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On k-cordial labelling

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Abstract

Hovey [*Discrete Math.* 93 (1991), 183–194] introduced simultaneous generalizations of harmonious and cordial labellings. He defines a graph G of vertex set $V(G)$ and edge set $E(G)$ to be k-cordial if there is a vertex labelling f from $V(G)$ to $\mathbb{Z}_k$, the group of integers modulo k, so that when each edge xy is assigned the label $(f(x)+f(y)) \pmod{k}$, the number of vertices (respectively, edges) labelled with i and the number of vertices (respectively, edges) labelled with j differ by at most one for all i and j in $\mathbb{Z}_k$. In this paper we give some necessary conditions for a graph to be k-cordial for certain k. We also give some new families of 4-cordial graphs.

1 Introduction

All graphs in this paper are finite, simple and undirected. We follow the basic notation and terminology of graph theory as in [5] and of graph labelling as in [7].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labelling $f : V(G) \longrightarrow \mathbb{Z}_k$ induces an edge labeling $f^+ : E(G) \longrightarrow \mathbb{Z}_k$, defined by $f^+(xy) = f(x)+f(y)$, for all edges $xy \in E(G)$. For $i \in \mathbb{Z}_k$, let $n_i(f) = |\{v \in V(G) \mid f(v) = i\}|$ and $m_i(f) = |\{e \in E(G) \mid f^+(e) = i\}|$. A labelling f of a graph G is called k-cordial if $|n_i(f) - n_j(f)| \leqslant 1$ and $|m_i(f) - m_j(f)| \leqslant 1$ for all $i, j \in \mathbb{Z}_k$. A graph G is called k-cordial if it admits a k-cordial labelling. Using this terminology a cordial graph can be designated 2-*cordial*.

This definition of k-cordial labellings of graphs was introduced by Hovey [11] as a generalization of both harmonious and cordial labellings. Harmonious labellings were introduced by Graham and Sloane [9]. They defined a graph G of q edges to be harmonious if there is an injection f from $V(G)$ to $\mathbb{Z}_q$ such that the induced function

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f^* from $E(G)$ to $\mathbb{Z}_q$, defined by $f^*(xy) = f(x) + f(y)$ for all edges $xy \in E(G)$, is a bijection. Cordial labellings were introduced by Cahit [3] who called a graph G cordial if there is a vertex labelling $f : V(G) \longrightarrow \{0, 1\}$ such that the induced labelling $f^* : E(G) \longrightarrow \{0, 1\}$, defined by $f^*(xy) = |f(x) - f(y)|$, for all edges $xy \in E(G)$ and with the following inequalities holding: $|n_0(f) - n_1(f)| \leqslant 1$ and $|m_0(f) - m_1(f)| \leqslant 1$, where $n_i(f)$ (respectively, $m_i(f)$) is the number of vertices (respectievly, edges) labelled with i.

For the purpose of further putting the current work into context, we mention some variations of harmonious labelling which are related to our work. Chang et al. [4] have investigated subclasses of harmonious graphs. They defined a graph G of q edges to be *strongly c-harmonious* if there is a positive integer c and an injection f, called a strongly c-harmonious labelling, from $V(G)$ to $\{0, 1, 2, \dots, q-1\}$ such that the induced function f^+ from $E(G)$ to $\{c, c+1, \dots, c+q-1\}$, defined by $f^+(x, y) = f(x) + f(y)$ for all edges $xy \in E(G)$, is a bijection. If G is a tree, Grace [8] allows the vertex labels to range from 0 to q. Acharya and Hegde [1] introduced a stronger form of the strongly c-harmonious labelling by calling a (p, q) graph G *strongly k-indexable* if there is a positive integer k and an injection f, called a *strongly k-indexable labelling*, from $V(G)$ to $\{0, 1, 2, \dots p-1\}$ such that the set of edge labels induced by adding the vertex labels is $\{k, k+1, \dots, k+q-1\}$. Enomoto et al. [6] introduced *super edge-magic labellings*. Hedge and Shetty [10] showed that a graph has a strongly k-indexable labelling if and only if it has a super edge-magic labelling.

In the previously cited paper by Hovey [11], he obtained the following results: caterpillars are k-cordial for all k; all trees are k-cordial for $k = 2, 3, 4$ and 5; odd cycles with pendant edges attached are k-cordial for all k; cycles are k-cordial for all odd k; for k even, C_{2mk+j} is k-cordial when $0 \leqslant j < \frac{k}{2} + 2$ and when $k < j < 2k$; $C_{(2m+1)k}$ is not k-cordial for even k; K_n is 3-cordial; and K_{nk} is k-cordial if and only if $n = 1$. He also advanced the following conjectures: all trees are k-cordial for all k; all connected graphs are 3-cordial; and C_{2mk+j} is k-cordial if and only if $j \neq k$, where k and j are even and $0 \leqslant j < 2k$. The last conjecture was verified by Tao [12]. The reference [7] surveys the current state of knowledge for all variations of graph labellings appearing in this paper.

In the next section of this paper we derive some necessary conditions on k-cordial labelings for certain k. In Section 3, we give some relations between k-cordial labellings and some other labellings, and we show that some families of graphs are 4-cordial.

2 Necessary conditions on k-cordial graphs

In this section, we give some necessary conditions for a graph to be k-cordial for certain values of k.

Note that if G is a (p, q) graph having k-cordial labelling f, then

$$\left\lfloor \frac{p}{k} \right\rfloor \leqslant n_i(f) \leqslant \left\lceil \frac{p}{k} \right\rceil \quad \text{and} \quad \left\lfloor \frac{q}{k} \right\rfloor \leqslant m_i(f) \leqslant \left\lceil \frac{q}{k} \right\rceil$$

for all $i \in \mathbb{Z}_k$.

Lemma 1 *If G is a (p,q) k-cordial graph with $p \equiv 0 \pmod{k}$, then $G + \overline{K}_n$ is k-cordial.*

Proof. Let $V(\overline{K}_n) = \{v_1, v_2, \ldots, v_n\}$ and let f be a k-cordial labeling of G. Define $g : V(G + \overline{K}_n) \longrightarrow \mathbb{Z}_k$ as $g(v) = f(v)$ if $v \in V(G)$ and $f(v_i) = (i-1) \pmod{k}$, $1 \leqslant i \leqslant n$. Clearly g is a k-cordial labelling of $G + \overline{K}_n$. □

Theorem 1 *Let k be even and G be a d-regular (p,q) graph with $p, q \equiv 0 \pmod{k}$. If G is k-cordial, then $q \equiv 0 \pmod{2k}$.*

Proof. Let G be k-cordial having k-cordial labelling f with $p, q \equiv 0 \pmod{k}$; then $\sum_{e \in E(G)} f^+(e) = \sum_{v \in V(G)} \deg(v) f(v) \pmod{k}$, which implies that

$$\frac{q}{k} \sum_{i=0}^{k-1} i \;=\; d\,\frac{p}{k} \sum_{i=0}^{k-1} i \pmod{k}.$$

So $q(k-1)/2 \equiv 0 \pmod{k}$, and hence $q \equiv 0 \pmod{2k}$. □

Corollary 1 *Let k be even. Then*

(i) C_n is not k-cordial if $n \equiv k \pmod{2k}$ [11]*;*

(ii) $C_n \times P_2$ is not k-cordial if $n \equiv k \pmod{2k}$.

The following result gives a necessary condition for a graph to be k-cordial for k having the form 2^t where t is a positive integer.

Theorem 2 *If G is an Eulerian k-cordial graph with $q \equiv 0 \pmod{k}$ edges, and if the degree of every vertex of G is divisible by 2^t $(t \geqslant 1)$ and $k \mid 2^t$, then $q \equiv 0 \pmod{2k}$.*

Proof. The structure of the proof is the same as in Theorem 1.

Corollary 2 *(i)* [3] *If G is an Eulerian graph of size $q \equiv 2 \pmod{4}$, then G is not 2-cordial.*
(ii) If G is a graph with size $q \equiv 4 \pmod{8}$, and with the degree of every vertex of G being divisible by 4, then G is not 4-cordial. In particular, C^2_{4n+2} is not 4-cordial for all n.

Proof. This is immediate. □

Corollary 3 *Let G be a (p,q) odd degree graph with $p \equiv 0 \pmod{2^t}$ such that $2^t \mid \deg(v) + 1$ for every vertex $v \in V(G)$, and let k be a proper divisor of 2^t. If G is k-cordial, then $p + q \equiv 0 \pmod{2k}$.*

Proof. Let G be k-cordial having k-cordial labelling f. Since $p \equiv 0 \pmod{k}$, by Lemma 1, $G + K_1$ is k-cordial of size $p + q \equiv 0 \pmod{k}$ and 2^t divides the degree of its vertices; then applying Theorem 2, we obtain $p + q \equiv 0 \pmod{2k}$. □

3 4-cordial graphs

Hovey [11] observed that a graph G is harmonious if and only if G is $|E(G)|$-cordial. One direction of this result is not strictly true. The graph $3K_2$ is 3-cordial (by labelling the first component of the given graph by 0 and 1, the second by 0 and 2, and the third by 1 and 2) but not harmonious in the sense of the definition of harmonious graph due to Graham and Sloane [9]. His result may be restated as: every harmonious (p, q) graph is q-cordial, and the converse is true if $q \geqslant p - 1$. In this section we give some relations between k-cordial labelling and some other labellings, and we obtain some new families of 4-cordial graphs.

Theorem 3 *If G is a (p, q) strongly c-harmonious graph with $p = q$, $p = q + 1$ or $p = q - 1 = 2t + 1$, then G is 2-cordial.*

Proof. Let f be a strongly c-harmonious labeling of G and define $g : V(G) \longrightarrow \mathbb{Z}_2$ by

$$g(v) \;=\; f(v) \pmod 2, \text{ for every } v \in V(G).$$

If G is strongly c-harmonious with $p = q$ or $p = q+1$, then clearly $|n_0(g) - n_1(g)| \leqslant 1$ and if G has $p = q - 1 = 2t + 1$, then since there are exactly $t + 1$ odd integers in the set $\{0, 1, \ldots, 2t + 1\}$, we also have $|n_0(g) - n_1(g)| \leqslant 1$ in this case as well. Finally, observe that since $g^+(v) = f^+(v) \pmod 2$, and f^+ is a bijection onto an interval of positive integers, then $|m_0(g) - m_1(g)| \leqslant 1$. □

The following lemma is straightforward and we omit the proof.

Lemma 2 *If G is a (p, q) strongly c-harmonious graph, then G is k-cordial for every $k \geqslant q$.*

We define a graph G to be *perfectly cordial* if it is k-cordial for all k. The conjecture of Hovey [11] about trees may be restated as "Every tree is perfectly cordial".

Theorem 4 *Every strongly t-indexable graph is perfectly cordial.*

Proof. Let G be a strongly t-indexable graph. Reducing the vertex and edge labels modulo k, we obtain a k-cordial labeling of G. □

Note that the converse of the above theorem is not true; for example, the reader may show that the graph $K_5 - e$ (K_5 with an edge deleted) is perfectly cordial but not strongly k-indexable for any k since a necessary condition for a (p, q) graph to be strongly k-indexable is $q \leqslant 2p - 3$. Another example: K_4 is k-cordial for all $k \geqslant 3$, but not strongly k-indexable for any k. Examples of perfectly cordial graphs are $C_{2n+1} \times P_m$, P_n^2, and many other families of graphs in [2] and [7].

For any $n \geqslant 3$, let C_n^{+1} denote the graph formed by adding a pendant edge to one vertex of a cycle of order n. Although Hovey [11] showed that all odd cycles with pendant edges attached are k-cordial for all k, this is not always true for even cycles. However, we prove the following result:

Theorem 5 *C_{2k}^{+1} is not $(2k+1)$-cordial for all $k \geqslant 2$.*

Proof. Let $V(C_{2k}^{+1}) = \{v_1, v_2, \ldots, v_{2k}\} \cup \{u\}$, where $E(C_{2k}^{+1}) = \{v_iv_j \mid i-j \equiv \pm 1 \pmod{2k}\} \cup \{v_1u\}$. Suppose that C_{2k}^{+1} has $(2k+1)$-cordial lavbelling f; then both f and f^+ are bijections, and $\sum_{e\in E(C_{2k}^{+1})} f^+(e) = \sum_{i=0}^{2k} i = k(2k+1)$. On the other hand,

$$\sum_{e\in E(C_{2k}^{+1})} f^+(e) = f(u) + f(v_1) + 2\sum_{i=0}^{2k} f(v_i) = f(v_1) - f(u) + 2k(2k+1).$$

Then $f(v_1) - f(u) \equiv 0 \pmod{2k+1}$, which contradicts the fact that f is a bijection. Hence C_{2k}^{+1} is not $(2k+1)$-cordial. □

Theorem 6 *K_n is 4-cordial if and only if $n \leqslant 6$.*

Proof. If $n \geqslant 8$, then K_n is not 4-cordial, by Hovey [11]. If $n = 7$, suppose that K_7 has 4-cordial labeling f; then there exists $i \in \mathbb{Z}_4$ such that $n_i(f) = 1$. If $i = 0$ or 2, then $m_2(f) = 4$, which is absurd. If $i = 1$ or 3, then $m_0(f) = 4$, which again is a contradiction. Hence K_7 is not 4-cordial. If $1 \leqslant n \leqslant 6$, let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$; then K_n has 4-cordial labelling f described as follows: $f(v_i) = (i-1) \pmod 4$, $1 \leqslant i \leqslant 5$, and $f(v_6) = 2$. □

Theorem 7 *C_n^2 is 4-cordial if and only if $n \not\equiv 2 \pmod 4$.*

Proof. Necessity follows from Corollary 2. For sufficiency, let $V(C_n^2) = \{v_1, v_2, \ldots, v_n\}$. If $3 \leqslant n \leqslant 5$, then $C_n^2 = K_n$, which is 4-cordial by Theorem 6. Let $f : V(C_n^2) \longrightarrow \mathbb{Z}_4$. If $n \equiv 0$ or $1 \pmod 4$, $n \geqslant 8$, define $f(v_i) = (i-1) \pmod 4$, $1 \leqslant i \leqslant n$. If $n \equiv 3 \pmod 4$, let $n = 4m+3$, $m \geqslant 1$. We label the vertices of the cycle by the successive labels: $\left(\prod_{i=1}^{m-1}(0\ 1\ 2\ 3)\right)(2\ 3\ 1\ 3\ 1\ 0\ 2)$. The reader may verify that in each case f is a 4-cordial labeling. □

Let $K_{m,n}$ be the complete bipartite graph with $V(K_{m,n}) = \{u_i \mid 1 \leqslant i \leqslant m\} \cup \{v_j \mid 1 \leqslant j \leqslant n\}$ and $E(K_{m,n}) = \{u_iv_j \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$.

Theorem 8 *$K_{m,n}$ is 4-cordial if and only if m or $n \not\equiv 2 \pmod 4$.*

Proof. For necessity, suppose that $K_{m,n}$, with m and $n \equiv 2 \pmod 4$ is 4-cordial, with 4-cordial labeling f; then

$$\sum_{e\in E(K_{m,n})} f^+(e) = \left(\sum_{v\in V(K_{m,n})} \deg(v) f(v)\right) \pmod 4,$$

and

$$\begin{aligned}
\frac{3mn}{2} &= \left(n\sum_{i=1}^{m} f(u_i) + m\sum_{j=1}^{n} f(v_j)\right) \pmod 4, \\
\text{then } \frac{3mn}{2} &= \left(n\sum_{i=1}^{m} f(u_i) + m\left(\frac{3(m+n)}{2} - \sum_{i=1}^{m} f(u_i)\right)\right) \pmod 4, \\
\text{then } \frac{3mn}{2} &= \left((n-m)\sum_{i=1}^{m} f(u_i) + \frac{3m(m+n)}{2}\right) \pmod 4, \\
\text{and hence } \frac{3mn}{2} &\equiv 0 \pmod 4,
\end{aligned}$$

and so $mn \equiv 0 \pmod 8$, a contradiction.

Conversely, define $f : V(K_{m,n}) \longrightarrow \mathbb{Z}_4$ as follows:

$$\begin{aligned}
f(u_i) &= (i-1) \pmod 4, \quad 1 \leqslant i \leqslant m, \\
f(v_j) &= (m-1+j) \pmod 4, \quad 1 \leqslant j \leqslant n.
\end{aligned}$$

Let $A = (a_{ij})$ be a matrix of order $m \times n$, where $a_{ij} = f(u_i) + f(v_j)$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant m$. The matrix A represents the edge labels of $K_{m,n}$. Note that the elements of A in each row or column are consecutive non-negative integers modulo 4. We have the following cases:

Case 1: $m \equiv 0 \pmod 4$
In this case, each column of A contains an equal number of edges labelled i, $i \in \mathbb{Z}_4$, so $K_{m,n}$ is 4-cordial for all n in this case.

Case 2: $m \equiv 1 \pmod 4$
Again, each column of A contains an equal number of edges labelled i, $i \in \mathbb{Z}_4$, in addition to one edge labelled j, $j \in \mathbb{Z}_4$. These additional edge labels in each column are consecutive, so we may write them as: $j,\ j+1,\ j+2,\ j+3,\ j,\ j+1, \ldots$, and again $K_{m,n}$ is 4-cordial for all n.

Case 3: $m \equiv 2 \pmod 4$
Once again, each column of A contains an equal number of edges labelled i, $i \in \mathbb{Z}_4$, in addition to two edges labelled j and $j+1$, $j \in \mathbb{Z}_4$. We may write these additional edge labels in all columns as: $j, j+1; j+1, j+2; j+2, j+3; j+3, j; \ldots$. If $n \not\equiv 2 \pmod 4$, then $K_{m,n}$ is 4-cordial .

Case 4: $m \equiv 3 \pmod 4$
As above, each column of A contains an equal number of edges labelled i, $i \in \mathbb{Z}_4$, in addition to three edges labelled j, $j+1$ and $j+2$, $j \in \mathbb{Z}_4$. We may write these additional edge labels in all columns as:

$$j, j+1, j+2; j+1, j+2, j+3; j+2, j+3, j; j+3, j, j+1; \ldots.$$

Again, $K_{m,n}$ is 4-cordial for all n. □

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