common framework.

A *(simplicial) complex* on a finite set X is a collection $\mathcal{C}$ of subsets of X, closed under containment. Each set in $\mathcal{C}$ is called a *face* of the complex, and the maximal faces (with respect to containment) are called *facets* or *bases*. The *dimension* of a complex $\mathcal{C}$ is the maximum cardinality of a face. The *f–vector* (or *face–vector*) of a d–dimensional complex $\mathcal{C}$ is $(f_0, f_1, \ldots, f_d)$, where f_i is the number of faces of cardinality i in $\mathcal{C}$. Finally the *f–polynomial* of a d–dimensional complex $\mathcal{C}$ is the generating function $f_{\mathcal{C}}(x) = \sum_i f_i x^i$ for the f–vector $(f_0, f_1, \ldots, f_d)$ of the complex.

For each of the previously described graph polynomials, there is a complex for which the graph polynomial is a simple evaluation of the f–polynomial. The independence complex $\mathcal{I}(G)$ of graph G is the complex on the vertex set V of G whose faces are the independent sets of G. The independence polynomial is merely the f–polynomial of the independence complex. The complex on the edge set of G whose

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faces are those sets of edges whose removal leaves G connected is called the *cographic* matroid of G, $\mathrm{Cog}(G)$, and the reliability polynomial of G is in fact

$$\mathrm{Rel}(G,p) = p^m f_{\mathrm{Cog}(G)}\left(\frac{1-p}{p}\right),$$

where m is the number of edges of G. Given a linear order $<$ of the edges E of G, the *broken circuit complex*, $\mathrm{BC}(G,<)$ is the complex on E whose faces are subsets of the edges that don't contain a broken circuit (that is, a circuit minus its $<$-least edge). Then

$$\pi(G,x) = x^n f_{\mathrm{BC}(G,<)}(-1/x),$$

where n is the number of vertices of G.

One of the most startling applications of simplicial complexes to graph theory is undoubtedly Lovász's proof [25] of the chromatic number of Kneser graphs. His argument centers on the *neighbourhood complex* $\mathcal{N}(G)$ of a graph, whose vertices are the vertices of the graph and whose faces are subsets of vertices that have a common neighbour. We define a univariate polynomial, which we call the *neighbourhood polynomial* of graph G:

$$\mathrm{neigh}_G(x) = \sum_{U \in \mathcal{N}(G)} x^{|U|}.$$

While there have been a number of articles that explored properties of neighbourhood complexes [2, 16, 24, 25], the neighbourhood polynomial has not been previously investigated.

As an example, in a four-cycle $\{a,b,c,d\}$, the empty set trivially has a common neighbour (as the graph has at least one vertex) while each of the single vertices has a neighbour. Each set $\{a,c\}$ and $\{b,d\}$ has two common neighbours, but one suffices, and there is no subset of three vertices that have a common neighbour. Thus the neighbourhood complex is $\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,d\}\}$ and the associated neighbourhood polynomial is

$$\mathrm{neigh}_{C_4}(x) = 1 + 4x + 2x^2.$$

Note that this is the same as the neighbourhood polynomial for a path with four vertices, so, as might be expected, these polynomials are not unique to the underlying graph.

As another example, note that

$$\mathrm{neigh}_{K_n}(x) = (1+x)^n - x^n,$$

as every subset of the vertices of a complete graph except the entire vertex set has a common neighbour. Similarly, the neighbourhood polynomial of the complete bipartite graph $K_{m,n}$ is $(1+x)^m + (1+x)^n - 1$ (as the first two terms have overcounted the empty set).

The facets (i.e., maximal faces) of the neighbourhood complex of a graph G are simply the neighbourhoods of each vertex. Thus the number of facets is unusually small for a complex—it is at most the number of vertices in the graph and moreover, they can be enumerated in polynomial time. This contrasts sharply with all of the other graph polynomials mentioned earlier. The difficulty in determining the neighbourhood complex and polynomials lies in the recognition of the overcounting of subsets.

Let $N_1, \ldots, N_k$ be the maximal (with respect to containment) neighbourhoods of the vertices of a graph G with n vertices and m edges. In general, $k \leq n$ as some vertices may have the same neighbourhoods, or one might be a subset of the other. A set belongs to the neighbourhood complex of G if and only if it is a subset of one of the N_i's. Thus inclusion-exclusion can be used to enumerate such sets, and to generate the neighbourhood polynomial. Unfortunately, the algorithm is exponential, so it is of little use in calculations. On closer examination, and assuming that G has no isolated vertices, a first order approximation for the neighbourhood polynomial is

$$\sum_{v\in V}(1+x)^{\mathrm{deg}(v)} - x\sum_{v\in V}(\mathrm{deg}(v)-1) - (n-1) = \sum_{v\in V}(1+x)^{\mathrm{deg}(v)} - x(2m-n) - (n-1)$$

where $\mathrm{deg}(v)$ is the degree of v. This corresponds to counting all subsets of each of the facets (i.e. neighbourhoods) and correcting for counting the empty set n times and each vertex v $\mathrm{deg}(v)$ many times. Those sets that are in the neighbourhoods of two or more vertices are also overcounted but it is more difficult to see how to easily correct the formula. But for one class of graphs, the formula is correct as it stands. We say G is C_4*-free* if G does not contain C_4 as a subgraph (not necessarily induced). If G is C_4-free then two vertices can have at most one common neighbour and therefore the only non-empty sets that have more than one common neighbour are the singletons. This argument proves the following result.

Theorem 1 *Let G be C_4-free with n vertices and m edges. Then*

$$neigh_G(x) = \sum_{v\in V}(1+x)^{deg(v)} - x(2m-n) - (n-1).$$

An immediate observation is the following.

Corollary 2 *The neighbourhood polynomial for a C_4-free graph depends only on the degree sequence of the graph and can be calculated in polynomial time.*

Theorem 1 gives us the neighbourhood polynomials for many graphs, including:

- If $G = C_n$, a cycle of length $n > 4$, then $\mathrm{neigh}_G(x) = 1 + nx + nx^2$;
- If G is an r-regular graph of girth at least 5, $\mathrm{neigh}_G(x) = n(1+x)^r - n(r-1)x - (n-1)$. In particular, for the Petersen graph, $\mathrm{neigh}_G(x) = 1+10x+30x^2+10x^3$;
- If G is a tree, then $\mathrm{neigh}_G(x) = \sum_{v\in V}(1+x)^{\mathrm{deg}(v)} - x(n-1) - (n-1)$.

2 Graph Operations and Neighbourhood Polynomials

What is the effect that various graph operations have on the neighbourhood polynomial? The first is trivial, relying solely on the fact that the only sets of the disjoint union of two graphs having a common neighbour are the sets in each of the graphs that have a common neighbour, with the empty set being the only such set the graphs have in common.

Lemma 3 *Let G and H be graphs and $G \cup H$ be the disjoint union of G and H. Then*

$$neigh_{G \cup H}(x) = neigh_G(x) + neigh_H(x) - 1.$$

2.1 Joins

Let G_a be the *join* of G and a (i.e. a is adjacent to every vertex of G). Then it is easy to verify directly that $\mathrm{neigh}_{G_a}(x) = (1+x)^{|V(G)|} + x \cdot \mathrm{neigh}_G(x)$. The *join* of two graphs G and H is formed from their disjoint union by adding in all the edges between G and H. The formula for the neighbourhood polynomial of the join of two graphs is only slightly more complicated.

Theorem 4 *Let G and H be graphs with $n_G = |V(G)|$ and $n_H = |V(H)|$. Then*

$$neigh_{G+H}(x) = (1+x)^{n_H} neigh_G(x) + (1+x)^{n_G} neigh_H(x) - neigh_G(x) neigh_H(x).$$

Proof: Any set $S = S_G \cup S_H$, with $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$, that has a common neighbour v, either has $v \in G$ and v a common neighbour of S_G in G or $v \in H$ and v a common neighbour of S_H in H. Conversely, any set of the form $S = S_G \cup S_H$, with $S_G \subseteq V(G)$ and $S_H \subseteq V(H)$, where either S_G has a common neighbour in G or S_H has a common neighbour in H, is a subset of the vertices of $G + H$ that has a common neighbour. The only subsets that are counted twice in this process are those $S = S_G \cup S_H$ where both S_G has a common neighbour in G and S_H has a common neighbour in H. Now $(1+x)^{n_H}\mathrm{neigh}_G(x)$ is the generating function for all sets of the form $S = S_G \cup S_H$ with $S_G \subseteq V(G)$ and S_H having a common neighbour in H. Likewise, $(1+x)^{n_G}\mathrm{neigh}_H(x)$ is the generating function for all sets of the form $S = S_G \cup S_H$ with $S_H \subseteq V(H)$ and S_G having a common neighbour in G. Finally, $\mathrm{neigh}_G(x)\mathrm{neigh}_H(x)$ is the generating function for all the subsets $S = S_G \cup S_H$ where both S_G has a common neighbour in G and S_H has a common neighbour in H (the overcounted amount). It follows that

$$\mathrm{neigh}_{G+H}(x) = (1+x)^{n_H}\mathrm{neigh}_G(x) + (1+x)^{n_G}\mathrm{neigh}_H(x) - \mathrm{neigh}_G(x)\mathrm{neigh}_H(x).$$

∎

For example, consider the join of two paths: $f_{P_n}(x) = 1 + nx + (n-2)x^2$ and $f_{P_m}(x) = 1 + mx + (m-2)x^2$ and therefore

$$\begin{aligned} f_{P_n+P_m}(x) &= (1+x)^m(1+nx+(n-2)x^2) \\ &+(1+x)^n(1+mx+(m-2)x^2) \\ &-(1+nx+(n-2)x^2)(1+mx+(m-2)x^2). \end{aligned}$$

2.2 Cartesian Products

The graph $G\Box H$ is the *Cartesian product* of graphs G and H. Its vertex set is $V(G) \times V(H)$ where $(u,v) \in V(G\Box H)$ is adjacent to $(u',v') \in V(G\Box H)$ if either $u = u'$ and vv' is an edge of H or $v = v'$ and uu' is an edge of G. It can easily be seen that $G\Box H$ contains $|V(G)|$ copies of H (i.e. (a,H) for all $a \in V(G)$) and also into $|V(H)|$ copies of G.

Theorem 5 *Let G and H be graphs with $|V(G)| = n_G$, $|E(G)| = m_G$, $|V(H)| = n_H$ and $|E(H)| = m_H$. Then*

$$\begin{aligned} neigh_{(G\Box H)}(x) &= 1 + n_H(neigh_G(x) - 1) + n_G(neigh_H(x) - 1) \\ &+ \sum_{(a,y)\in V(G)\times V(H)} ((1+x)^{deg_G(a)} - 1)((1+x)^{deg_H(y)} - 1) \\ &-2m_G m_H x^2 - n_G n_H x. \end{aligned}$$

Proof: Let $|V(G)| = n_G$, $|E(G)| = m_G$, $|V(H)| = n_H$ and $|E(H)| = m_H$. In computing $\text{neigh}_{G\Box H}(x)$, from each copy of G and of H there is a contribution of $\text{neigh}_G(x)$ and $\text{neigh}_H(x)$, respectively, for a total of $n_H\text{neigh}_G(x) + n_G\text{neigh}_H(x)$. The empty set has been counted once in each copy, so $n_G + n_H - 1$ needs to be subtracted. Each singleton has been counted twice — (a,x) is counted once in the copy (a,H) and once in (G,x) — so $n_G n_H x$ needs to be subtracted.

Any other set, S, with a common neighbour must contain two points, say (a,z) and (b,y) where $a \neq b$ and $z \neq y$. The common neighbour of S must be (a,y) or (b,z) (or both). If S contains more than 2 elements, say $(c,y) \in S$, then $c \neq a$, as otherwise there is no common neighbour of S. Then the common neighbour of S must be (a,y) and all other vertices of S are of the form (d,y) or (a,w). Moreover, $S' = \{d \mid (d,y) \in S\}$ has common neighbour a and $S'' = \{w \mid (a,w) \in S\}$ has the common neighbour y. All such S contribute $((1+x)^{\deg_G(a)} - 1)((1+x)^{\deg_H(y)} - 1)$. If $\{a,b\} \in E(G)$ and $\{y,z\} \in E(H)$ then the sets $\{(a,y),(b,z)\}$ and $\{(a,z),(b,y)\}$ have both been counted twice; the first with (a,z) and (b,y) as common neighbours, the second with (a,y) and (b,z). Thus $2m_G m_H x^2$ needs to be subtracted. This count includes all the sets with common neighbours. Thus

$$\begin{aligned} \text{neigh}_{(G\Box H)}(x) &= 1 + n_H(\text{neigh}_G(x) - 1) + n_G(\text{neigh}_H(x) - 1) \\ &+ \sum_{(a,y)\in V(G)\times V(H)} ((1+x)^{\deg_G(a)} - 1)((1+x)^{\deg_H(y)} - 1) \end{aligned}$$

$$-2m_G m_H x^2 - n_G n_H x.$$

■

Theorem 5 can be used recursively to find the neighbour polynomial for the hypercube, Q_n. However, the direct counting method of the proof is straightforward.

Corollary 6 $neigh_{Q_n}(x) = 1 + 2^n((1+x)^n - 1) - 2^{n-1}\binom{n}{2}x^2 - (n-1)2^n x.$

Proof: The neighbourhood of any of the 2^n vertices contributes $(1+x)^n$ to the polynomial. In the sum of all these, the empty set has been counted 2^n times. If v is a singleton then it has been counted $n \cdot \deg(v)$ times where $\deg(v)$ is the degree of v. Every pair in a neighbourhood (and there are $2^n\binom{n}{2}$ of these) is counted twice (as they lie in exactly two C_4s). This proves the formula. ■

3 Roots of Neighbourhood Polynomials

It is natural to inquire about the nature and location of the roots of any graph polynomial; such investigations have been carried out for matching polynomials [18], chromatic polynomials [3, 6, 9, 10, 11, 23, 26, 27, 28, 30, 31, 32], reliability [4, 5, 30, 29], and independence polynomials [7, 8, 12, 13, 17, 19, 20, 21], to name but a few.

One of the most interesting questions concerns the closure of the roots. The closure of chromatic roots has recently been shown to be the entire complex plane [31], and a similar result holds for independence polynomials. The closure of the roots of reliability polynomials is not known, though it contains $\{z \in \mathbb{C} : |z-1| \leq 1\}$ and some roots barely outside the disk [29]. The roots of matching polynomials are not dense in the plane, as they are always real (see [18]).

For neighbourhood polynomials, in the last section, we conjecture that the roots are dense in the complex plane. Here we show that the closure contains all of the complex plane except possibly for a disc of radius 1. To do so, we begin by finding an explicit formula for the neighbourhood polynomials for a recursive family of graphs.

Proposition 7 *Consider the repeated join of P_4, as follows. Let $G_1 = P_4$ and let $G_m = G_{m-1} + G_{m-1}$, for $m > 1$. Then*

$$neigh_{G_n}(x) = \prod_{i=0}^{n-1}(z^{2^i} + (z-1)^{2^i}),$$

where $z = (1+x)^2$.

Proof: It is easy to check that the number of vertices of G_m is $n_{G_m} = 2^{m+1}$. The formula from Theorem 4 gives

$$\begin{aligned}\mathrm{neigh}_{G_m}(x) &= 2(1+x)^{2^{m+1}}\mathrm{neigh}_{G_{m-1}}(x) - (\mathrm{neigh}_{G_{m-1}}(x))^2 \\ &= \mathrm{neigh}_{G_{m-1}}(x)\left(2(1+x)^{2^{m+1}} - \mathrm{neigh}_{G_{m-1}}(x)\right).\end{aligned}$$

(We note that regardless of the actual graph used for G_1 the roots of $\mathrm{neigh}_{G_{n+1}}(x)$ include all the roots of $\mathrm{neigh}_{G_n}(x)$ plus some others.)

Now, with $G_1 = P_4$ then

$$\begin{aligned}\mathrm{neigh}_{G_2}(x) &= (1+4x+2x^2)(2(1+x)^4 - (1+4x+2x^2)) \\ &= (2(x+1)^2 - x^2)(2(1+x)^4 - 2(x+1)^2 + x^2).\end{aligned}$$

Substituting $z = (1+x)^2$ we find that

$$\mathrm{neigh}_{G_2}(x) = (z + (z-1))(z^2 + (z-1)^2).$$

We proceed inductively. Suppose that

$$\mathrm{neigh}_{G_n}(x) = \prod_{i=0}^{n-1}(z^{2^i} + (z-1)^{2^i}).$$

Then

$$\mathrm{neigh}_{G_{n+1}}(x) = (\prod_{i=0}^{n-1}(z^{2^i} + (z-1)^{2^i})(2z^{2^n} - \prod_{i=0}^{n-1}(z^{2^i} + (z-1)^{2^i}). \tag{1}$$

Consider the expansion of the formal product $\prod_{i=0}^{n-1}(a^{2^i} + b^{2^i})$. Each term a^k occurs exactly once (consider the base 2 representation of k) therefore

$$\prod_{i=0}^{n-1}(a^{2^i} + b^{2^i}) = \sum_{i=0}^{2^n-1} a^i b^{2^n-1-i}.$$

Note also that the latter term arises in

$$a^{2^n} - b^{2^n} = (a-b)\cdot\sum_{i=0}^{2^n-1} a^i b^{2^n-1-i}.$$

Substituting $z = z$, $b = z-1$ we get that

$$z^{2^n} - (z-1)^{2^n} = \sum_{i=0}^{2^n-1} z^i (z-1)^{2^n-1-i},$$

that is,

$$z^{2^n} - \left(\sum_{i=0}^{2^n-1} z^i (z-1)^{2^n-1-i}\right) = (z-1)^{2^n}.$$

Using this, we find from (1) that

$$\mathrm{neigh}_{G_{n+1}}(x) = \left(\prod_{i=0}^{n-1}(z^{2^i} + (z-1)^{2^i}\right)\left(2z^{2^n} - \prod_{i=0}^{n-1}(z^{2^i} + (z-1)^{2^i}\right)$$

$$
\begin{aligned}
&= \left(\prod_{i=0}^{n-1}(z^{2^i}+(z-1)^{2^i}\right)\left(z^{2^n}+\left(z^{2^n}-\prod_{i=0}^{n-1}(z^{2^i}+(z-1)^{2^i}\right)\right)\\
&= \left(\prod_{i=0}^{n-1}(z^{2^i}+(z-1)^{2^i}\right)\left(z^{2^n}+(z-1)^{2^n}\right)\\
&= \prod_{i=0}^{n}(z^{2^i}+(z-1)^{2^i}).
\end{aligned}
$$

This completes the inductive proof. ■

Theorem 8 *The closure of the roots of neighbourhood polynomials contains the complex plane except possibly for the unit disc centered at $z=-1$.*

Proof: From the previous proposition the roots of neighbourhood polynomials contain the set of $\pm r^{1/2}-1$ as r ranges over all of the roots of the polynomials $z^{2^n}+(z-1)^{2^n}$ $(n\geq 0)$. What is the closure of the roots of $\{z^{2^n}+(z-1)^{2^n} : n\geq 0\}$? We note first that any such root lies on the line $Re(z)=1/2$, since $z^{2^n}+(z-1)^{2^n}=0$ implies $|z|=|z-1|$, implying that any such z is equidistant from the points $z=0$ and $z=1$. In fact, $\frac{z}{z-1}$ must be a 2^n–th root ω of -1, and the union of all such roots are dense in the unit circle. Taking $\omega=a+bi$ with $a^2+b^2=1$, with a little bit of algebra we see that

$$
z \;=\; \frac{1}{2}-\frac{b}{2(1-a)}i.
$$

The closure of the values for a is $[-1,1]$, and as $b=\pm\sqrt{1-a^2}$, the closure of the values for $\frac{b}{2(1-a)}$ is $(-\infty,\infty)$. It follows that closure of the roots of the polynomials $z^{2^n}+(z-1)^{2^n}$ contains the line $Re(z)=1/2$ (in fact, it is equal to it). As $z=(1+x)^2$, it follows that if r is a root of $z^{2^n}+(z-1)^{2^n}$, then $\pm r^{1/2}-1$ is a root of $\mathrm{neigh}_{G_{n+1}}(x)$.

Now, if we replace each vertex of G_n with an independent set of size m (that is, we form the lexicographic product $G_n[\overline{K_m}]$), then it is not hard to see that

$$
\mathrm{neigh}_{G_n[\overline{K_m}]}(x)=\mathrm{neigh}_{G_n}((1+x)^m-1)
$$

as the sets of vertices that have common neighbours in $G_n[\overline{K_m}]$ arise precisely from sets of vertices of G_n that have a common vertex, with each vertex v replaced by a nonempty subset of the set of m independent vertices that replace v.

Thus, if we take any root r of $z^{2^n}+(z-1)^{2^n}$, then we have seen that $\pm r^{1/2}-1$ is a root of $\mathrm{neigh}_{G_{n+1}}(x)$, and the m–th roots of $\pm r^{1/2}-1$ are roots of neighbourhood polynomials. The closure of r ranges over the line $Re(z)=1/2$, in particular,

$$
\bigcup_{n\geq 0}\{\pm r^{1/2}-1 : r \text{ is a root of } z^{2^n}+(z-1)^{2^n}\} \qquad (2)
$$

contains complex numbers of all moduli greater than or equal to $1-1/\sqrt{2}$.

Let c be any complex number of modulus greater than 1, and let $\varepsilon > 0$. We need to show that there is a root of a neighbourhood polynomial in the disc of radius ε centered at c. Now there is a positive δ such that if the argument and moduli of a complex number c' are within δ of c then c' lies in the disc. Now all sufficiently large $m \geq M$, an m–th root of any non–zero complex number will have its argument within δ of the argument of c. We can also choose M large enough so that we can find a root R of a neighbourhood polynomial with the modulus of $R+1$ lies between $(|c|-\delta)^M$ and $(|c|+\delta)^M$. (Both $(|c|+\delta)^M$ and $(|c|+\delta)^M - (|c|-\delta)^M$ grow arbitrary large as $M \to \infty$, and the previous paragraph implies that we can choose a root R of the neighbourhood polynomial of one of the G_n's whose modulus is arbitrarily close to any number in $[1-1/\sqrt{2}, \infty)$.) Then one of the m–th roots of $R+1$, ρ, will lie in the disc.

Thus the closure of the roots of $\text{neigh}_{G_n[\overline{K_m}]}(1+x)$ contains $\{z \in \mathbb{C} : |z| \geq 1\}$. It follows immediately that the closure of the roots of $\text{neigh}_{G_n[\overline{K_m}]}(x)$ contain $\{z \in \mathbb{C} : |z+1| \geq 1\}$, and we are done. ■

While we do not know if the closure of the roots of neighbourhood polynomials are dense in the unit disc centered at $z = -1$, we do know that this disc does contain some roots. Some arise from the proof of the previous theorem from the roots of (2) (in particular, $-1 + 1/\sqrt{2}$ is in the closure of the roots). For $n \geq 5$, $\text{neigh}_{C_n}(x) = 1 + nx + nx^2$ has as its roots

$$-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4}{n}}$$

and these approach -1 and 0 from the right and left, respectively. This example brings to mind a few other questions about the roots. When are the roots real? What is the closure of the real roots of neighbourhood polynomials? All of the real roots we have seen so far are bounded (between -1 and 0). There are, of course, no nonnegative roots as neighbourhood polynomials have positive coefficients. We can show that the real roots of neighbourhood polynomials are not bounded.

Proposition 9 *The set of real roots of neighbourhood polynomials is unbounded.*

Proof: Given a graph G_n with n vertices, consider the neighbourhood polynomial for $G + \overline{K_m}$. By Theorem 4 we have

$$\text{neigh}_{G+\overline{K_m}}(x) = (1+x)^n + \left((1+x)^k - 1\right)\text{neigh}_G(x).$$

Consider $G = C_n$ for $n \geq 5$ and let $k = n-3$. Then

$$\text{neigh}_{C_n+\overline{K_m}}(x) = (1+x)^n + \left((1+x)^{n-3} - 1\right)(1 + nx + nx^2).$$

Setting $x = -n-2$ we find that

$$\begin{aligned}&\text{neigh}_{C_n+\overline{K_m}}(-n-2) = \\ &\quad \big[8(n+1)^{n-1} - 15(n+1)^{n-2} + 7(n+1)^{n-3} + (-1)^{n-1}8(n+1)^3 \\ &\qquad + (-1)^n 8(n+1)^2 + (-1)^{n-1}15(n+1) + (n-1) + (-1)^n 7\big](-1)^{n-2}.\end{aligned}$$

Setting $x = -n - 1$ we find that

$$\begin{aligned} \text{neigh}_{C_n+\overline{K_m}}(-n-1) &= (-n)^n + \big((-n)^{n-3} - 1)(n^3 + n^2 + 1) \\ &= (-1)^{n-3}(n^{n-1} + n^{n-3} + \ldots\big). \end{aligned}$$

So for n sufficiently large, the neighbourhood polynomial of $C_n + \overline{K_m}$ changes sign on $[-n-1, -n-2]$. It follows that there is a negative real root in this interval. Thus the negative real roots of neighbourhood polynomials are unbounded. ∎

4 Problems

The previous section raises some interesting questions regarding the roots of the neighbourhood polynomials.

Problem 1 *Is the closure of the roots of neighbourhood polynomials the entire complex plane* $\mathbb{C}$*?*

Problem 2 *Is the closure of the real roots of neighbourhood polynomials the entire negative real axis?*

Problem 3 *For which graphs does the neighbourhood polynomial have only real roots?*

Considerations of real roots have been useful (for example, for matching polynomials) in proving that various combinatorial sequences are *unimodal.* A sequence $\langle b_0, \ldots, b_d \rangle$ of real numbers is unimodal if there is a $k \in \{0, \ldots, d\}$ such that

$$b_0 \leq b_1 \leq \cdots \leq b_{k-1} \leq b_k \geq b_{k+1} \geq \ldots \geq b_d.$$

Various combinatorial sequences are known to be unimodal, such as the binomial coefficients $\langle \binom{n}{i} \rangle$, the Stirling numbers of the first kind (c.f. [15]) and the coefficients of the Gaussian polynomials. Other sequences have long been conjectured to be unimodal: the absolute value of the coefficients in the chromatic polynomial [26, 22], various coefficients in expansions of the reliability polynomial [14] and various sequences related to matroids (c.f. [33]). Most of these ask about the unimodality of various f–vectors of families of complexes (broken circuit complexes and matroidal complexes).

A result of Erdös et al. [1] shows that the independence polynomial can be far from unimodal. Are neighbourhood polynomials unimodal?

In fact, neighbourhood polynomials need not be unimodal. Consider the graph consisting of disjoint complete graphs of order $m \geq 4$, K_m, and a cycle, C_n of order $n \geq 5$, with a bridge between them. Then the neighbourhood polynomial of the resulting graph $G_{m,n}$ has the form

$$\begin{aligned}
&\mathrm{neigh}_{G_{m,n}}(x) = \\
&\quad 1+(n+m)x+\left[\binom{m}{2}+n+2+(m-1)\right]x^2+\left[\binom{m}{3}+\binom{m-1}{2}+1\right]x^3 \\
&\quad +\left[\binom{m}{4}+\binom{m-1}{3}\right]x^4+\ldots+\left[\binom{m}{m-1}+\binom{m-1}{m-2}\right]x^{m-1}+x^m
\end{aligned}$$

and we see that for sufficiently large n ($n \geq \frac{1}{6}m^3-\frac{1}{2}m^2-\frac{5}{3}m+1$) the coefficient of x^2 will exceed both the coefficients of x and x^3. Thus for such graphs, the neighbourhood polynomials need not be unimodal.

Problem 4 *Determine when the coefficients of a neighbourhood polynomial are unimodal.*

A result of Newton (c.f. [15]) implies that if all the roots of a polynomial with positive coefficients are real, then the sequence of coefficients is in fact unimodal.

Another question we raise is the complexity of calculating the neighbourhood polynomial of a graph. Most graph polynomials are known to be intractable, so we suspect this is the case for neighbourhood polynomials. On the other hand, the degree and the largest and smallest coefficients can be calculated quickly (the largest nonzero coefficient is the number of distinct neighbourhoods of vertices).

Problem 5 *Is determining the neighbourhood polynomial # P complete?*

Problem 6 *Is determining the value of the neighbourhood polynomial at* $x = 1$ *(i.e. counting the number of subsets of vertices that have a common neighbour) # P complete?*

We remark that if one wishes to *approximate* the neighbourhood polynomial in polynomial time, one can use the fact that you can calculate the bottom and top coefficients quickly and then use the well known *Kruskal–Katona* bounds (c.f. [14]) for simplicial complexes (these give a lower bound for the number of faces of size $i-1$ in terms of the number of faces of size i).

Problem 7 *Determine necessary and sufficient conditions for a polynomial with positive integer coefficients to be a neighbourhood polynomial of a graph.*

We end with some remarks about a generalization of neighbourhood complexes and neighbourhood polynomials. Let G be a graph on vertex set V with distinguished subsets of vertices M and W (M and W may overlap, be identical or even empty). We say that a subset U of M has a *witness* in W if there is a common neighbour of U in W, that is, there is a vertex $w \in W$ such that w is adjacent to each vertex in U; such a vertex w is called a *witness* in W for set U. Vertices have to be considered in two guises: one as part of a subset with a common neighbour, or as one of the common neighbours.

The set of subsets of M that have a witness in W form a complex, as a witness for a set is also a witness for its subsets. We call this complex the (G, M, W)–*neighbourhood complex* of G, and denote it as $\mathcal{N}_{M,W}(G)$. The *generalized neighbourhood polynomial* of graph G (written as $\text{neigh}_{G,M,V}(x)$) is simply the f–polynomial of this complex. To make this more specific, for a vertex v of G, let $N(v)$ denote the set of neighbours of v. For a subset U of M, define

$$C(U) = \cap_{y \in U} N(y),$$

that is, $C(U)$ is the set of *common neighbours* of U. Note that $C(\emptyset) = V$.

Definition 10 *Given a graph G, the generalized neighbourhood polynomial of G is*

$$neigh_{G,M,W}(x) = \sum_{U \subset M,\, C(U) \cap W \neq \emptyset} x^{|U|}.$$

In the case where $U = W = V$, the complex and polynomial coincide with the neighbourhood complex and polynomial, respectively, of the graph G. It is not hard to see that the class of generalized neighbourhood complexes of graphs coincides with the class of all (finite) simplicial complexes.

We originally considered generalized neighbourhood complexes in our quest for a recursion for calculating neighbourhood polynomials. We have been able to prove the following recursions:

Proposition 11 *Let G be a graph on vertex V. Let M and W be subsets of V with $v \in M$. Then*

$$neigh_{G,M,W}(x) = neigh_{G,M-v,W}(x) + x \cdot neigh_{G,M-v,W \cap N(v)}(x).$$

For $M = \emptyset$, we have

$$neigh_{G,\emptyset,W}(x) = \begin{cases} 1 & \text{if } W \neq \emptyset; \\ 0 & \text{if } W = \emptyset. \end{cases}$$

■

Proposition 12 *Let G be a graph on vertex V. Let M and W be subsets of V with $v \in W$. Then*

$$neigh_{G,M,W}(x) = neigh_{G,M,W-w}(x) + neigh_{G,M \cap N(v),v}(x) - neigh_{G,M \cap N(v),W-v}(x).$$

For $W = \emptyset$, we have

$$neigh_{G,M,\emptyset}(x) = 0$$

and for $W = \{v\}$ we have

$$neigh_{G,M,\{v\}}(x) = (1 + x)^{N(v) \cap M}.$$

■

Problem 8 *Is there a recursion for calculating neighbourhood polynomials?*

References

[1] Y. Alavi, P. Malde, A.J. Schwenk and P. Erdős, The vertex independence sequence of a graph is not constrained, *Congr. Numer.* 58 (1987), 15–23.

[2] F.G. Arenas and M.L. Puertas, The Neighborhood Complex of an Infinite Graph, *Divulgaciones Mathematicás* 8 (2000), 69–74.

[3] F. Brenti, G.F. Royle and D.G. Wagner, Location of zeros of chromatic and related polynomials of graphs, *Canad. J. Math.* 46 (1994), 55–80.

[4] J.I. Brown and C.J. Colbourn, On the log concavity of reliability and matroidal sequences, *Adv. Appl. Math.* 15 (1994), 114–127.

[5] J.I. Brown and C.J. Colbourn, Roots of the reliability polynomial, *SIAM J. Disc. Math.* 5 (1992), 571–585.

[6] J.I. Brown, Chromatic polynomials and order ideals of monomials, *Discrete Math.* 189 (1998), 43–68.

[7] J.I. Brown, K. Dilcher, and R.J. Nowakowski, Roots of Independence Polynomials of Well Covered Graphs, *J. Algebraic Combin.* 11 (2000), 197–210.

[8] J.I. Brown and R.J. Nowakowski, Bounding the Roots of Independence Polynomials, *Ars Combin.* 58 (2001), 113–120.

[9] J.I. Brown, C.A. Hickman, A.D. Sokal and D.G. Wagner, On the chromatic roots of generalized theta graphs, *J. Combin. Theory Ser. B* 83 (2001), 272–297.

[10] J.I. Brown and C.A. Hickman, On chromatic roots of large subdivisions of graphs, *Discrete Math.* 242 (2002), 17–30.

[11] J.I. Brown and C.A. Hickman, On Chromatic Roots with Negative Real Part, *Ars Combin.* 63 (2002), 211–221.

[12] J.I. Brown, C.A. Hickman, and R.J. Nowakowski, On the Location of Roots of Independence Polynomials, *J. Algebraic Combin.* 19 (2004), 273–282.

[13] J.I. Brown, C.A. Hickman, and R.J. Nowakowski, The Independence Fractal of a Graph, *J. Combin. Theory Ser. B* 87 (2003), 209–230.

[14] C.J. Colbourn, *The Combinatorics of Network Reliability*, Oxford University Press, Oxford and New York, 1987.

[15] L. Comtet, *Advanced Combinatorics*, Reidel Pub. Co., Boston, 1974.

[16] P. Csorba, C. Lange, I. Schurr and A. Wassmer, Box complexes, neighborhood complexes and the chromatic number, *J. Combin. Theory Ser. A* 108 (2004), 159–168.

[17] D.C. Fisher and A.E. Solow, Dependence Polynomials, *Discrete Math. 82* (1990), 251–258.

[18] C. Godsil, *Algebraic Combinatorics*, Chapman & Hall, New York, 1993.

[19] I. Gutman, An identity for the independence polynomials of trees, *Publ. Inst. Math. (Belgrade)* 50 (1991), 19–23.

[20] I. Gutman, Some analytic properties of the independence and matching polynomials, *Match.* 28 (1992), 139–150.

[21] C. Hoede and X. Li, Clique polynomials and independent set polynomials of graphs, *Discrete Math.* 25 (1994), 219–228.

[22] S. G. Hoggar, Chromatic polynomials and logarithmic concavity, *J. Combin. Theory Ser. B* 16 (1974), 248–254.

[23] B. Jackson, A zero-free interval for chromatic polynomials of graphs, *Combin. Probab. Comput.* 2 (1993), 325–336.

[24] M. Khale, The neighborhood complex of a random graph, *J. Combin. Theory Ser. A* 114 (2007), 380–387.

[25] L. Lovàsz, Kneser's conjecture, chromatic number and homotopy, *J. Combin. Theory Ser. A* **25** (1978) 319-324.

[26] R. Read, An introduction to Chromatic Polynomials, *J. Combin Theory* **4** (1968), 52-71.

[27] R.C. Read and G.F. Royle, Chromatic Roots of Families of Graphs, in: *Graph Theory, Combinatorics and Applications* Vol. 2, (Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk, Eds.), Wiley, New York, 1991 1009–1029.

[28] R.C. Read and W.T. Tutte, Chromatic polynomials, in: *Selected Topics in Graph Theory* 3, (L.W. Beineke and R.J. Wilson, Eds.), Academic Press, New York, 1988, pp. 15–42.

[29] G. Royle and A.D. Sokal, The Brown-Colbourn conjecture on zeros of reliability polynomials is false, *J. Combin. Theory Ser. B* 91 (2004), 345–360.

[30] A.D. Sokal, Bounds on the complex zeros of (di)chromatic polynomials and Potts-model partition functions, *Combin. Probab. Comput.* 10 (2001), 41–77.

[31] A.D. Sokal, Chromatic roots are dense in the whole complex plane, *Combin. Probab. Comput.* 13 (2004), 221–261.

[32] V. Thier, Graphen and Polynome, Diploma thesis, TU München, 1983.

[33] D.J.A. Welsh, *Matroid Theory*, Academic Press, London, 1976.

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