AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **42** (2008), Pages 83–90

Degree sum conditions in graph pebbling

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Abstract

Given a graph G on n vertices and a distribution, D, of pebbles on the vertices of G, we define a *pebbling move* to be the removal of two pebbles from a given vertex and the placement of one on an adjacent vertex. If D has n pebbles and if after a sequence of pebbling moves we can place a pebble on any specified vertex then we call G Class 0. We give a sufficient degree sum condition for G to be Class 0.

1 Introduction

We let $G = (V, E)$ be a simple graph with vertex set $V := V(G)$ and edge set $E := E(G)$ where $|V| = n$. For sets $A, B \subset V$ and $A \cap B = \emptyset$, we use $G[A, B]$ to denote the bipartite subgraph of G containing all edges with one end-vertex in each of A and B. We define the degree of a vertex v, denoted $d(v)$, to be the number of edges incident with v and denote its set of neighbors by $N(v)$. The minimum degree, maximum degree and independence number of a graph G are denoted $\delta(G), \Delta(G)$ and $\alpha(G)$, respectively. We let $\sigma_k(G) = min\{d(x_1)+\ldots+d(x_k)|x_1, \ldots, x_k \text{ are independent in } G\}$.

Given a distribution D of pebbles on the vertices of G, which may be thought of as an assignment of integer weights to the vertices of G, we say that a *pebbling move* consists of removing two pebbles from a vertex and then placing one pebble on an adjacent vertex. The number of pebbles placed on a vertex v is denoted $D(v)$. Given a target vertex r, known as the *root vertex*, we say that r can be *reached*, or *pebbled*, if after a sequence of pebbling moves it is possible to place a pebble on r. The *pebbling number* of $G, \pi(G)$, is the least integer m such that, regardless of how m pebbles are distributed on the vertices of G, after a sequence of pebbling moves it is possible to reach any vertex. It is easy to see that $\pi(G) > n - 1$ since placing each of $n - 1$ pebbles on a distinct vertex leaves one vertex, r, without a pebble and no pebbling moves possible. Graphs for which $\pi(G) = n$ are known as Class 0 graphs and this class is the object of our consideration. It is obvious that such graphs must be connected and in fact must be 2-connected. The latter is seen true if we let x be a cut-vertex of G, let the components of $G(V \setminus \{x\}, E)$ be G_1, G_2 and $v_i \in G_i$ and consider the following distribution in G of n pebbles, $D(v_1) = 3, D(v_2) = 0, D(x) = 0$ and $D(v) = 1$ for all other vertices. The distribution does not allow v_2 to be pebbled.

In [4], the problem of determining necessary and sufficient conditions for a graph G to be Class 0 is given. Most results in this direction, including those surveyed in [4], focus on conditions on the diameter and connectivity of G. A result in [5], which we discuss below, gives a sufficient condition in regards to the number of edges of G. Here, we give a sufficient degree sum condition, which is best possible, for G to be Class 0.

Theorem 1 *If $\sigma_2(G) \geq n$, then G is Class 0.*

The proof of Theorem 1 is essentially the same as the proof of Theorem 2 in Czygrinow and Hurlbert [2], so we do not present it here. However, as a result we obtain the following.

Corollary 2 *If $\delta(G) \geq \lceil \frac{n}{2} \rceil$, then G is Class 0.*

In [2], it was incorrectly claimed that if $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$ then G is Class 0. The error in [2] occurred in the proof of the lower bound on $\delta(G)$. To see this, consider when n is odd the following graph G - which has minimum degree $\lfloor \frac{n}{2} \rfloor$, but is not Class 0. Let G be the graph of two complete graphs of order $\lceil \frac{n}{2} \rceil$ intersecting in a single vertex. This graph contains a cut-vertex, so it cannot be Class 0. Thus we must necessarily have $\delta(G) \geq \lceil \frac{n}{2} \rceil$. The proof that this bound is sufficient to guarantee membership in Class 0 holds as given in [2] with $\lfloor \frac{n}{2} \rfloor$ replaced by $\lceil \frac{n}{2} \rceil$, but also follows immediately from Theorem 1.

We are able to offer the following new result.

Theorem 3 *Let G be a graph on $n \geq 6$ vertices. If for each maximal independent set, S, of G we have*

$$\sum_{v \in S} d(v) \geq (|S| - 1)(n - |S|) + 2$$

then G is Class 0.

Using this result, we can see that the complete multipartite-graph, $K_{p_1,\ldots,p_t}$, with partite sets $P_1, \ldots, P_t$ where $|P_i| = p_i$, $t \geq 2$ and $1 \leq p_1 \leq p_2 \leq \ldots \leq p_t$, is Class 0 as long as $\Sigma p_i \geq 6$, except in the case $t = 2$ and $p_1 = 1$. Notice that the only maximal independent sets are $P_1, \ldots, P_t$.

We also obtain the following corollary due to Pachter, Snevily and Voxman [5].

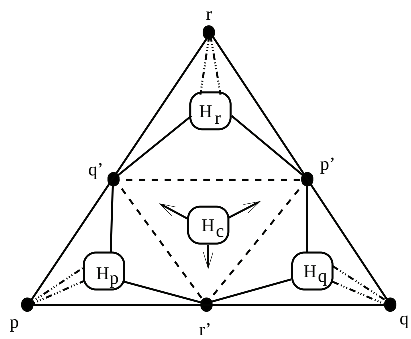

Figure 1: The family $\mathcal{F}$

Corollary 4 [5] *If G is a graph on $n \geq 4$ vertices and $|E(G)| \geq \binom{n-1}{2} + 2$, then G is Class 0.*

For a survey of results in graph pebbling, we refer the reader to [3] and [4].

2 Proof of Main Result

To see that the conditions given in Theorem 1 and Theorem 3 are best possible, consider the following construction. Let G' be any graph on $n-k$ vertices with vertex set $\{x_1, \ldots, x_{n-k}\}$. To G' we add vertices $\{x_{n-k+1}, \ldots, x_n\}$ with each vertex in $\{x_{n-k+1}, \ldots x_{n-1}\}$ adjacent to each vertex in $\{x_1, \ldots, x_{n-k}\}$ and x_n adjacent only to x_1; denote this graph by G. In G, the set of vertices $x_{n-k+1}, \ldots, x_n$ forms an independent set of size k with degree sum $(k-1)(n-k)+1$, yet G is not Class 0 as it contains a cut-vertex, x_1.

To prove Theorem 3 we first present a result from [1]. To do this, we must first describe a class of graphs $\mathcal{F}$. The class, $\mathcal{F}$, we give is a correction to the one given in [1], yet the corresponding result (and its proof) still holds.

We refer the reader to Figure 1. To begin, each $F \in \mathcal{F}$ has a six-cycle $C_6 = pr'qp'rq'p$. For each vertex p, q and r there exists a subgraph (possibly the empty graph), denoted H_p, H_q and H_r, respectively. In each component of H_p there exists a vertex adjacent to p (denoted by double dotted lines), and each vertex in H_p is adjacent to q', r' (denoted by solid lines). Similar statements hold for H_q and H_r. Further, in each F there exists a subgraph, denoted H_c, in which each vertex is adjacent to at least two of $\{p', q', r'\}$ (denoted by arrowed lines). At least two of the edges $p'q', q'r', r'p'$ exist (denoted by dashed lines). Edges may not exist between any pair H_i, H_j. This describes all possible edges of a member of $\mathcal{F}$.

Theorem 5 [1] *If a graph G on $n \geq 6$ vertices has diameter two, connectivity at*

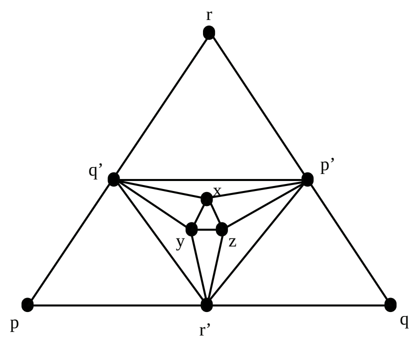

Figure 2: A graph F with diameter two, connectivity two, but not Class 0

least two and $\pi(G) > n$*, then* $G \in \mathcal{F}$*.*

The class of graphs given in [1] differs from the class $\mathcal{F}$. It differs from $\mathcal{F}$ only in H_c, all other aspects are the same. In [1], H_c was essentially described as consisting of two parts $H_{c'}$ and $H_{c''}$, where $H_{c'}$ consists of those vertices in H_c adjacent to vertices p' and q' only and $H_{c''}$ consists of all other vertices in H_c. It was specified that edges did not exist between $H_{c'}$ and $H_{c''}$, however we show that this need not be the case and this leads to our description of $\mathcal{F}$ as given above.

To see that the description of $\mathcal{F}$ given here is more broad than the one given in [1], we give a graph F which has diameter two and connectivity two, is not Class 0 and is not a member of the family of graphs described in [1]. We refer the reader to the graph given in Figure 2. By inspection, it is easy to see that F has diameter two and connectivity two. The graph F is not Class 0 since if we let $D(p) = D(q) = 3, D(r) = D(p') = D(q') = D(r') = 0$ and $D(x) = D(y) = D(z) = 1$ then it is impossible to pebble r.

A consequence of Theorem 5, as shown in [1], is that if a graph G has diameter two and connectivity at least three then G is Class 0. The discussion following the statement of Theorem 3 gives an instance of a graph shown to be Class 0 by Theorem 3, but not by this consequence - namely the graph $K_{2,n-2}$.

We now give two preparatory propositions towards the proof of Theorem 3. Given a graph $F \in \mathcal{F}$ and a set of vertices, R, in $V(F)$ we say that R has *Property 1* if R contains a vertex in each of $H_p \cup p, H_q \cup q$ and $H_c \cup H_r \cup r$ and does not contain any element of $\{p', q', r'\}$.

Proposition 6 *If* $F \in \mathcal{F}$ *and* $\alpha(F) \geq 3$*, then for each* s*,* $3 \leq s \leq \alpha(F)$*, there is an independent set of size* s *in* F *with Property 1.*

Proof: Let $F \in \mathcal{F}$ and S be an independent set in $V(F)$ of size at least three.

Assume that S does not have Property 1 and we will show that there exists an independent set T with $|T| = |S|$ having Property 1.

If S does not contain any vertex from $\{p', q', r'\}$, then as there are no edges between $H_p \cup p$ and $H_q \cup q$, $H_p \cup p$ and $H_c \cup H_r \cup r$, and $H_q \cup q$ and $H_c \cup H_r \cup r$, we may remove a single vertex from S and replace it by any vertex from the set which it does not intersect, while at the same time ensuring the number of sets it intersects increases, to form S'. If S' has Property 1 then we let $T = S'$. Otherwise, we repeat this procedure to form S'', at which point we are guaranteed that S'' has Property 1 and so we let $T = S''$.

Otherwise, S does contain a vertex from $\{p', q', r'\}$. The set S may contain at most two vertices from $\{p', q', r'\}$, as this set induces at least two edges. First consider if S contains precisely one such vertex. Let's say $p' \in S$, then as p' is adjacent to each vertex in $H_q \cup q$ and $r' \notin S$ we may replace p' by any vertex in $H_q \cup q$ to form an independent set, S'. Similarly, if $q' \in S$, or $r' \in S$, then a similar procedure may be performed to form an independent set, S'. The set S' has $|S'| = |S|$ and does not contain any vertex in $\{p', q', r'\}$, so by the previous case either S' has Property 1 or we may find a suitable set T.

Finally, we consider the case in which S contains two vertices from $\{p', q', r'\}$. Regardless of the choice of the two, every remaining vertex in the graph will be adjacent to at least one of the two. Thus the set S cannot contain any other vertices from G. This contradicts that the size of S is at least three. □

For a positive integer a, we define a positive integer partition of length t of a to be a vector $\mathbf{a} = (a_1, \ldots, a_t)$ such that $a_1 + \ldots + a_t = a$ and for $1 \le i \le t$ we have $a_i \in \mathbb{Z}^+$.

Proposition 7 *Let a, b, t be positive integers with $b \ge a \ge t$. Let $\mathbf{a}, \mathbf{b}$ be positive integer partitions of length t of a and b, respectively. If for $1 \le i \le t$ we have $b_i \ge a_i$ then*

$$f(\mathbf{a}, \mathbf{b}) = \Sigma_{i=1}^{t} a_i(b_i - a_i) \tag{1}$$

is maximized when for some i we have $a_i = a - (t-1), b_i = b - (t-1)$, and so $a_j = b_j = 1$ for all $j \ne i$ and $f(\mathbf{a}, \mathbf{b}) = (a - (t-1))(b - a)$.

We delay the proof of Proposition 7, a purely number theoretic result, until after the proof of our main result which we now give.

The proof of Theorem 3 is based on arguments given in [2].

Proof of Theorem 3: Let G be given according to the conditions of Theorem 3. If $\alpha(G) = 1$, then $G = K_n$ and the result holds trivially. If $\alpha(G) = 2$, then the condition of Theorem 1 holds and G is Class 0.

Thus we may assume that $\alpha(G) \ge 3$ and suppose G is not Class 0. We begin by showing that G must belong to $\mathcal{F}$. Let $x, y \in V(G)$ such that $xy \notin E(G)$ and let S be any maximal independent set containing both x and y. As S is independent, the maximum degree of a vertex in S is $n-|S|$. This fact and the hypothesis imply that we have,

$$\begin{aligned} d(x)+d(y) &\geq (|S|-1)(n-|S|)+2-(|S|-2)(n-|S|) \\ &= (n-|S|)+2. \end{aligned}$$

The pigeonhole principle implies that x and y must share at least two common neighbors, and so the diameter of G is at most two. The diameter is at least two since $\alpha(G) \geq 3$, and so the diameter must be equal to two. We can also reach the conclusion that between any pair of non-adjacent vertices there exists at least two vertex disjoint x,y-paths. Now consider $x, y \in V(G)$ such that $xy \in E(G)$, we seek to find an x,y-path distinct from the edge xy. This will show that between any two vertices in G there exists two vertex disjoint paths and so, by a theorem of Whitney [6], G is 2-connected. As G is connected at least one of x and y has another neighbor, say x does and call x's neighbor u. If $uy \in E(G)$ then uy is the second path we seek. Thus we may assume that $uy \notin E(G)$ and let S' be a maximal independent set containing both u and y. Then, as above, we may show that u and y have at least two neighbors in common, one of which, say v, is distinct from x. We then have $xuvy$ as an x,y-path distinct from the edge xy. Thus G is 2-connected.

As G has diameter 2, is 2-connected and, by assumption, is not Class 0, then by Theorem 5 G is in $\mathcal{F}$. Now consider a maximal independent set S such that $|S| = \alpha(G)$. We apply Proposition 6 to S to obtain an independent set T with $|T| = \alpha(G)$ and T has Property 1. Let's say that i vertices from T are in $H_p \cup p$, j vertices from T are in $H_q \cup q$ and k vertices from T are in $H_c \cup H_r \cup r$. We then have the following,

$$\begin{aligned} \sum_{u\in T, u\in H_p\cup p} d(u) + \sum_{v\in T, v\in H_q\cup q} d(v) + \sum_{w\in T, w\in H_c\cup H_r\cup r} d(w) &\leq i(|H_p\cup p|+2-i)+j(|H_q\cup q|+2-j) \\ &\quad +k(|H_c\cup H_r\cup r|+2-k) \\ &= 2(i+j+k)+i(|H_p\cup p|-i)+j(|H_q\cup q|-j) \\ &\quad +k(|H_c\cup H_r\cup r|-k) \\ &= 2\alpha(G)+i(|H_p\cup p|-i)+j(|H_q\cup q|-j) \\ &\quad +k(|H_c\cup H_r\cup r|-k). \end{aligned}$$

Note that $i,j,k \geq 1$, $i+j+k = \alpha(G)$ and $|H_p \cup p|+|H_q \cup q|+|H_r + H_c + r| = n-3$. We may now apply Proposition 7 with $a = \alpha(G), b = (n-3)$ and $t = 3$. As a result, the sum on the right-hand side of the above inequality is at most $2\alpha(G) + (\alpha(G) - 2)(n-3-\alpha(G))$. However, this quantity is less than $(\alpha(G)-1)(n-\alpha(G))+2$ when $n > 4$. That is, we obtain a contradiction to the degree sum condition. Thus G is Class 0. □

We now present the proof of Corollary 4.

Proof of Corollary 4: For $n = 4, 5$, we can see by inspection that the claim is true. Now, for $n \geq 6$ let $G = (V, E)$ be as given and consider any independent set S in G. By the edge count we see that G has at most $n - 3$ non-edges. The set S contains exactly $\binom{|S|}{2}$ non-edges and so $G[S, V \setminus S]$ contains at most $n - 3 - \binom{|S|}{2}$ non-edges. We then have,

$$\begin{aligned} \sum_{v \in S} d(v) &\geq |S|(n - |S|) - (n - 3 - \binom{|S|}{2}) \\ &\geq (|S| - 1)(n - |S|) + 2. \end{aligned}$$

Thus, by Theorem 3, G is Class 0. □

We now present the proof of Proposition 7.

Proof of Proposition 7: Let a, b, t be positive integers with $b \geq a \geq t$ and let $\mathbf{a}, \mathbf{b}$ be any positive integer partitions of length t, respectively, for which $b_i \geq a_i$, $1 \leq i \leq t$.

First suppose that $a = b$. In this case, $\Sigma a_i = \Sigma b_i$ and so $\Sigma(b_i - a_i) = 0$. As $b_i \geq a_i$ we must have that $b_i = a_i$ for all i. Thus $f(\mathbf{a}, \mathbf{b}) = 0$ and the conclusion holds true trivially.

We may now consider when $b > a$. As $b > a$, there is an i for which $b_i > a_i$. Let $d_i = b_i - a_i > 0$. If for some $j \neq i$ we have $a_j \geq a_i$ then choose the largest such a_j. If there is more than one choice, then of these choose the one with the largest such j. We may then replace $\mathbf{b}$ by $\mathbf{b_1} = (b_1, \ldots, b_i - d_i, \ldots, b_j + d_i, \ldots b_t)$. We then have that $f(\mathbf{a}, \mathbf{b_1}) \geq f(\mathbf{a}, \mathbf{b})$ since

$$\begin{aligned} f(\mathbf{a}, \mathbf{b_1}) - f(\mathbf{a}, \mathbf{b}) &= a_i(b_i - d_i - a_i) + a_j(b_j - a_j + d_i) - [a_i(b_i - a_i) + a_j(b_j - a_j)] \\ &= d_i(a_j - a_i) \geq 0, \quad \text{since } a_j \geq a_i. \end{aligned}$$

Repeating this procedure until it is no longer possible allows us to replace $\mathbf{b_1}$ by some $\mathbf{b_2}$, so that in $\mathbf{b_2}$ there exists a unique j for which $b_j > a_j$. Fix this j.

In $\mathbf{b_2}$ for $i \neq j$ we have $a_i = b_i$. If we have for some i, $a_i, b_i > 1$ then we perform the following operation. Replace $\mathbf{a}$ and $\mathbf{b_2}$ by $\mathbf{a_3} = (a_1, \ldots, a_i - (a_i - 1), \ldots, a_j + (a_i - 1), \ldots, a_t)$ and $\mathbf{b_3} = (b_1, \ldots, b_i - (b_i - 1), \ldots, b_j + (b_i - 1), \ldots, b_t)$, respectively. We then have that $f(\mathbf{a_3}, \mathbf{b_3}) > f(\mathbf{a}, \mathbf{b_2})$ since,

$$\begin{aligned} f(\mathbf{a_3}, \mathbf{b_3}) - f(\mathbf{a}, \mathbf{b_2}) &= (a_j + (a_i - 1))(b_j - a_j) - a_j(b_j - a_j) \\ &= (a_i - 1)(b_j - a_j) \\ &> 0. \end{aligned}$$

Repeating this procedure until it is no longer possible allows us to replace $\mathbf{a}, \mathbf{b_2}$ by some $\mathbf{a}^*, \mathbf{b}^*$ so that there exists a unique j for which $b_j > a_j$ and for all $i \neq j$ we have $a_i = b_i = 1$ and $f(\mathbf{a}^*, \mathbf{b}^*) \geq f(\mathbf{a}, \mathbf{b})$. The conclusion now readily holds. □

Acknowledgment

The authors wish to thank the anonymous referee for his/her valuable suggestions.

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(Received 20 July 2007; revised 25 Sep 2007)